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# Invariant Manifolds for Semilinear Partial Differential Equations

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## 1 INTRODUCTION

When studying the behaviour of a dynamical system in the neighbourhood of an equilibrium point the first step is to construct the stable, unstable and centre manifolds. These are manifolds that are invariant under the flow relative to a neighbourhood of the equilibrium point and carry the solutions that decay or grow (or neither) at certain rates. These ideas have a long history, see for instance Poincaré [32] and Hadamard [11]. Sophisticated recent results can be found in Fenichel [7], Hirsch, Pugh and Shub [17] and Kelley [22].

There are two parallel traditions in the history of invariant manifolds. One dates back to Hadamard [11] and the other to Liapunov [26] and Perron [31]. The fundamental idea is the same: if the linearized system at an equilibrium point has subspaces with asymptotic rates that are well enough separated, then they can be used to get curved versions (manifolds) of these subspaces for the nonlinear system. However, the techniques that are used to produce these manifolds have different flavours. The Liapunov–Perron approach is functional-analytic and the strategy is to find the manifolds as fixed points of a certain integral equation. The Hadamard approach is more directly geometrical, using the splitting between different subspaces to gain estimates on the projections of the flow in the different directions.

The purpose of this paper is to prove some of these theorems in the context of semilinear partial differential equations. This amounts to proving the theorems in an infinite-dimensional context with assumptions weak enough to allow differential operators in the linear part. Henry [15] has many results of this kind but he assumes that the semigroup of the linearized operator is analytic. This allows the use of fractional powers of that operator and hence derivatives in the nonlinearity, but prevents application to some important problems such as the nonlinear wave equation. We shall base our theory within the context of  $C_0$ -semigroups (see Pazy [30]) and give some applications that are not covered by Henry's theory. In the context of partial differential equations, results are also given by Carr [4], Chow and Hale [5], and Keller [23]. These also use  $C_0$ -semigroups but restrict to cases where the resulting centre manifold is finite-dimensional. Our results will cover cases where the centre manifold is infinite-dimensional, as we show in section 4 for the Nonlinear Klein–Gordon equation. All of this work falls well within the Liapunov–Perron tradition.

The most comprehensive set of results is due to Ball [1]. Our hypotheses are very similar to Ball's, in fact somewhat stronger, and the resulting theorems are close. However, the proofs are completely different. Ball adopts the strategy of Hale [12], which is the Liapunov–Perron approach. Our approach is to construct invariant cones in the Banach space to carry the information about the splitting into subspaces associated with different asymptotic rates. This is essentially the Hadamard style of approach except that we do use the idea of globalizing the problem (see section 2) some form of which seems unavoidable if 'centre' directions (those associated to eigenvalues of zero real part) are included.

There are inherent difficulties in constructing invariant cones in a Banach space due to the lack of an inner product. Our strategy has been to renorm the spaces and then apply the variation of constants formula to verify the invariance of relevant cones. This technique is of interest in itself as it allows one to perform estimates on the projected flows for small time and not just as  $t \rightarrow \infty$ . Attracting invariant manifolds that are global but finite-dimensional (so-called inertial manifolds) have been the focus of much attention recently,

### 1. Introduction

see for instance Foias, Sell and Temam [10] and Foias *et al.* [9]. In [9] and in Mallet-Paret and Sell [27] the authors use an invariant cone technique in Hilbert space to produce these inertial manifolds. It is hoped that some of our results will aid in extending this to the context of Banach spaces.

A difficulty that someone encounters when entering the field of invariant manifolds is in understanding exactly what a centre, centre-stable, or centre-unstable manifold is. Their definitions are usually only given obliquely through the statements of the theorems. We have taken the approach of defining clearly each of the invariant manifolds and then couching the theorems as statements of existence. The plethora of properties that these manifolds enjoy are listed after the theorems.

Many of the ideas in this paper were learnt by the authors from C. Conley, in particular the idea of mixing invariant cones with topological arguments. A good example of an approach of this kind is the paper of McGehee [28] which contains a stable manifold theorem for maps. Conley (and McGehee) was thinking in finite dimensions and the main difficulty in our work has been extending these ideas to infinite dimensions. The basic technique for obtaining invariant cones is a renorming followed by an application of Gronwall's lemma.

Consider the semilinear equation

$$u_t = Au + f(u) \quad (1.1)$$

where  $u \in X$ , a Banach space, and  $t$  represents time. Throughout this paper we shall assume the following about  $A$  and  $f$ .

(H1)  $A: X \rightarrow X$  is a closed, densely defined linear operator that generates a  $C_0$ -semigroup on  $X$ , call it  $S(t)$  (see Pazy [30]).

(H2) The spectrum of  $A$ ,  $\sigma(A) = \sigma^s \cup \sigma^c \cup \sigma^u$  with

$$\sigma^s = \{\lambda \in \sigma(A): \operatorname{Re} \lambda < 0\}$$

$$\sigma^c = \{\lambda \in \sigma(A): \operatorname{Re} \lambda = 0\}$$

$$\sigma^u = \{\lambda \in \sigma(A): \operatorname{Re} \lambda > 0\}$$

where  $\sigma^s$ ,  $\sigma^c$  and  $\sigma^u$  are spectral sets (open and closed subsets of  $\sigma(A)$ ), two of which are bounded.

(H3) The nonlinearity  $f$  is defined on  $X$  and is locally Lipschitz. Also  $f(0) = 0$  and for all  $\varepsilon > 0$ , there exists a neighbourhood  $U = U(\varepsilon)$  of 0 such that  $f$  has Lipschitz constant  $\varepsilon$  in  $U$ .

*Remarks* (1) The hypothesis (H3) says that  $f(u)$  is higher order. Since we do not require differentiability, we shall only get weak results about the smoothness of the invariant manifolds, namely that they are Lipschitz.

(2) In many applications, the manifolds being Lipschitz is sufficient. We shall prove some stability and instability results in this paper for which this suffices.

be viewed as a compact perturbation of one whose spectrum can be estimated directly.  
 (2) The terms dissipative and conservative do not conform to any standard usage. We use them as they are suggestive of the applications. We shall be interested in the FitzHugh-Nagumo equation which falls under case (D) and a nonlinear wave equation which is (C).

By the assumptions on  $A$  and  $f$  we get existence and uniqueness of mild solutions to (1) for small time, see Pazy, p. 185 [30]. Mild solutions are solutions to the associated integral equation:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s)) \, ds.$$

In the following 'solution' will always mean 'mild solution'. If  $u_0 \in X$  there is a solution  $u \in C([0, T]; X)$  for some  $T > 0$  and  $u(0) = u_0$ . This  $u(t)$  gives us the usual conditions for a local semiflow. We shall use the notation.

$$\Phi_t(u_0) = u(t).$$

DEFINITIONS If  $V \subset U$ , we say that  $V$  is *positively invariant* relative to  $U$  if for each  $v \in V$

$$\bigcup_{s \in [0, t]} \Phi_s(v) \subset U \quad \text{implies} \quad \bigcup_{s \in [0, t]} \Phi_s(v) \subset V$$

for all  $t > 0$ .

A *backwards solution* branch for  $u_0 \in X$  on the interval  $[-t, 0]$  for some  $t > 0$  is a continuous curve  $u: [-t, 0] \rightarrow X$ , with the property that

$$\Phi_s(u(-s)) = u_0,$$

for  $s \in [0, t]$ . When a particular backwards solution branch is under consideration, we will write  $\Phi_{-s}(u_0) = u(-s)$ .

$V_+$  is *negatively invariant* relative to  $U$  if for every  $v \in V$  one has:  
 If a backwards solution branch for  $v$  exists then there is a  $t_0 > 0$  and a backwards solution branch,  $\Phi_{-s}(v)$ , for  $0 \leq s < t_0$ , with  $t_0$  maximal, so that

$$\bigcup_{s \in [0, t]} \Phi_{-s}(v) \subset U \quad \text{implies} \quad \bigcup_{s \in [0, t]} \Phi_{-s}(v) \subset V$$

for all  $0 \leq t < t_0$ .

$V$  is *invariant* relative to  $U$  if it is both positively and negatively invariant relative to  $U$ .

The definition of negative invariance is somewhat weak in that other solution branches need not lie in  $V$ . This is unfortunate but under the current assumptions it is impossible to gain control of all backwards solution branches as there can be a drastic breakdown of backwards uniqueness.

However, for such applications as bifurcation calculations some smoothness is needed. To obtain smoothness results from smoothness assumptions on  $f$  one can apply the results of this paper inductively, we shall postpone the details of this argument to a future paper.

(3) (H3) is satisfied if  $f \in C^1$  and  $Df(0) = 0$ .

Under the hypothesis (H2), there are invariant (under  $A$ ) subspaces associated to  $\sigma^s$ ,  $\sigma^c$  and  $\sigma^u$ , call these  $X^s$ ,  $X^c$ ,  $X^u$ , and  $X = X^s \oplus X^c \oplus X^u$ . The association is that

$$\sigma^s = \sigma(A|_{X^s}), \quad \sigma^c = \sigma(A|_{X^c}), \quad \sigma^u = \sigma(A|_{X^u}),$$

see Taylor [38]. Set  $X^{cs} = X^c \oplus X^s$  and  $X^{cu} = X^c \oplus X^u$ .

Let us introduce some notation. With  $*$  =  $s, c, u, cs, cu$ , let  $\pi^*: X \rightarrow X^*$  be the natural projection,  $A^* = A|_{X^*}$  and  $S^*(t) = S(t)|_{X^*}$ . Note that the projections  $\pi^*$  are continuous since two of the spectral sets are bounded. Furthermore  $A^*$  generates  $S^*(t)$ .

We shall give two further sets of assumptions. The first set we shall call the *dissipative case* (D).

(D1)  $\dim X^c < +\infty$

(D2)  $\dim X^u < +\infty$

(D3) There exists  $M > 0$  and  $\sigma > 0$  such that

$$\|S^*(t)\| \leq Me^{-\sigma t}$$

for all  $t > 0$  ( $\|\cdot\|$  here denotes the operator norm).

The second set is called the *conservative case* (C).

(C1)  $\dim X^u < +\infty$ .

(C2)  $\dim X^s < +\infty$ .

(C3) A generates a  $C_0$ -group  $S(t)$  and for all  $\rho > 0$  there exists  $M > 0$  such that  $\|S^c(t)\| \leq Me^{\rho t}$ , for all  $t$ .

Remarks (1) The assumption (D3) implies that  $\sigma^s \subset \{\lambda: \operatorname{Re} \lambda < -\sigma\}$  and (C3) implies similarly that the spectral set  $\sigma^c$  lies within the imaginary axis. Instead of (D3) and (C3) it would be more satisfying to make these assumptions about  $\sigma^s$  and  $\sigma^c$  respectively. However, these estimates are assumptions on the spectra of the semigroups and the relationship between the spectrum of a semigroup and that of its generator is a notoriously complex one. There are conditions on the semigroup that give an appropriate spectral mapping theorem, such as norm continuity, see Hille and Phillips [16], or the stronger assumption of analyticity, see Kato [21]. There are also assumptions on the resolvent that work, see Slemrod [35]. We take a different point of view, however, and in the applications use the fact that in semilinear problems the semigroup can often

If the precondition that the orbit lies in  $U$  is omitted, then the set is called *imply positively invariant*, negatively invariant or invariant, respectively. We will show that in a neighbourhood,  $U$  of 0 there are manifolds which are invariant relative to  $U$ :

$W^u$ , the local unstable manifold of 0  
 $W^s$ , the local stable manifold of 0  
 $W^c$ , a local centre manifold of 0  
 $W^{eu}$ , a local centre-unstable manifold of 0  
 $W^{cs}$ , a local centre-stable manifold of 0.

These manifolds can be used to give a 'curvilinear' coordinate system in  $X$  in a neighbourhood of the equilibrium  $u = 0$ .

**DEFINITIONS** Given a neighbourhood,  $U$ , of 0 we define

$W^s = \{u \in U; \Phi_t(u) \in U \text{ for all } t \geq 0 \text{ and } \Phi_t(u) \rightarrow 0 \text{ exponentially as } t \rightarrow \infty\}$   
 $W^u = \{u \in U; \text{ some backward branch } \Phi_t(u) \text{ exists for all } t < 0 \text{ and lies in } U, \text{ further, } \Phi_t(u) \rightarrow 0 \text{ exponentially as } t \rightarrow -\infty\}.$

Sometimes we write  $W^s(U)$  and  $W^u(U)$  to emphasize their dependence on the neighbourhood  $U$ .

From these definitions,  $W^s$  and  $W^u$  exist as sets, called the stable and unstable sets, respectively. It is trivial to check that they are both invariant relative to  $U$ . The purpose of the theorems is to show that they are indeed manifolds.

The remaining manifolds all involve centre directions. These are not determined as consisting of initial data whose resulting solutions satisfy growth conditions but rather as data whose solutions do *not* satisfy certain growth conditions. We cannot prove that the set of all such points forms a manifold as in general it will not, but it will contain a manifold. We take the approach of defining what such a manifold would be and then proving its existence.

**DEFINITIONS** Given a neighbourhood,  $U$ , of 0 a *centre-stable manifold* is a Lipschitz manifold  $Y \subset U$  such that

- (a)  $Y$  is invariant relative to  $U$ .
- (b)  $\pi^{cs}(Y)$  contains a neighbourhood of 0 in  $X^{cs}$ ,
- (c)  $Y \cap W^u = \{0\}$ .

A *centre-unstable manifold* is a Lipschitz manifold  $Y \subset U$  such that

- (a)  $Y$  is invariant relative to  $U$ ,
- (b)  $\pi^{cu}(Y)$  contains a neighbourhood of 0 in  $X^{cu}$ ,
- (c)  $Y \cap W^s = \{0\}$ .

**Remark** Conditions (a) and (b) say that we are dealing with an invariant manifold of adequate dimension. For a centre-unstable manifold, condition (c) says that it consists of  $u_0$  for which  $\Phi_t(u_0)$  does not tend exponentially to 0 as  $t \rightarrow +\infty$ . We do not know exactly what  $\Phi_t(u_0)$  does as  $t \rightarrow +\infty$ , it may or may not tend to 0 but it does not do so exponentially.

**DEFINITIONS** Given a neighbourhood,  $U$ , of 0, a *centre manifold* is a Lipschitz manifold  $Y \subset U$  such that

- (a)  $Y$  is invariant relative to  $U$ .
- (b)  $\pi^c(Y)$  contains a neighbourhood of 0, in  $X^c$ ,
- (c)  $Y \cap W^u = \{0\}$  and  $Y \cap W^s = \{0\}$ .

**THEOREM 1.1** Under the assumptions (H1–3) and either (C) or (D) there exists an open neighbourhood  $U$  of 0 in  $X$  such that

- (i)  $W^s$  is a Lipschitz manifold which is tangent to  $X^s$  at 0, that is, there exists a Lipschitz continuous function

$$h^s: \pi^s(U) \rightarrow X^{cu} \text{ whose graph is } W^s, \\ h^s(0) = 0 \text{ and } h^s \text{ is differentiable at 0 with } Dh^s(0) = 0.$$

- (ii) There is a centre-unstable manifold,  $W^{cu}$ , in  $U$  which is tangent to  $X^{eu}$  at 0. In fact, there exists a Lipschitz function

$$h^{cu}: \pi^{cu}(U) \rightarrow X^s \text{ whose graph is } W^{cu}, \\ h^{cu}(0) = 0 \text{ and } Dh^{cu}(0) = 0.$$

**THEOREM 1.2** Under the same assumption as Theorem 1.1 there exists an open neighbourhood  $U$  of 0 in  $X$  such that

- (i)  $W^u$  is a Lipschitz manifold which is tangent to  $X^u$  at 0, that is, there exists a Lipschitz continuous function

$$h^u: \pi^u(U) \rightarrow X^{cs} \text{ whose graph is } W^u, \\ h^u(0) = 0 \text{ and } Dh^u(0) = 0.$$

- (ii) There is a centre-stable manifold,  $W^{cs}$ , in  $U$  which is tangent to  $X^{cs}$ . In fact, there exists a Lipschitz function

$$h^{cs}: \pi^{cs}(U) \rightarrow X^s \text{ whose graph is } W^{cs}, \\ h^{cs}(0) = 0 \text{ and } Dh^{cs}(0) = 0.$$

**THEOREM 1.3** Under the same assumptions as Theorem 1.1 there exists an open neighbourhood  $U$  of 0 in  $X$  with a centre manifold,  $W^c$ , which is tangent to  $X^c$  at

0. In fact, there is a Lipschitz function

$$h: \pi(U) \rightarrow X^s \oplus X^u \text{ whose graph is } W^c, \\ h'(0) = 0 \text{ and } Dh'(0) = 0.$$

There are some further properties of these invariant manifolds which may be of interest. Their validity will be clear once the above theorems have been proved.

(P1)  $W^c = W^{cu} \cap W^{cs}$ . If  $X^u = \{0\}$  then  $W^{cu} = W^c$ , etc.

(P2) The centre-unstable manifold that we construct,  $W^{cu}$  is attracting in the sense that if  $u_0 \in U$  and  $\Phi_t(u_0) \in U$  for all  $t > 0$  then the  $\omega$ -limit set of  $u_0$  lies in  $W^{cu}$ .

(P3) The centre-stable manifold that we construct,  $W^{cs}$ , has a repulsion property. There is a sufficiently small neighbourhood  $V$  so that if  $u_0 \notin W^{cs}$  but  $u_0 \in V$  then  $\Phi_t(u_0)$  leaves  $V$  in some positive time. So if  $W^u \neq \emptyset$  there is a solution that leaves any sufficiently small neighbourhood of 0 in positive time, thus 0 is unstable. Note that if  $\Phi_t(u_0) \in V$  for all  $t \geq 0$  then we may conclude that  $u_0 \in W^{cs}$ .

**Remarks** If  $S(t)$  is a group, as in case (C), then  $W^{cu}$  as constructed below is  $W^{cs}$  in backward time and so  $W^{cu}$  enjoys the property (P3) in backward time. Furthermore, since  $W^c = W^{cs} \cap W^{cu}$ , any point in a sufficiently small neighbourhood of 0 which does not lie on  $W^c$ , the centre manifold constructed here, must leave that neighbourhood in forward time or in backward time. In both cases (C) and (D) one has backward existence for points on  $W^{cu}$  ( $\dim X^{cu} < \infty$  in case (D)). Thus, any centre manifold for which there are arbitrarily small neighbourhoods of 0, each of which intersects that manifold in an invariant set, must coincide with  $W^c$  on a sufficiently small neighbourhood of 0.

The name 'repulsion' perhaps seems inappropriate, however, if the system were linear then points not lying in  $X^{cs} = W^{cs}$  actually recede from that manifold exponentially fast in forward time and in a sense this is true in the nonlinear case also.

(P4) From the definition of  $W^s$  it follows that it is invariant relative to  $U$ . However, from the proof it will be seen that  $W^s$  is positively invariant provided  $U$  is chosen appropriately. Similarly  $W^u$  is negatively invariant.

A question of interest is whether these manifolds are unique or not, i.e. whether there are other manifolds that satisfy the same properties. For the stable and unstable manifolds, their definition makes them unique, given the neighbourhood  $U$ . They are both defined as the set of points satisfying a certain property. It is obvious that if  $V \subset U$ ,  $W^s(V) \subset W^s(U)$ , a natural question is whether  $W^s(V) = W^s(U) \cap V$ . A condition on  $V$  can be given that guarantees this equality.

The uniqueness of  $W^{cu}$  and  $W^{cs}$  is a much more complex issue and a rather important one. In general they are not unique, see e.g. Carr, p. 28 [4]. The abstract theorem (section 2) supplies a unique manifold when applied to obtain  $W^{cu}$  and  $W^{cs}$ . However, this is deceptive as the application of the abstract theorem requires a modification of the original equation and different modifications may supply different manifolds. At the end of section 3 we shall give a discussion of the uniqueness of all these invariant manifolds.

In section 2, we give a general approach to proving invariant manifold theorems. We use arguments on invariant cones to construct two complementary manifolds in two different directions that are characterized by complementary exponential rates. These manifolds are actually global but this feature is also deceptive as the local character is hidden in an assumption of small Lipschitz constant on the nonlinear term.

In section 3, each of the above theorems is deduced from these general theorems. Applications to actual partial differential equations are given in section 4.

## 2 GENERAL RESULTS

Let  $X$  be a Banach space and consider the evolution equation in  $X$

$$u_t = Au + g(u) \quad (2.1)$$

Assume

(G1)  $A$  generates a  $C_0$ -semigroup of operators  $S(t)$ .

(G2)  $X = X^- \oplus X^+$ , where  $X^-$  and  $X^+$  are closed and invariant under  $A$  and  $\dim X^+ < \infty$ .

Let  $A^\pm = A|_{X^\pm}$  and assume

(G3)  $A^-$  generates a  $C_0$ -semigroup  $S^-(t)$  on  $X^-$  satisfying

$$\|S^-(t)\| \leq M_1 e^{\alpha t},$$

for  $t \geq 0$ , where  $M_1 > 0$  and  $\alpha$  is a real constant.

(G4)  $A^+$  generates a  $C_0$ -group  $S^+(t)$  on  $X^+$  satisfying

$$\|S^+(t)\| \leq M_2 e^{\beta t},$$

for  $t \leq 0$ , where  $M_2 > 0$  and  $\beta > \alpha$ .

(G5)  $g(0) = 0$  and  $g$  is Lipschitz continuous with Lipschitz constant less than  $(\beta - \alpha)/(4M_1^2 \max\{\|\pi^+\|, \|\pi^-\|\})$ , where  $\pi^\pm$  are the natural projections onto  $X^\pm$  respectively.

We shall renorm the spaces  $X^-$  and  $X^+$  so that  $M_1$  and  $M_2$  can be dropped.

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LEMMA 2.1 The spaces  $X^-$  and  $X^+$  can be renormed so that in (G3) and (G4)  $M_1$  and  $M_2$  can be taken to be 1.

*Proof* Let the norm in  $X^-$  be denoted by  $|\cdot|$  and the operator norm by  $\|\cdot\|$ . Define a new norm,  $|\cdot|_1$ , on  $X^-$  by

$$|v|_1 = \sup_{t \geq 0} e^{-\alpha t} |S^-(t)v| \quad \text{for } v \in X^-.$$

Clearly,  $|v| \leq |v|_1$  and by (G3)  $|v|_1 \leq M_1 |v|$ , so  $|\cdot|$  and  $|\cdot|_1$  are equivalent norms on  $X^-$ . Let  $\|\cdot\|_1$  be the operator norm on  $X^-$  equipped with  $|\cdot|_1$ . Then for  $v \in X^-$  and  $\tau \geq 0$

$$\begin{aligned} |S^-(\tau)v|_1 &= \sup_{t \geq 0} e^{-\alpha t} |S^-(t)S^-(\tau)v| \\ &= \sup_{t \geq 0} e^{-\alpha(t+\tau)} |S^-(t+\tau)v| e^{\alpha\tau} \\ &= \sup_{t \geq \tau} e^{-\alpha t} |S^-(t)v| e^{\alpha\tau} \leq |v|_1 e^{\alpha\tau}. \end{aligned}$$

Hence,  $\|S^-(\tau)\|_1 \leq e^{\alpha\tau}$  for  $\tau \geq 0$ . The proof for  $X^+$  is identical and will be omitted.

*Remark* From now on we shall identify  $X$  with  $X^- \times X^+$  and give it the norm  $|v| + |w|$  where  $(v, w) \in X^- \times X^+$  and the individual norms are those supplied by Lemma 2.1. If we let  $g^\pm = \pi^\pm g$  then in the new norm an easy calculation shows that  $g^\pm$  both have Lipschitz constants less than  $(\beta - \alpha)/4$ . Therefore we can choose a number  $\varepsilon < (\beta - \alpha)/4$  to be the Lipschitz constant of both  $g^\pm$ .

We will show that under the above assumptions there exist two globally defined invariant manifolds intersecting only at 0. These manifolds are graphs of Lipschitz functions, and are characterized by the exponential rate at which solutions on them approach zero as  $t \rightarrow +\infty$  or as  $t \rightarrow -\infty$ .

Note that nothing is assumed about the signs of  $\alpha$  and  $\beta$ . The assumption in (G4) that  $A^+$  generates a  $C_0$ -group is not restrictive since it is assumed in (G1) that  $\dim X^+ < +\infty$ .

From the above remark, a number  $\gamma$  can be found so that

$$-\beta + 2\varepsilon < \gamma < -\alpha - 2\varepsilon.$$

In the following,  $\Phi_{(u_0)}$  denotes the solution  $u(t)$  of (2.1) satisfying  $u(0) = u_0$ .

DEFINITION Let

$$W^- = \{u \in X : e^{\gamma t} \Phi_{(u)}(u) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

and

$$W^+ = \{u \in X : \text{a backward branch } \Phi_{(u)}(u) \text{ exists for all } t \leq 0 \text{ and } e^{\gamma t} \Phi_{(u)}(u) \rightarrow 0 \text{ as } t \rightarrow -\infty\}.$$

THEOREM 2.1 Assume (G1)–(G5), then there exists a Lipschitz function  $h^- : X^- \rightarrow X^+$  such that  $W^- = \text{graph}(h^-)$  and  $h^-(0) = 0$ .

THEOREM 2.2 Assume (G1)–(G5), then there exists a Lipschitz function  $h^+ : X^+ \rightarrow X^-$  such that  $W^+ = \text{graph}(h^+)$  and  $h^+(0) = 0$ .

*Remark* The definitions of  $W^-$  and  $W^+$  automatically make them invariant sets where invariance means invariance relative to  $X$ . The point of these theorems is to show that they are, in fact, manifolds.

The basic ingredients in the proofs of these theorems are Gronwall's Lemma, the Contraction Mapping Theorem and an argument related to the Wazewski Principle. The underlying idea that certain cones and moving cones are positively invariant is due to the difference in growth rates in  $X^-$  and  $X^+$ . In particular we will show that  $K = \{(v, w) \in X^- \times X^+ : |v| \leq |w|\}$  is positively invariant under the flow  $\Phi_t$  (see Fig. 1).

One expects that manifolds over  $X^+$  lying in  $K$  are compressed together by  $\Phi_t$ . This turns out to be true and the forward flow is, in fact, a contraction with fixed point  $W^+$ . One can also guess that for all  $v \in X^-$  there is a point  $w(v) \in X^+$  so that  $\Phi_t(v, w(v))$  remains in  $K^c$  (complement of  $K$ ) for  $t > 0$ . This is proved using Brouwer degree. The set of points  $(v, w(v))$  will form  $W^-$ .

First we prove some technical lemmas.

LEMMA 2.2 Let  $a, b : [0, T] \rightarrow [0, \infty)$  be continuous. Suppose that there is a positive number,  $A$  such that

$$a(t) \leq a(0) + A \int_0^t a(s) ds, \quad 0 \leq t \leq T \quad (2.2)$$

then

$$a(t) \leq a(0)e^{At}, \quad 0 \leq t \leq T. \quad (2.3)$$

If there is a positive number,  $B$  such that

$$b(t + \tau) \leq b(t) + B \int_t^{t+\tau} b(s) ds. \quad (2.4)$$

for all  $\tau$  such that  $-T \leq -t \leq \tau \leq 0$ , then

$$b(t) \geq b(0)e^{-Bt}, \quad 0 \leq t \leq T. \quad (2.5)$$

*Proof* (2.2)  $\rightarrow$  (2.3) is a special case of Gronwall's inequality, see Hartman [13]. (2.4)  $\rightarrow$  (2.5) follows from the previous case by putting  $a(-\tau) = b(t + \tau)$  for fixed  $t > 0$  and  $-t \leq \tau \leq 0$ , and then letting  $\tau = -t$  in the resulting inequality.

Write (2.1) as a system, where  $(v, w) \in X^- \times X^+$ .

$$\begin{aligned} v_t &= A^- v + g^-(v, w) \\ w_t &= A^+ w + g^+(v, w) \end{aligned} \quad (2.6)$$

The mild solution of (2.1) can be decomposed, using the invariance of  $X^\pm$ , continuity of  $\pi^\pm$  and the uniqueness of this mild solution, into the mild solution of (2.6). The formulae for this mild solution give

$$v(t) = S^-(t)v(0) + \int_0^t S^-(t-s)g^-(v(s), w(s))ds$$

for  $t \geq 0$ , and

$$w(t+\tau) = S^+(\tau)w(t) + \int_0^\tau S^+(\tau-s)g^+(v(t+s), w(t+s))ds$$

for  $\tau \geq -t$ , since  $S^+$  forms a  $C_0$ -group.

Taking norms and using the hypotheses (G3)–(G5) and the remark following Lemma 2.1, we obtain the estimates

$$|v(t)|e^{-\alpha t} \leq |v(0)| + \varepsilon \int_0^t (|v(s)| + |w(s)|)e^{-\alpha s} ds, \quad (2.7)$$

for  $0 < t$ , and

$$|w(t+\tau)|e^{-\beta(t+\tau)} \leq |w(t)|e^{-\beta t} + \varepsilon \int_0^\tau (|v(t+s)| + |w(t+s)|)e^{-\beta(t+s)} ds \quad (2.8)$$

for  $-t \leq \tau \leq 0$ . These together with Lemma 2.2 yield:

LEMMA 2.3 Let  $(v, w)$  be a mild solution to (2.6). Suppose

$$|w(s)| \leq k_1 |v(s)| \quad \text{for } 0 \leq s \leq t, \quad (2.9)$$

then

$$|v(t)| \leq |v(0)| \exp((\alpha + \varepsilon(1 + k_1))t). \quad (2.10)$$

Suppose

$$|v(s)| \leq k_2 |w(s)| \quad \text{for } 0 \leq s \leq t, \quad (2.11)$$

then

$$|w(t)| \geq |w(0)| \exp((\beta - \varepsilon(1 + k_2))t). \quad (2.12)$$

We can now give the lemma which justifies Fig. 1. Define

$$K_\lambda = \{(v, w) \in X^- \times X^+ : \lambda |v| \leq |w|\} \quad \text{for } \lambda > 0.$$

From (G5) we can introduce positive parameters  $\mu < 1 < \nu$  so that

$$\varepsilon < (\beta - \alpha)/(2 + \nu + \mu^{-1}). \quad (2.13)$$

## 2. General results

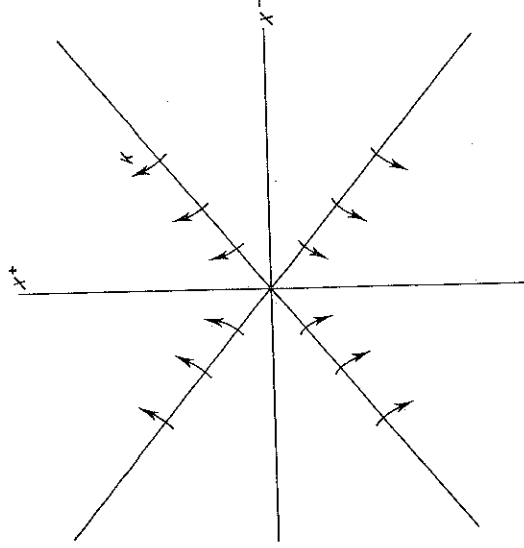


Fig. 1

We further restrict  $\mu$  and  $\nu$  so that

$$\varepsilon(1 + \mu^{-1}) - \beta < \gamma < -\varepsilon(1 + \nu) - \alpha. \quad (2.14)$$

LEMMA 2.4 For  $\lambda \in [\mu, \nu]$ ,  $K_\lambda$  is positively invariant.

*Proof* Let  $\xi > 0$  be so small that  $\varepsilon < (\beta - \alpha)/(2 + 1 + \xi + (\lambda - \xi)^{-1})$ . It suffices to consider  $\dot{u}_0 \in \partial K_\lambda$  with  $u_0 \neq 0$ , then  $|w_0| = \lambda |v_0|$ . For small positive  $t$  we have

$$|w(s)| \leq (\lambda + \xi) |v(s)| \quad \text{and} \quad |v(s)| \leq (\lambda - \xi)^{-1} |w(s)|, \quad 0 \leq s \leq t.$$

Applying Lemma 2.3 yields

$$\frac{|v(t)|}{|w(t)|} \leq \frac{|v_0|}{|w_0|} \exp((\alpha - \beta + \varepsilon(2 + 1 + \xi + (\lambda - \xi)^{-1}))t) < \lambda^{-1}.$$

*Remark* Since we now know that  $K_\lambda$  is positively invariant for  $\lambda \in [\mu, \nu]$ , if  $(v_0, w_0) \in K_\lambda$  then we have, by Lemma 2.3,

$$|w(t)| \geq |w(0)| \exp((\beta - \varepsilon(1 + \lambda^{-1}))t) \quad \text{for } t > 0. \quad (2.15)$$

A stronger invariance is valid as shown by the following result for moving cones, see Fig. 2 as a guide.

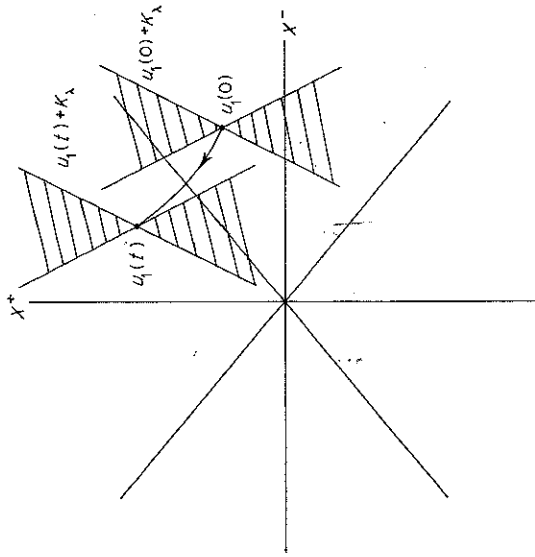


Fig. 2

LEMMA 2.5 If  $u_1$  and  $u_2$  are two solutions to (2.1) with  $u_2(0) \in u_1(0) + K_\lambda$  for some  $\lambda \in [\mu, v]$ , then

$$u_2(t) \in u_1(t) + K_\lambda \quad \text{for all } t \geq 0.$$

*Proof* We may suppose that  $u_2(0) - u_1(0) \in \partial K_\lambda$  and that  $u_2(0) \neq u_1(0)$ . We have

$$\begin{aligned} v_2(t) - v_1(t) &= S^-(t)(v_2(0) - v_1(0)) \\ &\quad + \int_0^t S^-(t-s)(g^-(v_2(s), w_2(s)) - g^-(v_1(s), w_1(s))) ds \end{aligned}$$

and using (G5) we get

$$\begin{aligned} |v_2(t) - v_1(t)| &\leq |v_2(0) - v_1(0)|e^{at} \\ &\quad + e \int_0^t (|v_2(s) - v_1(s)| + |w_2(s) - w_1(s)|)e^{a(t-s)} ds \end{aligned} \quad (2.16)$$

for  $t > 0$ .

Similarly for  $-t \leq \tau \leq 0$

$$\begin{aligned} |w_2(t + \tau) - w_1(t + \tau)| &\leq |w_2(t) - w_1(t)|e^{b\tau} \\ &\quad + e \int_t^0 (|v_2(t+s) - v_1(t+s)| + |w_2(t+s) - w_1(t+s)|)e^{b(t-s)} ds \end{aligned} \quad (2.17)$$

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Choose  $\xi > 0$  as in the proof of Lemma 2.4. Again for small  $t > 0$ ,  $|w_2(s) - w_1(s)| \leq (\lambda + \xi)|v_2(s) - v_1(s)|$  and  $|v_2(s) - v_1(s)| \leq (\lambda - \xi)^{-1}|w_2(s) - w_1(s)|$  for  $0 \leq s \leq t$ . Using these in (2.16) and (2.17) and applying Lemma 2.2 gives  $|v_2(t) - v_1(t)| < \lambda^{-1}|w_2(t) - w_1(t)|$ , which completes the proof.

*Remark* As before, if  $u_2(0) \in u_1(0) + K_\lambda$  for some  $\lambda \in [\mu, v]$  we now have

$$|w_2(t) - w_1(t)| \geq |w_2(0) - w_1(0)| \exp((\beta - \varepsilon(1 + \lambda^{-1}))t) \quad \text{for } t > 0. \quad (2.18)$$

We are now prepared to construct  $W^-$  and  $W^+$  and prove Theorems 2.1 and 2.2.

*Proof of Theorem 2.1* We construct  $W^-$  as follows. Fix  $v_0 \in X^-$  and let  $B = \{w_0 \in X^+ : |w_0| \leq \mu|v_0|\}$ . By Lemma 2.4 the positive evolution of  $\{v_0\} \times \partial B$  lies in  $K_\mu$  and, in fact, if  $G_t = \{w_0 \in B : |w(t)| \leq \mu|v(t)|\}$ , then  $G_t \subset G_s$  for  $0 \leq s \leq t$ . Now let  $t > 0$  and define

$$\psi_s(w_0) = \begin{cases} \frac{|w(s)|v_0|}{\mu|w(s)|v_0|} & \text{if } |w(s)| < \mu|v(s)| \\ \frac{|w(s)|v_0|}{\mu|w(s)|v_0|} & \text{if } |w(s)| \geq \mu|v(s)| \end{cases}$$

for  $w_0 \in B$  and  $0 \leq s \leq t$ .

This map can be described geometrically. First follow the point  $(v_0, w_0)$  with the flow to  $(v(s), w(s))$  then map it into  $B$  in one of two ways. If  $(v(s), w(s)) \in K_\mu$  we project it into  $X^+$  using  $\pi^+$  and then into  $\partial B$  along the ray from the origin in  $X^+$ . If  $(v(s), w(s)) \notin K_\mu$  we project it into  $\{v \in X^+ : |v| = |v_0|\} \times B$  along the ray from the origin in  $X$  and then project into  $B$  by applying  $\pi^+$ . The formula shows that these agree if  $(v(s), w(s)) \in \partial K_\mu$ . This map is similar to the one that is constructed in the Wazewski principle.

Now, the interior of  $B$  is a convex open set containing 0 and it lies in the finite-dimensional space  $X^+$ . Also,  $\psi_s(w_0)$  is continuous for  $(s, w_0) \in [0, t] \times B$  and so the Brouwer degree  $d(\psi_s(\cdot), B, 0)$  is defined and constant provided  $\psi_s(w_0) \neq 0$  for all  $w_0 \in \partial B$  and  $s \in [0, t]$ . Since  $K_\mu$  is positively invariant, we know this to be the case. Since  $\psi_0 = I$ , the identity, we have

$$1 = d(I, B, 0) = d(\psi_t, B, 0).$$

Hence, there exists  $w \in B$  such that  $\psi_t(w) = 0$  (in fact a similar argument shows that  $\psi_t(B) = B$ ). This means that  $G_t \neq \emptyset$  for all  $t > 0$  and so  $\{G_t\}_{t \geq 0}$  satisfies the finite intersection property. Hence,  $G_\infty = \bigcap_{t \geq 0} G_t \neq \emptyset$ .

We must show that this intersection consists of a unique point. Suppose  $w_1, w_2 \in G_\infty$ , then  $(v_0, w_2) \in (v_0, w_1) + K_\mu$  and by Lemma 2.5 this holds for the forward evolution of these points, i.e.  $(v_2(t), w_2(t)) \in (v_1(t), w_1(t)) + K_\mu$ . By (2.18) we have

$$|w_2(t) - w_1(t)| \geq |w_2 - w_1| \exp((\beta - \varepsilon(1 + \mu^{-1}))t) \quad \text{for } t > 0.$$



By definition of  $w_1$  and  $w_2$

$$\mu|v_2(t)| \geq |w_2(t)| \quad \text{and} \quad \mu|v_1(t)| \geq |w_1(t)|.$$

It follows from (2.10) that for  $i = 1, 2$ , and  $t \geq 0$ ,

$$|v_i(t)| \leq |v_0| \exp((\alpha + \varepsilon(1 + \mu))t).$$

Thus

$$\begin{aligned} 2|v_0| \exp((\alpha + \varepsilon(1 + \mu))t) &\geq |v_2(t)| + |v_1(t)| \\ &\geq \mu^{-1}(|w_2(t)| + |w_1(t)|) \geq \mu^{-1}|w_2(t) - w_1(t)| \\ &\geq \mu^{-1}|w_2 - w_1| \exp((\beta - \varepsilon(1 + \mu^{-1}))t) \quad \text{for } t \geq 0. \end{aligned}$$

The choice of  $\mu$  in (2.13) now implies that  $w_2 = w_1$ . The function  $h^-: X^- \rightarrow X^+$  defined by  $h^-(v_0) = G_\infty$  is, therefore, single-valued.

We need to show that  $W^- = \text{graph}(h^-)$ . We have from (2.10) that  $|v(t)| \leq |v_0| \exp((\alpha + \varepsilon(1 + \mu))t)$  for  $t > 0$  and since  $|w(t)| \leq \mu|v(t)|$ , then by (2.14)

$$e^{\mu t} \Phi_i(v_0, h^-(v_0)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This shows that  $\text{graph}(h^-) \subset W^-$ .

If  $w_1 \neq h^-(v_0)$ , then  $(v_0, w_1) \in (v_0, h^-(v_0)) + K_\mu$ . As above,  $|w_1(t) - w(t)| \geq |w_1 - h^-(v_0)| \exp((\beta - \varepsilon(1 + \mu^{-1}))t)$  for  $t \geq 0$  and  $e^{\mu t}|w_1(t) - w(t)| \rightarrow 0$  as  $t \rightarrow \infty$ . This shows that  $W^-$  is the graph of  $h^-$ . Clearly,  $h^-(0) = 0$ .

It remains to show that  $h^-$  is Lipschitz continuous. But this is clear since if  $(v_2, h^-(v_2)) \in (v_1, h^-(v_1)) + K_\mu$  for some  $v_1 \neq v_2$ , then by the previous argument, not both positive trajectories can approach zero when multiplied by  $e^{\mu t}$ , as  $t \rightarrow \infty$ . Hence  $h^-$  has Lipschitz constant  $\mu$ . This completes the proof of Theorem 2.1.

*Proof of Theorem 2.2* Let  $h: X^+ \rightarrow X^-$  be a Lipschitz function with Lipschitz constant  $\nu^{-1}$  and  $h(0) = 0$ . Let  $H$  denote the graph of  $h$ , then  $H \subset K_\nu$ . We will show that  $\Phi_i(H)$  is the graph of a  $\nu^{-1}$ -Lipschitz function for all  $t \geq 0$  and that this converges to  $W^+$  as  $t \rightarrow \infty$ .

STEP 1  $\pi^+ \Phi_i(H) = X^+$  for each  $t > 0$ .

*Proof* Fix  $t > 0$ , let  $w_0 \in X^+$  and choose  $R > |w_0|$  such that

$$R \exp((\beta - \varepsilon(1 + \nu^{-1}))t) > |w_0|. \quad \text{Let } B_R = \{w \in X^+ : |w| \leq R\}.$$

Notice that  $\pi^+ \Phi_s(h(w), w)$  is continuous in  $(s, w) \in [0, t] \times B_R$ . Also, for  $|w| = R$ , (2.15) implies that

$$|\pi^+ \Phi_s(h(w), w)| \geq R \exp((\beta - \varepsilon(1 + \nu^{-1}))s) > |w_0| \quad \text{for } 0 \leq s \leq t.$$

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Hence, by the homotopy invariance of Brouwer degree,

$$\begin{aligned} d(\pi^+ \Phi_s(h(\cdot), \cdot), B_R, w_0) &= d(\pi^+ \Phi_0(h(\cdot), \cdot), B_R, w_0) \\ &= d(I, B_R, w_0) = 1. \end{aligned}$$

Since  $w_0$  was arbitrary, this completes Step 1.

STEP 2  $\Phi_i(H)$  is the graph of a  $\nu^{-1}$ -Lipschitz function, for  $t \geq 0$ .

*Proof* Let  $u_1$  and  $u_2$  lie in  $\Phi_i(H)$ . Then there exist points  $u_1^0$  and  $u_2^0$  in  $H$  so that  $u_1 = \Phi_i(u_1^0)$  and  $u_2 = \Phi_i(u_2^0)$ . Because  $H$  is the graph of a  $\nu^{-1}$ -Lipschitz function,  $u_2^0 \in u_1^0 + K_\nu$ . By Lemma 2.5,  $u_2 \in u_1 + K_\nu$ . This completes Step 2.

Let  $\tilde{h}$  be the function whose graph is  $\Phi_i(H)$  and define  $T_i$  by  $T_i(h) = \tilde{h}$ . Let  $Y = \{h \in C(X^+, X^-) : h(0) = 0, h \text{ is } \nu^{-1}\text{-Lipschitz}\}$  then  $Y$  is a complete metric space when given the norm

$$\|h\|_{\text{Lip}} = \sup_{w \neq 0} |h(w)|/|w|.$$

Step 2 shows that  $T_i: Y \rightarrow Y$ .

STEP 3  $T_i$  is a contraction on  $Y$  for  $t$  sufficiently large.

*Proof* Let  $h_i \in Y$  and  $\tilde{h}_i = T_i(h_i)$  for  $i = 1, 2$ . Let  $w \in X^+$ , then there are points  $w_1, w_2 \in X^+$  such that  $(h_i(w), w) = \Phi_i(h_i(w_1), w_1)$  for  $i = 1, 2$ . Since  $K_\nu$  is invariant (2.15) gives

$$|w| \geq |w_1| \exp((\beta - \varepsilon(1 + \nu^{-1}))t) \quad \text{for } i = 1, 2. \quad (2.19)$$

Also since  $(\tilde{h}_2(w), w) \neq \tilde{h}_1(w)$  and  $K_\mu$  provided  $\tilde{h}_2(w) \neq \tilde{h}_1(w)$ , Lemma 2.5 implies

$$\Phi_s(h_2(w_2), w_2) \neq \Phi_s(h_1(w_1), w_1) + K_\mu \quad \text{for } 0 \leq s \leq t. \quad (2.20)$$

Using (2.16) this leads to

$$|\tilde{h}_2(w) - \tilde{h}_1(w)| \leq |h_2(w_2) - h_1(w_1)| \exp((\alpha + \varepsilon(1 + \mu))t). \quad (2.21)$$

We also have

$$\begin{aligned} |h_2(w_2) - h_1(w_1)| &\leq |h_2(w_2) - h_2(w_1)| + |h_2(w_1) - h_1(w_1)| \\ &\leq \nu^{-1}|w_2 - w_1| + |h_2(w_1) - h_1(w_1)| \\ &\leq \mu \nu^{-1}|h_2(w_2) - h_1(w_1)| + |h_2(w_1) - h_1(w_1)| \end{aligned}$$

by (2.20), and therefore

$$|h_2(w_2) - h_1(w_1)|(1 - \mu \nu^{-1}) \leq |h_2(w_1) - h_1(w_1)|$$

We conclude this section with a proposition useful for proving parts of Theorems 1.1 and 1.2 concerning the tangency of  $W^u$  and  $W^{eu}$ . First, note the following:

LEMMA 2.6 For any  $t > 0$  and  $\delta_1 > 0$  there exists  $\delta > 0$  such that if  $(v_0, w_0) \in K_v$  and  $|w(t)| \leq \delta$  then

$$|v(s)| + |w(s)| \leq \delta_1 \quad \text{for } 0 \leq s \leq t.$$

Proof Since  $K_v$  is positively invariant, (2.15) applies and gives

$$|w(t)| \geq |w(s)| \exp((\beta - \alpha(1 + v^{-1}))(t - s))$$

or

$$|w(s)| \leq |w(t)| \max\{1, \exp((\beta - \alpha(1 + v^{-1}))t)\}, \quad 0 \leq s \leq t$$

also

$$|v(s)| \leq v^{-1} |w(s)|, \quad 0 \leq s \leq t.$$

Using the notation found in the proof of Theorem 2.2, define  $Y_0 = \{h \in Y : h \text{ is differentiable at } 0 \text{ and } Dh(0) = 0\}$ . It is easy to show that  $Y_0$  is closed in  $Y$ .

PROPOSITION Assume  $g$  differentiable at 0 and  $Dg(0) = 0$ , then  $T_t : Y_0 \rightarrow Y_0$  for all  $t > 0$  and therefore the fixed point,  $h^+$ , of  $T_t$  lies on  $Y_0$ .

Proof Fix  $n \in \mathbb{N}$  and  $t > 0$ . Let  $\bar{\varepsilon} > 0$  satisfy  $\bar{\varepsilon} < (\beta - \alpha)/(2 + n + n^{-1})$ . There exists a neighbourhood  $U$  of 0 such that  $|g(u)| \leq \bar{\varepsilon}|u|$  for all  $u \in U$ . Let  $h \in Y_0$ , then by perhaps choosing  $U$  to be smaller we can assume that  $U \cap \text{graph}(h) \subset K_v$ . By Lemma 2.6, there exists  $\delta > 0$  such that if  $(v_0, w_0) \in K_v$  with  $|w(t)| \leq \delta$ , then  $(v(s), w(s)) \in U$  for  $0 \leq s \leq t$ . We suppose that  $|w| \leq \delta$ , then there exists a point  $(v_0, w_0) \in \text{graph}(h)$  such that  $w(t) = w$  and  $(v(t), w(t)) \in \text{graph}(h)$ . Since  $\text{graph}(h) \subset K_v$ ,  $(v(s), w(s)) \in U$  for  $0 \leq s \leq t$ . Now Lemma 2.4 applies with  $\varepsilon = \bar{\varepsilon}$  and with  $v = n$  to show that

$$\{(v, w) \in \text{graph}(h) : |w| \leq \delta\} \subset K_v$$

that is,

$$\sup_{|w| \leq \delta} \frac{|\bar{h}(w)|}{|w|} \leq \frac{1}{n}.$$

Since  $n$  was arbitrary we have shown that  $D\bar{h}(0) = 0$ .

### 3 PROOFS OF THE MAIN THEOREMS

In order to apply the general results of Section 2 to prove Theorems 1.1 and 1.2, we must modify the nonlinearity in (1.1), so that it is globally Lipschitz continuous with a small Lipschitz constant.

Combined with (2.21) this yields

$$|\bar{h}_2(w) - \bar{h}_1(w)| \leq v(v - \mu)^{-1} |h_2(w_1) - h_1(w_1)| \exp((\alpha + \varepsilon(1 + \mu)t).$$

Now (2.19) implies that

$$|\bar{h}_2(w) - \bar{h}_1(w)|/|w| \leq v(v - \mu)^{-1} \times \exp((\alpha - \beta + \varepsilon(2 + \mu + v^{-1}))t) \|h_2 - h_1\|_{L^p}.$$

So  $\|T_t(h_2) - T_t(h_1)\|_{L^p} \leq v(v - \mu)^{-1} \exp((\alpha - \beta + \varepsilon(2 + \mu + v^{-1}))t) \|h_2 - h_1\|_{L^p}$ . This completes Step 3 in view of the restrictions on  $\mu$  and  $v$  in (2.13).

We can now conclude that  $T_t$  has a unique fixed point,  $h_t \in Y$ , for each  $t$  sufficiently large.

STEP 4  $h_t$  is independent of  $t$  and  $\text{graph}(h_t)$  is invariant.

Proof If  $\tau > 0$  then  $T_t(T_\tau(h_t)) = T_\tau(T_t(h_t)) = T_\tau h_t$  if  $t$  is sufficiently large. This means that  $T_\tau h_t$  is the unique fixed point of  $T_t$  and so  $T_\tau h_t = h_t$ . That is,  $h_t$  is a fixed point of  $T_\tau$  for all  $\tau > 0$ . This completes Step 4. From now on denote  $h_t$  by  $h^+$ .

Since  $\Phi_t(\text{graph}(h^+)) = \text{graph}(h^+)$  for all  $\tau > 0$ , a branch  $\Phi_t(u)$  exists in  $\text{graph}(h^+)$  for all  $t < 0$  and  $u \in \text{graph}(h^+)$ . To show that  $W^+ = \text{graph}(h^+)$  we must show the following.

STEP 5  $e^{v\tau} \Phi_t(u) \rightarrow 0$  as  $t \rightarrow -\infty$  if and only if  $u \in \text{graph}(h^+)$ .

Proof Since  $\text{graph}(h^+) \subset K_v$ , if  $(\bar{v}(t), \bar{w}(t)) = \Phi_t(\bar{v}, \bar{w})$  for  $t < 0$  where  $\bar{v} = h^+(w)$  and we have taken a backward solution branch in  $\text{graph}(h^+)$ , then

$$|\bar{w}(0)| \geq |\bar{w}(t)| \exp(-(\beta - \alpha(1 + v^{-1}))t)$$

by (2.12). Hence,

$$e^{v\tau} |\bar{w}(0)| \leq |\bar{w}(0)| \exp(\gamma + \beta - \alpha(1 + v^{-1}))t$$

for  $t < 0$ . Since  $|\bar{v}(t)| \leq v^{-1} |\bar{w}(t)|$  and  $\gamma + \beta - \alpha(1 + v^{-1}) > 0$  by (2.14) we have  $e^{v\tau} \Phi_t(u) \rightarrow 0$  as  $t \rightarrow -\infty$  for  $u \in \text{graph}(h^+)$ .

Now suppose that  $u_0 = (v_0, w_0) \notin \text{graph}(h^+)$  and that  $(v(t), w(t)) = \Phi_t(u_0)$  exists for all  $t < 0$ . Let  $\bar{u}_0 = (h^+(w_0), w_0)$  then  $u_0 \notin K_v + K_v$  so, by Lemma 2.5,  $\Phi_t(u_0) \notin \Phi_t(\bar{u}_0) + K_v$  for all  $t < 0$ , i.e.,  $|\bar{w}(t) - w(t)| \leq v|\bar{v}(t) - v(t)|$  for all  $t < 0$ . As in similar arguments above, (2.16) and Gronwall's inequality gives  $|v(0) - \bar{v}(0)| \exp((\alpha + \varepsilon(1 + v))t) \leq |v(t) - \bar{v}(t)|$  for  $t < 0$ . This means that  $|v(t) - \bar{v}(t)| e^{v\tau} \rightarrow 0$  as  $t \rightarrow -\infty$  since  $\gamma + \alpha + \varepsilon(1 + v) \leq 0$  by (2.14), and hence that  $u_0 \notin W^+$ . This completes Step 5 and the proof of Theorem 2.2.

Fix  $\eta > 0$  and let  $\delta > 0$  be chosen so that  $f$  has a Lipschitz constant less than  $\eta/12$  in  $B(0, 2\delta)$ , the ball centred at 0 of radius  $2\delta$ . Define  $\psi: X \rightarrow \mathbb{R}$  by

$$\psi(u) = \begin{cases} 1 & |u| \leq \delta \\ 2 - |u|/\delta & \delta \leq |u| \leq 2\delta \\ 0 & |u| > 2\delta \end{cases}$$

and set  $g(u) = \psi(u)f(u)$ . It follows that  $g = f$  in  $B(0, \delta)$  and it is easy to show that  $g$  is globally Lipschitz with constant  $\varepsilon < \eta/4$ . Clearly we may take  $\varepsilon > 0$  as small as we please by our choice of  $\delta$ .

*Proof of Theorem 1.1 in case (D)* We shall apply theorems 2.1 and 2.2 with  $X^- = X^s$  and  $X^+ = X^{cu}$ . Set  $\alpha = -\sigma$  from (D3) and fix  $\beta$  so that  $-\sigma < \beta < 0$ . From (D1) and (D2) there is a constant  $M' = M'(\beta)$  so that

$$\|S^{cu}(t)\| \leq M'e^{\beta t}$$

for  $t \leq 0$ . Now renorm as in Lemma 2.1 so that  $M' = M = 1$  (M from (D3)). Note that  $\beta$  can be taken to be arbitrarily close to zero but  $M'$  depends on  $\beta$  and consequently the renorming depends on the choice of  $\beta$ .

Next we set  $\eta = (\beta - \alpha)/4$  in the modification of  $f$  given above. The hypotheses of Theorems 2.1 and 2.2 are now satisfied and yield invariant manifolds  $W^-$  and  $W^+$ . The modified system agrees with (1.1) inside the neighbourhood  $B(0, \delta)$ . Set  $U$  to be a product neighbourhood in  $X$  lying inside  $B(0, \delta)$ , then we obtain the manifolds of Theorem 1.1 as  $W^- \cap U$  and  $W^+ \cap U$ . These are both Lipschitz manifolds invariant relative to  $U$  given by Lipschitz functions. Call these functions, which are restrictions of  $h^-$  and  $h^+$ ,  $h^s$  and  $h^{cu}$ , respectively. It follows from Theorems 2.1 and 2.2 that  $h^s(0) = 0$  and  $h^{cu}(0) = 0$ .

We shall now show that if  $\tilde{h}^-$  is constructed using another renorming and modification, there is a neighbourhood  $V$  of 0, in which the old equation holds, and  $\text{graph}(\tilde{h}^-) = W^- \cap V$ . Suppose there exists  $v_0 \in \text{dom } \tilde{h}^-$  such that  $\tilde{h}^-(v_0) \neq h^-(v_0)$ . Now  $|\Phi_t(v_0, \tilde{h}^-(v_0))| \leq |(v_0, \tilde{h}^-(v_0))|e^{(\alpha+2\varepsilon)t}$  and  $V$  can be chosen so that this trajectory stays in  $V$  for all  $t \geq 0$ . Since there is a unique point in  $V$  with first co-ordinate  $v_0$  for which this exponential estimate holds, then  $\tilde{h}^-(v_0) = h^-(v_0)$ .

From these considerations we can conclude that  $W^-$  is tangent to  $X^-$  at 0, i.e.  $Dh^-(0) = 0$ . This follows by allowing  $\mu \rightarrow 0$  (recall  $h^-$  is  $\mu$ -Lipschitz) by having  $\varepsilon \rightarrow 0$  and shrinking  $U$  if necessary.

To see that  $W^{cu}$  is tangent to  $X^{cu}$ , the proposition following the proof of theorem 2.1 is used. By (H3)  $Df(0) = 0$  and, hence  $Dg(0) = 0$ . The proposition then gives that  $Dh^+(0) = 0$  and so  $Dh^{cu}(0) = 0$ .

The above arguments yield the Lipschitz manifolds  $W^+ \cap U$  and  $W^- \cap U$  and give their properties. We must show that

$$W^s = W^- \cap U$$

and  $W^+ \cap U$  is a centre-unstable manifold.

If  $U$  is chosen to be a suitable product neighbourhood, it follows from (2.10) that trajectories on  $W^- \cap U$  stay in  $U$  for all  $t \geq 0$ . By Theorem 2.1 they decay at the rate  $e^{(\alpha+2\varepsilon)t}$  and so lie on  $W^s$ . Now suppose that  $u_0 \notin W^-$  but  $u_0 \in W^s \cap U$ . Then  $\Phi_t(u_0) \in U$  for all  $t \geq 0$  and  $\Phi_t(u_0) \rightarrow 0$  exponentially. Suppose  $|\Phi_t(u_0)| \leq M'e^{\beta t}$  for some  $\zeta < 0$  and all  $t \geq 0$ . Renorm  $X^{cu}$  so that

$$\|S^{cu}(t)\| \leq e^{\beta t} \quad \text{for } t \leq 0$$

and change the modification described at the beginning of this section so that  $\beta$  and  $\varepsilon$  satisfy  $\beta > \zeta + 2\varepsilon$ . This shrinks the neighbourhood on which the modified equation restricts to the original equation. However if  $t \geq T$  and  $T$  is large enough we can assume that  $\Phi_t(u_0)$  lies in a neighbourhood in which the equation agrees with the unmodified one. Let  $v_T = \pi^-(\Phi_T(u_0))$  and apply (2.18) to  $(v_T, h^-(v_T))$  and  $\Phi_T(u_0)$

$$|\pi^+(\Phi_t(v_T, h^-(v_T))) - \pi^+(\Phi_{t+T}(u_0))| \geq |\pi^+(v_T, h^-(v_T)) - \pi^+(\Phi_T(u_0))|e^{(\beta-2\varepsilon)t}.$$

But both  $\Phi_t(v_T, h^-(v_T))$  and  $\Phi_{t+T}(u_0)$  converge to 0 at a faster exponential rate than  $\beta - 2\varepsilon$  which contradicts this inequality. This shows that  $W^s = W^- \cap U$ .

It is easy to see that  $W^+ \cap U$  does not intersect  $W^s$  except at 0 since they live in complementary cones. From Step 5 in the proof of Theorem 2.2, each point  $u_0 \in W^{cu}$  has a backward solution branch for the modified system for all  $t \leq 0$ . It therefore has one for the original problem unless it first leaves  $U$ . This proves that the  $W^{cu}$  is a centre-unstable manifold according to the definition in section 1.

*Proof of Theorem 1.2 in case (D)* Here we set  $X^- = X^{cs}$  and  $X^+ = X^u$ . Since  $\sigma^u$  is finite, its real part is bounded below, choose  $\beta > 0$  so that  $\beta < \min\{\text{Re } \lambda: \lambda \in \sigma^u\}$ . Let  $\alpha$  be any number with  $0 < \alpha < \beta$ . (D1) and (D3) imply

$$\|S^{cs}(t)\| \leq Me^{\alpha t} \quad (3.1)$$

for some  $M = M(\alpha)$ . Renorm so that  $M = 1$  and modify as above with  $\eta = \beta - \alpha$ . (G1)–(G5) are now satisfied. Applying theorems 2.1 and 2.2 yields Lipschitz manifolds  $W^-$  and  $W^+$ . Let  $U$  be a product neighbourhood inside  $B(0, \delta)$  where the original equation applies. Consider  $W^- \cap U$  and  $W^+ \cap U$ , these are easily seen to be the graphs of Lipschitz functions  $h^{cs}$  and  $h^u$  which are restrictions of  $h^-$  and  $h^+$ , respectively. It follows from theorems 2.1 and 2.2 that  $h^{cs}(0) = 0$  and  $h^u(0) = 0$ .

The tangency of  $W^+$  to  $X^u$  is immediate from the proposition in section 2, hence  $Dh^u(0) = 0$ . The tangency of  $W^-$  is the hardest of all.

Suppose  $W^-$  is not tangent to  $X^{cs}$ , then for some  $\lambda \in (0, 1)$  there exists a sequence  $\{u_n\} \subset W^- \cap K_\lambda$  such that  $u_n \rightarrow 0$  as  $n \rightarrow \infty$ . We may assume that  $\{u_n\} \subset U' \subset U$  where  $U'$  is so small that  $K_{\lambda/2} \cap U'$  is positively invariant relative to  $U'$  and the moving cone invariance holds with  $K_{\lambda/2}$  (see Lemma 2.5) while in  $U'$ . This can be done because we may take the Lipschitz constant of  $g$  to be

arbitrarily small in  $U'$ . If  $\Phi_{\lambda}(u_n) \in U'$  for all  $t \geq 0$  we would have a contradiction to (2.15). So we may find a positive sequence  $\{t_n\}$  and a number  $\delta' > 0$  such that  $w_n = \pi^+ \Phi_{\lambda}(u_n)$  satisfies  $|w_n| = \delta'$  and  $|\pi^+ \Phi_{\lambda}(u_n)| < \delta'$  for  $t < t_n$ ,  $n = 1, 2, \dots$ . Furthermore, since  $u_n \rightarrow 0$ , it follows that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $\bar{u}_n \in W^+ \cap U$  be chosen so that  $\pi^+ \bar{u}_n = w_n$ . Since  $\bar{u}_n \in K$ , (see proof of Theorem 2.2),  $\Phi_{\lambda}(u_n) \in W^- \subset K^c$ ,  $\lambda < v$  and  $|\pi^+ \bar{u}_n| = |w_n| = \delta'$  we have  $|\pi^+ \bar{u}_n - \pi^+ \Phi_{\lambda}(u_n)| \geq \delta'(u^{1-\nu} - v^{-1}) = \delta' > 0$  for all  $n = 1, 2, \dots$ . Now  $\Phi_{\lambda}(u_n) \in \Phi_{-\lambda}(u_n) + K_{\lambda/2}^c$  and  $\Phi_{\lambda}(u_n) \in \Phi_{-\lambda}(u_n) + K_{\lambda/2}$  for  $0 \leq t \leq t_n$ , so  $|\pi^+ \bar{u}_n - \pi^+ \Phi_{-\lambda}(u_n)| \leq 4\lambda^{-1} |\pi^+ \Phi_{-\lambda}(u_n)|$ . The left-hand side is bounded below by  $\delta' e^{-(\alpha+2\epsilon)t_n}$  and the right-hand side is bounded above by  $4\lambda^{-1} \delta' e^{-(\beta-2\epsilon)t_n}$  and  $0 < \alpha + 2\epsilon < \beta - 2\epsilon$  (compare with (2.15)–(2.17)). Since  $t_n \rightarrow \infty$  this yields a contradiction. We have thus established (ii).

It remains to show that these manifolds are indeed the desired ones. From Theorem 2.2 we see that  $W^+ \cap U \subset W^n$ . Again the most difficult part is to show that  $W^n \subset W^+ \cap U$ . Firstly we need the uniqueness of unstable manifolds.

We must show that if  $\hat{h}^+$  is constructed after a different renorming and/or modification there is a neighbourhood  $\hat{U}$  of 0 on which the original equation holds and

$$\text{graph}(\hat{h}^+) \cap \hat{U} \subset \text{graph}(h^+).$$

Let  $0 < \delta < \alpha$  and renorm  $X^s$ , using Lemma 2.1, so that

$$\|S^s(t)\| \leq e^{\delta t}$$

for all  $t \geq 0$ . Fix  $\epsilon$  such that  $0 < \epsilon < \delta$  and modify  $f$  outside a product neighbourhood  $\hat{U}$  so that  $g$  has Lipschitz constant  $\epsilon$ . Furthermore choose  $\hat{U}$  so that

$$\text{graph}(h^+) \cap \hat{U} \subset K_1 \cap \hat{U}$$

which can be done due to the tangency of  $W^+$  which was proved above.

Now for  $u \in \text{graph}(h^+) \cap \hat{U}$ , we have

$$|\pi^+ \Phi_{\lambda}(u)| \leq |\pi^+ u| \exp((\beta - 2\epsilon)t)$$

for all  $t < 0$ . It follows that  $\Phi_{\lambda}(u)$  remains in  $\hat{U}$  for  $t < 0$ . By step 5 in the proof of Theorem 2.2, this implies that  $u \in \text{graph}(\hat{h}^+)$  and hence that  $\text{graph}(\hat{h}^+)$  and  $\text{graph}(h^+)$  agree in  $\hat{U}$ .

The proof that  $W^n = W^+ \cap U$  can now be completed by the same argument as for the stable manifold. That is, we use the uniqueness to get a point on  $W^+$  that can be used to compare with a point on any backward trajectory.

The fact that  $W^- \cap U$  is a  $W^s$  is now trivial since  $W^-$  and  $W^+$  intersect only at 0.

*Proof of Theorems 1.1 and 1.2 in case (C)* Firstly we observe that the proof of Theorem 1.2 above goes through for this case also. To apply Theorems 2.1 and

2.2 the exponential growth condition (3.1) is supplied explicitly by (C3). The remainder of the proof did not depend on the decomposition  $X^{cs} \oplus X^u$ , which is the only thing that has changed.

To prove Theorem 1.1, simply reverse time and reapply Theorem 1.2.

To prove Theorem 1.3 we need some type of implicit function theorem. The following is appropriate, since we do not have smoothness of  $h^{cs}$  and  $h^{cu}$ .

**LEMMA 3.1** Let  $Z$  be a Banach space and suppose  $F: Z \rightarrow Z$  is Lipschitz continuous with Lipschitz constant  $k < 1$ , then  $I - F$  is one-to-one and onto. The proof is immediate once one observes that  $z_0 + F$  is a contraction on  $Z$  for each fixed  $z_0 \in Z$ .

*Proof of Theorem 1.3* Theorems 2.1 and 2.2 supplied us with globally defined Lipschitz functions  $h^+: X^{cu} \rightarrow X^s$  and  $h^-: X^{cs} \rightarrow X^u$  with Lipschitz constants  $\nu^{-1}$  and  $\mu$  respectively. Thus for fixed  $x \in X^c$ , the function  $F: X^s \times X^u \rightarrow X^s \times X^u$  defined by

$$F(y, z) = (h^+(x + z), h^-(x + y))$$

is Lipschitz continuous with Lipschitz constant  $k = \max\{\nu^{-1}, \mu\}$  where the norms on  $X^s$  and  $X^u$  are the renormings used in the theorems. The renorming on  $X^s$  inherited from  $X^{cs}$  (proof of Theorem 1.2) can be chosen to be the same as that on  $X^s$  itself (proof of Theorem 1.1), similarly for  $X^u$ . By the previous lemma there is a unique solution  $(y, z) = h^c(x)$  defined by the equation

$$(y, z) - F(y, z) = 0$$

that is,  $y = h^+(x + z)$  and  $z = h^-(x + y)$ . The usual argument shows that  $h^c$  is Lipschitz continuous with Lipschitz constant  $2k/(1 - k)$ . Note that in  $U$ ,  $\text{graph}(h^+) = W^{cu}$  and  $\text{graph}(h^-) = W^{cs}$ . The manifold  $\text{graph}(h^c)$  is therefore a centre manifold and since  $W^c = W^{cu} \cap W^{cs}$  its tangency follows from that of  $W^{cu}$  and  $W^{cs}$ .

*Proofs of the properties*

(P1) This is immediate from the constructions.

(P2)  $W^{cu}$  is constructed using the decompositions  $X^s \oplus X^{cu}$ . Let  $K$  be the associated cone. If  $u_0 \in K \cap U$  then it lies on the graph of a function in the space  $Y$ . The flow of the modified system is used to get a contraction. Therefore the graph of this Lipschitz function is pushed onto  $W^+$  by this flow. If  $\Phi_t(u_0)$  stays in  $U$  then it actually satisfies the unmodified problem and  $\Phi_t(u_0)$  is pushed onto  $W^{cu} = W^+ \cap U$ . If  $u_0 \notin K$  and  $u_0 \notin W^s$ , apply a moving cone argument to get  $\Phi_t(u_0) \in K$  for some  $t > 0$  and reapply the above. If  $u_0 \in W^s$  then the property is trivial.

(P3) Let  $V \subset U$  and  $K$  the cone associated with the splitting  $X^{cs} \oplus X^u$ . Choose  $\lambda > 1$  so that the moving cone argument can be carried out with  $K_\lambda$  as long as  $\Phi_t(u_0)$  stays inside  $U$ . Choose  $V$  small enough so that if  $u \in \partial U \cap K^c$  then  $(u + K_\lambda) \cap V = \emptyset$ .

If  $u_0 \in V \setminus W^{cs}$ , choose  $u_1 \in W^{cs} \cap V$  ( $V$  must be a product neighbourhood), so that  $u_0 \in u_1 + K_\lambda$ . By Lemma 2.5,  $\Phi_t(u_0) \in \Phi_t(u_1) + K_\lambda$ . If  $\Phi_t(u_1) \in U$  for all  $t \geq 0$  then estimate (2.18) shows that  $\Phi_t(u_0)$  leaves  $U$ . If  $\Phi_s(u_1) \in \partial U$  for some  $s > 0$  then  $\Phi_s(u_0) \notin V$  by construction of  $V$  and  $K_\lambda$ . This proves the property.

(P4)  $W^s$  lies in the complement of the cone  $K$ . If  $V$  is a neighbourhood of 0 that has the property that  $V \cap K^c$  is positively invariant relative to  $K^c$  then,  $W^s(V)$  is positively invariant. This follows because  $W^s(V) \subset W^s(U) \cap V \subset K^c$  and so  $W^s(V) \cap V \subset K^c$ , since  $W^s(V)$  is positively invariant relative to  $V$  a solution can only leave it in forward time by leaving  $V$ . But to leave  $V$  it must first leave  $K^c$  and this is impossible on  $W^s(V)$ .

To satisfy the condition that  $V \cap K^c$  be positively invariant relative to  $K^c$ , one can choose a product neighbourhood  $V = V_1 \times V_2$  where  $V_1 \subset X^{cu}$  has radius  $\varepsilon_1$  and  $V_2 \subset X^s$  has radius  $\varepsilon_2$ . Then choose  $\varepsilon_1$  and  $\varepsilon_2$  small but satisfying the condition  $\varepsilon_1 > \varepsilon_2$ .

Similar considerations apply to  $W^u$ .

#### Discussion of uniqueness

It was commented in section 1 that if  $V \subset U$  then  $W^s(V) \subset W^s(U) \cap V$ . To achieve the equality  $W^s(V) = W^s(U) \cap V$  it suffices to know that  $W^s(V)$  is positively invariant and this is satisfied by any  $V$  for which  $V \cap K^c$  is positively invariant relative to  $K^c$ , as in the proof of (P4).

The uniqueness of  $W^{cs}$  and  $W^{cu}$  is considerably more subtle but a similar condition to that above is helpful.  $U$  is the neighbourhood of 0 supplied by the appropriate theorem and  $K$  is the cone for the appropriate decomposition.

**PROPOSITION 3.1** Suppose  $V$  is a neighbourhood of 0 with  $V \subset U$  and  $V \cap K^c$  positively invariant relative to  $K^c$ . If  $Y_1$  and  $Y_2$  are both centre-stable manifolds for  $V$ , lying in  $K^c$  then there is a neighbourhood  $\hat{V} \subset V$  so that  $Y_1 = Y_2$  in  $\hat{V}$ .

*Proof* From their defining conditions  $\pi^{cs}(Y_1)$  and  $\pi^{cs}(Y_2)$  both contain neighbourhoods of 0 in  $X^{cs}$ . Let  $V_1$  be an open ball of radius  $\varepsilon_1$  contained in both of these. Let  $V_2$  be an open ball in  $X^u$  of radius  $\varepsilon_2$  such that  $V_1 \times V_2 \subset V$  but  $\varepsilon_2 > \varepsilon_1$ . Set  $\hat{V} = V_1 \times V_2$ .

For all  $v \in V_1$ , there is a  $w_1(v)$  and  $w_2(v)$  so that  $(v, w_1(v)) \in Y_1$  and  $(v, w_2(v)) \in Y_2$ . Assume that  $w_1(v) \neq w_2(v)$ . The usual cone argument now shows that  $\Phi_t(v, w_1(v))$  and  $\Phi_t(v, w_2(v))$  separate exponentially as  $t \rightarrow +\infty$  unless one leaves  $V$ . But neither can leave  $V$  since to do so would require first leaving  $K^c$  and this is

impossible since  $Y_1, Y_2 \subset K^c$  and both are positively invariant relative to  $V$ . This is a contradiction unless  $w_1(v) = w_2(v)$ . This shows that  $Y_1 = Y_2$  in  $\hat{V}$ .

**PROPOSITION 3.2** Suppose  $V$  is a neighbourhood of 0 with  $V \subset U$  and  $V \cap K$  negatively invariant relative to  $K$ . If  $Y_1$  and  $Y_2$  are both centre-unstable manifolds for  $V$  lying in  $K$  then there is a neighbourhood  $\hat{V} \subset V$  so that  $Y_1 = Y_2$  in  $\hat{V}$ .

*Proof* The proof is the same as above but working in backward time, which is possible in case (D) because  $X^{cu}$  is finite-dimensional, and in case (C) because there  $S(t)$  is a group.

As argued in the remarks following the statement of (P3), one has a uniqueness criterion for  $W^c$ :

**PROPOSITION** Any centre manifold for which there are arbitrarily small neighbourhoods of 0, each of which intersects that manifold in an invariant set, must coincide with  $W^c$ , as constructed here, on a sufficiently small neighbourhood of 0.

#### 4 EXAMPLES

##### Nonlinear Klein–Gordon equation

Consider the equation

$$u_{tt} = \Delta u + |u|^\gamma u - m^2 u \quad (4.1)$$

where  $\Delta$  is the Laplacian in  $\mathbb{R}^n$ ,  $n \geq 3$ , and  $\gamma > 0$ . It is well known that this equation possesses non-trivial stationary (independent of  $t$ ) solutions in  $H^1(\mathbb{R}^n)$  if  $m > 0$  and

$$\gamma < \frac{4}{n-2},$$

see for instance Strauss [36], Berestycki and Lions [2]. There are, in fact, infinitely many such solutions which are radially symmetric, at least one for any prescribed number of zeros in the radial variable, see Jones and Küpper [19]. An important question is whether they are stable or not. Keller [23] proved, that with added dissipation and some further assumptions on  $\gamma$  these are unstable and indeed have a finite-dimensional unstable manifold. Shatah [34] proved a very general result using energy methods but did not get such a complete picture of the behaviour of the solutions in a neighbourhood of the stationary solution. Here we shall improve Keller's result by showing that the addition of dissipation is unnecessary for instability. We also give a very complete

picture of the flow when we restrict to radial perturbations. We show that if the wave is non-degenerate (in the sense of Morse theory) then there is a unique centre manifold and the wave is stable relative to perturbations in that manifold. All other solutions leave any small neighbourhood in forward or backward time.

In the following  $H^1_r$  and  $L^2_r$  refer to the radial functions in  $H^1(\mathbb{R}^n)$  and  $L^2(\mathbb{R}^n)$ , respectively.

**THEOREM 4.1** If  $n \geq 3$  and  $\gamma < 2(n-2)$  then any non-trivial stationary solution  $\tilde{u}$  of (4.1) which is in  $H^1_r$  has a non-trivial unstable manifold when (4.1) is written as a dynamical system, on  $H^1 \times L^2$ .

We use the same set-up as Keller which goes back to Segal [33]. Equations (4.1) is written as a dynamical system:

$$\begin{aligned} u_t &= v \\ v_t &= \Delta u + f(u) \end{aligned} \quad (4.2)$$

where  $f(u) = |u|^\gamma u - m^2 u$ .

Rewrite (4.2) in the form

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = D \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ f(p + \tilde{u}) - f(\tilde{u}) - df(\tilde{u})p \end{pmatrix} \quad (4.3)$$

where  $u = p + \tilde{u}$  and  $v = q$ . The operator  $D$  is the linearization at the wave  $\tilde{u}$ .  $D$  is given by

$$D = \begin{pmatrix} 0 & I \\ \Delta + df(\tilde{u}) & 0 \end{pmatrix}. \quad (4.4)$$

We shall further decompose  $D$  as

$$D = A + B + K$$

where

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ df(\tilde{u}) & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & 0 \\ df(\tilde{u}) - df(0) & 0 \end{pmatrix}.$$

We see then that

$$A + B = \begin{pmatrix} 0 & I \\ \Delta + df(0) & 0 \end{pmatrix} \quad (4.5)$$

is the linearization at the trivial solution.

#### 4. Examples

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From Pazy [30], it is known that  $A$  generates a  $C_0$ -group on  $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ . Since  $B$  is bounded we have easily that  $A + B$  generates a  $C_0$ -group. The same argument shows that  $D$  generates a  $C_0$ -group. The full problem cast as (4.3) will now fit into our framework provided that the nonlinearity  $f(u)$  is a locally Lipschitz function on  $H^1(\mathbb{R}^n)$  into  $L^2(\mathbb{R}^n)$  and further that the Lipschitz constant of

$$f(p + \tilde{u}) - f(\tilde{u}) - df(\tilde{u})p$$

tends to zero as  $\|p\| \rightarrow 0$ . But this is exactly what is shown by Keller [23] p. 343 under the condition

$$\gamma < \frac{2}{n-2}.$$

In fact Keller's result is for a slightly more general nonlinearity and our results would work in the more general setting also but we have restricted ourselves to the pure power case for simplicity.

We need to show next that the linearized problem at the wave, namely the equation associated to (4.4), falls under the case (C). We firstly need to determine the spectrum of the operator  $D$ . If  $R_\lambda$  is the resolvent of the operator  $L = \Delta + df(\tilde{u})$  in  $L^2(\mathbb{R}^n)$  then one calculates easily that the resolvent of  $D$  is given by

$$Q_\lambda \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} R_{\lambda^2}(g + \lambda f) \\ f + \lambda R_{\lambda^2}(g + \lambda f) \end{pmatrix}. \quad (4.6)$$

It then easily follows that the spectrum of  $D$  is  $\pm$  the square root of that of  $L$ . The spectrum of  $L$  is real and since  $m^2 > 0$  it has only isolated eigenvalues of finite multiplicity in  $[0, \infty)$ . It follows that all of the spectrum of  $D$  lies on the imaginary axis except for a finite number of isolated eigenvalues of finite multiplicity, which lie on the real axis. The instability results from the fact that there are some eigenvalues on the positive real axis, see Strauss [37], (C1) and (C2) are thus verified. The difficulty lies in the verification of the estimate in (C3).

Since  $D$  generates a  $C_0$ -group  $S(t)$ ,  $D|_{X^c}$  does also. To obtain the estimate on the group  $S^c(t)$ , we shall compare  $S(t)$  with the group generated by the linear Klein-Gordon operator, as given in expression (4.5), call this  $S_\infty(t)$ .

**LEMMA 4.1**  $S(t) - S_\infty(t)$  is compact.

*Proof* Firstly we see that  $KS_\infty(t)$  is compact.  $K$  maps  $H^1 \times L^2$  into  $\{0\} \times L^2$  and so it suffices to show that the multiplication operator:

$$f(\tilde{u}) - df(0)$$

is compact from  $H^1$  into  $L^2$ . Let  $\phi_R$  be the characteristic function of the ball of radius  $R$  centred at 0. For any  $R$ , the multiplication operator  $\phi_R[df(\tilde{u}) - df(0)]$  is compact from  $H^1$  into  $L^2$ . Since  $\tilde{u} \rightarrow 0$  as  $|x| \rightarrow +\infty$

uniformly, see [36], and  $f$  is  $C^1$ ,  $\phi_R[d f(\bar{x}) - d f(0)] \rightarrow [d f(\bar{u}) - d f(0)]$  in  $L^\infty(\mathbb{R}^n)$  as  $R \rightarrow +\infty$  and then this limit holds in the operator norm from  $H^1$  to  $L^2$ . Since the compact operators are a closed subset in the norm topology, it follows that the original operator is compact.

Now  $H^1 \times L^2$  is reflexive, so  $S_\infty(t)^*$  is strongly continuous, see Pazy [30] p. 41. It follows that  $S_\infty(t)^* K^*$  is compact and strongly continuous. Vidav [39] p. 268 shows at the beginning of his Theorem 1a that this implies that  $S_\infty(t)^* K^*$  is norm continuous in  $t$ . Since we are in a Hilbert space  $KS_\infty(t)$  is norm continuous and compact. It is then an easy argument to get that

$$S(t-s)KS_\infty(s)$$

is norm continuous in  $s \in [0, t]$ .

The variation of constants formula gives us:

$$S(t) - S_\infty(t) = \int_0^t S(t-s)KS_\infty(s)ds$$

since  $S(t-s)KS_\infty(s)$  is compact and norm continuous in  $s$ , it follows that the integral on the right-hand side can be considered as a Riemann integral and the converges in the norm topology, it is therefore a compact operator and the proof of the lemma is complete.

It is well known that the linear Klein–Gordon equation preserves the energy:

$$\int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2 + m^2 u^2) dx$$

which is equivalent to the norm on  $(u, u_t) \in H^1 \times L^2$ , it therefore follows that

$$\|S_\infty(t)\| \leq 1,$$

where the norm is the energy norm.

This estimate is obviously not true of  $S(t)$ , but from the lemma we know that the essential spectrum of  $S(t)$  and  $S_\infty(t)$  agree, see Kato [20]. Therefore the only spectrum of  $S(t)$  outside the circle of radius 1 does not belong to the essential spectrum. Since  $S^*(t)$  is a restriction of  $S(t)$  the same is true of it. Now suppose that  $S^*(t)$  had some spectrum outside this circle. It would have to be point spectrum and therefore (see Pazy [30] p. 46) the generator of  $S^*(t)$  would have point spectrum in the right half-plane, but this is  $A^*$  and we have a contradiction. It follows that the spectral radius of  $S^*(t)$  is less than or equal to 1, for all  $t$ , and the estimates of (C3) follow easily.

*Proof of Theorem 4.1* We have now verified that the nonlinear Klein–Gordon equation fits into the framework we have set up and so we can apply the theorems that are stated in the introduction. In the neighbourhood of any non-trivial equilibrium the system (4.2) has a stable, unstable and centre manifold. Since

$$0 < \dim X^+ < \infty$$

there is a non-trivial (nonempty) finite-dimensional unstable manifold. The equilibrium is therefore unstable and the theorem is proved.

From the determination of the spectrum the stable manifold has the same dimension as the unstable one and the centre manifold is infinite-dimensional. We can give a much more precise picture of the flow in the neighbourhood of a critical point by studying the centre manifold.

We know that in a neighbourhood of  $\bar{U} \equiv (\bar{u}, 0) \in H^1 \times L^2$ , an infinite-dimensional centre manifold,  $W^c$ , exists. Let  $N$  be a neighbourhood of 0 in  $X^c$  and let  $h: N \rightarrow X^u \oplus X^s$  be the Lipschitz continuous function which is differentiable at zero whose graph (translated to  $\bar{U}$ ) is  $W^c$ . We are interested in the flow on  $W^c$  and for this reason we shall consider the ‘energy’ restricted to  $W^c$ . For  $U = (u, v) \in H^1 \times L^2$  the energy is given by

$$I(U) = \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla u|^2 + \frac{m^2}{2} |u|^2 - \frac{|u|^{\gamma+2}}{\gamma+2} + \frac{1}{2} |v|^2 \right).$$

Concerning the differentiability of  $I$ , we have:

LEMMA 4.2  $I \in C^2(H^1 \times L^2; \mathbb{R})$ .

*Proof* It suffices to show that, defining

$$J(u) = \int_{\mathbb{R}^n} |u|^{\gamma+2}$$

$J \in C^2(H^1; \mathbb{R})$ . Note that since  $\gamma > 0$ ,  $g(u) = |u|^{\gamma+2}$  is  $C^2$  in  $u$ . By the Krasnosel’skii lemma on the continuity of Nemytskii operators [24]

$g' \in C(L^p, L^q)$  and  $g'' \in C(L^p, L^s)$  where  $p = \gamma + 2$ ,  $q = (\gamma + 2)/(\gamma + 1)$  and  $s = (\gamma + 2)/\gamma$ .

We can then show that

$$\begin{aligned} J'(u)v &= \int_{\mathbb{R}^n} g'(u)v \\ \langle J''(u)v, w \rangle &= \int_{\mathbb{R}^n} g''(u)vw \end{aligned}$$

as follows:

$$\begin{aligned} |J(u+v) - J(u) - J'(u)v| &= \left| \int_{\mathbb{R}^n} \left( \int_0^1 \frac{\partial}{\partial s} g(u+sv) ds - g'(u)v \right) dx \right| \\ &= \left| \int_0^1 \int_{\mathbb{R}^n} (g'(u+sv)v - g'(u)v) dx ds \right| \\ &\leq \int_0^1 \int_{\mathbb{R}^n} |g'(u+sv) - g'(u)| |v| dx ds, \end{aligned}$$

Applying Hölder and Sobolev inequalities and the fact that  $g' \in C(L^p, L^q)$  we find that this is bounded above by

$$C_1 \|v\|_{H^1} \alpha(\|v\|_{H^1}).$$

The bound on  $\gamma$  ensures that  $H^1$  is embedded in  $L^p$ . Similarly, we obtain

$$\begin{aligned} |J'(u+w)v - J'(u)v - \langle J''(u)v, w \rangle| \\ \leq \int_0^1 \int_{\mathbb{R}^d} |g''(u+sw) - g''(u)| |v| |w| \, dx \, ds, \end{aligned}$$

applying the triple Hölder inequality and the fact that  $g'' \in C(L^p, L^2)$ , this is bounded above by

$$C_2 \|\tilde{v}\|_{L^p} \|w\|_{L^p} \alpha(\|w\|_{L^p}).$$

This completes the proof since by the Sobolev theorems the above is dominated by

$$C_3 \|v\|_{H^1} \|w\|_{H^1} \alpha(\|w\|_{H^1}).$$

Let  $U \in N$ , then

$$\begin{aligned} I(\tilde{U} + U + h(U)) &= I(\tilde{U}) + dI(\tilde{U})(U + h(U)) \\ &\quad + \frac{1}{2} \langle d^2 I(\tilde{U})(U + h(U)), U + h(U) \rangle \\ &\quad + \alpha(\|U + h(U)\|^2) \\ &= I(\tilde{U}) + \frac{1}{2} \langle d^2 I(\tilde{U})U, U \rangle + o(\|U\|^2) \end{aligned}$$

since  $h(U) = \alpha(\|U\|)$  as  $\|U\| \rightarrow 0$ . Here,  $\langle \cdot, \cdot \rangle$  denotes the duality pairing in  $H^1 \times L^2$ . Now  $\langle d^2 I(\tilde{U})U, U \rangle = -(Lu, u) + \|v\|^2$  for each  $U = (u, v) \in H^1 \times L^2$ .

We shall assume that in  $H^1$ ,  $L$  has only the trivial nullspace. Note that due to the spatial translation invariance of the set of equilibria for (4.1),  $L$  will always have a non-trivial nullspace in  $H^1$ . Our assumption is generic when attention is restricted to radially symmetric functions. Similarly, if we had been working in a bounded domain with homogeneous Dirichlet boundary conditions, then our assumption would be generically satisfied.

The formula (4.6) determines the relationship between the spectra of  $D$  and  $L$ . We need to see that the invariant subspace associated to the spectrum of  $D$  on the imaginary axis is built out of that associated to the spectrum of  $L$  on the negative real axis. In the following let  $Y$  denote the invariant subspace associated to the spectrum of  $L$  in  $(-\infty, 0)$ , also  $\pi: H^1 \times L^2 \rightarrow H^1$  is the natural projection.

LEMMA 4.3  $\pi(X^*) \subset Y$ .

*Proof* From the formula (4.6) and the operational calculus,  $\pi(X^*)$  is the range

of the projection operator:

$$\int_{\Gamma} R_{\lambda^2}(g + \lambda f) \, d\lambda.$$

where  $\Gamma$  is some curve surrounding the spectrum of  $D$  on the imaginary axis but avoiding the positive real axis. Now

$$\int_{\Gamma} R_{\lambda^2} g = \int_{\Gamma} \frac{R_{\lambda} g}{\lambda^2 \sqrt{\gamma}}$$

where the branch cut of the square root is the positive real axis and  $\tilde{\Gamma}$  is the image of  $\Gamma$  under the map  $z \rightarrow z^2$ . The term on the right-hand side lies in  $Y$  as  $\tilde{\Gamma}$  surrounds only that part of the spectrum of  $L$  associated with  $Y$ . The term  $\int_{\Gamma} \lambda R_{\lambda^2} f$  is dealt with similarly.

Now, in view of lemma 4.3 we have

$$-(Lu, u) \geq c_1 \|u\|_{H^1}^2$$

for some  $c_1 > 0$  for all  $U = (u, v) \in X^c$ . Thus, we see that

$$I(\tilde{U} + U + h(U)) - I(\tilde{U}) \geq c \|U\|^2 + o(\|U\|^2)$$

and so the restriction of  $I$  to  $W^c \equiv W^c \cap (H^1 \times L^2)$  has a strict local minimum at  $\tilde{U}$ . The conservation of  $I$  allows us to conclude that  $\tilde{U}$  is stable in  $W^c$ . In particular, if  $E - I(\tilde{U}) > 0$  is sufficiently small then  $\{U \in W^c : I(U) < E\}$  is a bounded invariant subset of  $W^c$  which contains  $\tilde{U}$ . Let  $N^s$  and  $N^u$  be small neighbourhoods of 0 in  $X^s$  and  $X^u$ , respectively, and  $N^c \equiv \pi^c(\{U \in W^c : I(U) < E\} - \{\tilde{U}\})$ . Then  $V \equiv (N^s \oplus N^u \oplus N^c + \{\tilde{U}\}) \cap (H^1 \times L^2)$  is a neighbourhood of  $\tilde{U}$  in  $H^1 \times L^2$  with the property that if  $u_0 \in V \setminus W^c$  then  $\Phi_t(u_0)$  leaves  $V$  either as  $t \rightarrow +\infty$  or as  $t \rightarrow -\infty$  (cf. (P3), and the subsequent remarks). The latter observation derives from the repulsion property (P3) in forward time and in backward time and the fact that  $W^c = W^{cs} \cap W^{cu}$ . We summarize the above in the following:

THEOREM 4.2 Let  $\tilde{U} = (\tilde{u}, 0)$  where  $\tilde{u} \in H^1$  is a stationary solution to (4.1) with  $\gamma < 2/(n-2)$ . Assume that  $\text{Ker}(L|_{H^1}) = \{0\}$  where  $L \equiv \Delta + d f'(\tilde{u})$ . Then  $\tilde{U}$  is stable with respect to perturbations in  $W^c$ . Moreover, there are arbitrarily small neighbourhoods of  $\tilde{U}$  for which any initial data which does not lie on  $W^c$  must leave that neighbourhood in forward or backward time.

#### FitzHugh–Nagumo equations

The FitzHugh–Nagumo equations are a system consisting of a reaction-diffusion equation coupled with an ordinary differential equation



$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \varepsilon(u - \gamma w). \end{aligned} \quad (4.7)$$

The nonlinear term  $f(u)$  is a cubic  $f(u) = u(u - a)(1 - u)$  with  $0 < a < 1/2$ . The parameters  $\varepsilon$  and  $\gamma$  are both assumed to be small and positive.

They arose originally as a simplification to the Hodgkin-Huxley equations for nerve propagation, see FitzHugh [8] and Nagumo, Arimoto and Yoshizawa [29]. The variable  $t$  represents time and  $x$  is distance along the nerve axon. Many authors have proved the existence of a fast travelling pulse solution to (4.7) when  $\varepsilon$  is small, see Carpenter [3], Hastings [14] and Langer [25].

It is shown in Jones [18] that if  $\varepsilon$  is small enough this pulse is stable. The main part of [18] is the spectral analysis. A theorem of Evans [6] of the form 'linearized stability implies nonlinear stability' completes the stability theorem. Here we will show how our results can be used to make this last step.

A travelling wave solution of (4.7) is a non-trivial solution that is a function of the single variable  $\xi = x - ct$ . If we change variables in (4.7) to  $(\xi, t)$ , it can be viewed as a time independent solution to

$$\begin{aligned} u_t &= u_{\xi\xi} + cu_{\xi} + f(u) - w \\ w_t &= cw_{\xi} + \varepsilon(u - \gamma w). \end{aligned} \quad (4.8)$$

We shall view (4.8) relative to the space

$$X = BU(\mathbb{R}, \mathbb{R}) \times BU(\mathbb{R}, \mathbb{R})$$

where  $BU(\mathbb{R}, \mathbb{R})$  is the space of bounded uniformly continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  with the supremum norm.

Suppose a travelling wave solution is given, that is, an equilibrium solution to (4.8), call it  $(\tilde{u}(\xi), \tilde{w}(\xi))$ . We shall use the following definition of stability.

**DEFINITION** The travelling wave  $(\tilde{u}, \tilde{w})$  is an asymptotically stable solution of (4.8) if there is a neighbourhood  $N \subset X$  of  $(\tilde{u}, \tilde{w})$  so that for any  $(u, w) \in N$ , there is a  $k \in \mathbb{R}$  such that

$$\|u(\xi + k, t) - \tilde{u}(\xi)\|_{\infty} + \|w(\xi + k, t) - \tilde{w}(\xi)\|_{\infty} \rightarrow 0$$

as  $t \rightarrow +\infty$  where  $(u(\cdot, t), w(\cdot, t))$  is the solution of (4.8) with  $u(\cdot, 0) = u$  and  $w(\cdot, 0) = w$ .

Note that since (4.7) is translation invariant, the translate of a stable travelling wave is stable. We shall use our theorems to prove that the spectral analysis in Jones [18] suffices to give the stability of the wave found near the singular limit.

**THEOREM 4.3** If  $\varepsilon$  is sufficiently small then  $(\tilde{u}, \tilde{w})$  is asymptotically stable.

We shall set the problem up in a fashion similar to the Klein-Gordon equation.

#### 4. Examples

Firstly linearize (4.8) at  $(0, 0)$ , setting  $Y = (p, r)$

$$Y_t = (A + B + C)Y \quad (4.9)$$

where

$$A = \begin{bmatrix} d^2 & 0 \\ d\xi^2 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} d & 0 \\ c \frac{d}{d\xi} & d \\ 0 & c \frac{d}{d\xi} \end{bmatrix}$$

and

$$C = \begin{pmatrix} df(0) & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix}.$$

Let  $D = A + B + C$ . The linearization at the wave is

$$Y_t = (D + K)Y \quad (4.10)$$

where

$$K = \begin{pmatrix} df(\tilde{u}) - df(0) & 0 \\ 0 & 0 \end{pmatrix}.$$

The full problem can be expressed as

$$\begin{pmatrix} p \\ r \end{pmatrix}_t = E \begin{pmatrix} p \\ r \end{pmatrix} + \begin{pmatrix} f(p + \tilde{u}) - f(\tilde{u}) - df(\tilde{u})p \\ 0 \end{pmatrix} \quad (4.11)$$

where  $E = D + K$ .

Suppose  $D$  generates a  $C_0$ -semigroup on  $X$ , then since  $K$  is bounded,  $D + K$  does also, see Pazy [30] p. 76. Set

$$F(p, r) = (f(p + \tilde{u}) - f(\tilde{u}) - df(\tilde{u})p, 0),$$

since  $f$  is assumed to be smooth,  $F$  is locally Lipschitz on  $X$  and the Lipschitz constant vanishes with  $\|p\|$ . The problem (4.11) thus fits into our framework, it can be shown that  $D$  generates a  $C_0$ -semigroup.

To see that  $D$  generates a  $C_0$ -semigroup, we proceed as follows. By an exercise of Henry [15] p. 23,  $-A$  is a sectorial operator.  $C$  is bounded and therefore  $A + C$  generates a  $C_0$ -semigroup. Let  $T_a$  be the translation operator on  $X$ , i.e.

$$T_a(u(\xi), w(\xi)) = (u(\xi + a), w(\xi + a))$$

$T_a$  is a bounded operator and  $T_{-a}$  is generated by  $B$ . If  $P(t)$  is the semigroup generated by  $A + C$  then an explicit calculation shows that  $A + B + C$  generates

$T_{-a}P(t)$ , which is a  $C_0$ -semigroup. The domain of  $A + B + C$  is the set:

$$\{(u, w): u', w' \in BU(\mathbb{R}, \mathbb{R}) \text{ and } (u', w') \in X\}.$$

Let us call the semigroup generated by  $D, S_\omega(t)$ . Then

$$S_\omega(t) = T_{-a}P(t).$$

We shall obtain an estimate on  $S_\omega(t)$  of the form

$$\|S_\omega(t)\| \leq M e^{-\omega t}, \quad (4.12)$$

for all  $t > 0$  and  $\varepsilon\gamma > \omega > 0$ . Now  $-(A + C)$  is sectorial, therefore  $P(t)$  is an analytic semigroup. It follows that we need only locate the spectrum of the generator to estimate the semigroup, see for instance Kato [21]. It is found easily that the spectrum of  $A + C$  is all on the real axis and the set  $(-\varepsilon\gamma, \infty)$  is in the resolvent set. We then have the estimate:

$$\|P(t)\| \leq M e^{-\omega t},$$

for some  $M > 0$  and  $\varepsilon\gamma > \omega > 0$ . Since translation is norm 1, the estimate (4.12) follows. To see what this gives for  $S(t)$ , we need the analogue of Lemma 4.1.

LEMMA 4.4  $S(t) - S_\omega(t)$  is compact.

Proof We firstly show that  $KS_\omega(s)$  is norm continuous in  $s > 0$ .  $K$  has two components, the second is 0 and the first is

$$\{df(u) - df(0)\}\pi_1$$

where  $\pi_1$  is projection on the first term. It suffices therefore to show that

$$\{df(u) - df(0)\}\pi_1 T_{-cs}P(s) = \{df(u) - df(0)\} T_{-cs}\pi_1 P(s)$$

is norm continuous in  $s > 0$ .  $T$  is used here to denote translation on both scalar and vector functions, a slight abuse of notation.

Since  $P(t)$  is an analytic semigroup,  $P(t)(p, r)$  lies in the domain of  $A$  if  $t > 0$  and by Henry [15] p. 21, Theorem 1.3.4 the quantity  $\|AP(t)(p, r)\|$  is bounded on  $[\tau, T]$  for any  $T > \tau > 0$ . It follows that  $\|p_{\xi\xi}\|$  is bounded on  $[\tau, T]$  for any  $T > \tau > 0$ , where  $p(\xi, t) = \pi_1 P(t)(p, r)$ . But then  $\|p_{\xi\xi}\|$  is bounded uniformly on  $[\tau, T]$  for any  $T > \tau > 0$ . Using the fact that an analytic semigroup is differentiable in  $t$  and the Mean Value Theorem, it is then easy to check that  $T_{-cs}\pi_1 P(t)(p, r) = p(\xi - ct, t)$  is norm continuous in  $t > 0$ .

By the variation of constants formula,

$$S(t) - S_\omega(t) = \int_0^t S(t-s)KS_\omega(s)ds$$

Using the Banach–Steinhaus theorem, since  $S(s)$  is strongly continuous and

$KS_\omega(s)$  is norm continuous, the integrand is norm continuous in  $s > 0$ . The same type of argument as in Lemma 4.1 shows that the integrand is compact for each  $s > 0$ , using the estimate (4.12). Since it is norm continuous, the integral is also compact and the lemma is proved.

We now have that, by Kato [20], the essential spectrum of  $S(t)$  and  $S_\omega(t)$  agree. From the analysis preceding the lemma, the spectrum of  $S_\omega(1)$  lies inside a circle of radius  $e^{-\omega}$ . The same is therefore true of the essential spectrum of  $S(1)$ . Recall that  $S^*(t)$  is the restriction of  $S(t)$  to the stable subspace  $X^s$ , since this is a restriction its only essential spectrum (at  $t = 1$ ) also lies inside the circle of radius  $e^{-\omega}$ . Suppose that  $S^*(1)$  had spectrum outside the circle of radius  $e^{-\omega}$ , since it would be point spectrum it would transfer to the generator. If  $\varepsilon$  is small enough, it is shown in Jones [18] that the only spectrum with real part greater than  $-\varepsilon\gamma$  for the generator, is a simple eigenvalue at zero. Therefore all the spectrum of  $S^*(1)$  lies inside the circle of radius  $e^{-\omega}$  and the estimate on this semigroup follows. We now have all the hypotheses of our theorems verified.

Proof of Theorem 4.3 There is a one-dimensional centre manifold at each travelling wave and a codimension-one stable manifold. Let  $U \in X$  denote a travelling wave. Then  $U(\cdot + k)$  is a translate. By property (P2)  $U(\cdot + k) \in W^u$ , since it is fixed by the semiflow. Therefore the curve of translates of  $U$  lies in the  $W^u$  we have constructed near  $U$ . By a dimension count, these curves must coincide.

Now let  $N$  be a neighbourhood of  $U$  in  $X$  which is separated by  $W^s$  and let  $k$  be so small that  $U(\cdot + k), U(\cdot - k) \in N$  with these being on opposite 'sides' of  $W^s$ . This is possible since  $W^u$  is a codimension-one manifold which is transverse to  $W^s$ . For  $V \in X$  and  $b > 0$  let  $B(V, b)$  denote the ball in  $X$  centred at  $V$  having radius

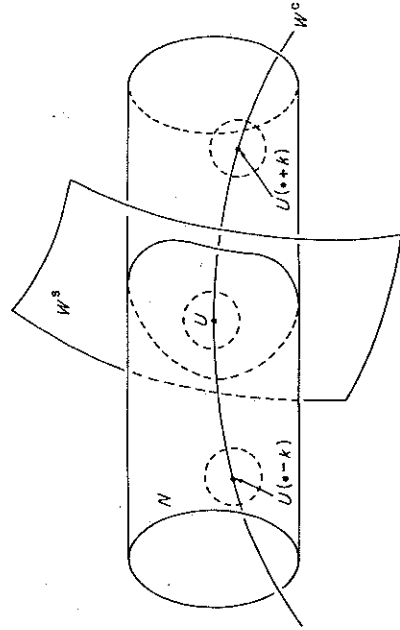


Fig. 3

. Since  $W^s$  is closed and since  $\{U(\cdot + s) : -k \leq s \leq k\}$  is compact, we may choose  $\epsilon > 0$  such that

$$B(U(\cdot + k), r) \cap W^s = \emptyset, \quad B(U(\cdot - k), r) \cap W^s = \emptyset$$

and

$$\bigcup_{-k \leq s \leq k} B(U(\cdot + s), r) \subset N,$$

(see Fig. 3).

Now let  $T_a : X \rightarrow X$  be the translation operator,  $T_a U(\cdot) = U(\cdot + a)$ . Since the sup-norm is translation invariant,  $T_a$  preserves the norm on  $X$  and so

$$T_a B(U, r) = B(U(\cdot + a), r).$$

Consider  $T_k B(U, r) = B(U(\cdot + k), r)$  and  $T_{-k} B(U, r) = B(U(\cdot - k), r)$ . These lie in different components of  $N \setminus W^s$ . Let  $V \in B(U, r)$  then  $\{V(\cdot + c) : -k \leq c \leq k\}$  is a closed curve that lies in  $N$ . Furthermore  $V(\cdot + k)$  and  $V(\cdot - k)$  lie in different components of  $N \setminus W^s$  and therefore there exists a  $c \in (-k, k)$  such that  $V(\cdot + c) \in W^s$ . But this is equivalent to saying that  $V$  lies on the stable manifold of  $U(\cdot - c)$ . This proves the theorem.

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## 2 Formally Symmetric Normal Forms and Genericity

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### INTRODUCTION

We consider several classes of dynamical systems on manifolds, like Hamiltonian systems, volume preserving systems, etc. On each of these classes there is a more or less natural topology. The word generic is used for properties of dynamical systems which hold for almost all elements, in the topological sense, of a given class. Formal definitions will be given below. Generic properties are known which imply a certain local simplicity of the system, e.g. that fixed points or equilibria are isolated. Still for such a system the global dynamics can be very complicated.