

## Approximately invariant manifolds and global dynamics of spike states

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**Abstract.** We investigate the existence of a true invariant manifold given an approximately invariant manifold for an infinite-dimensional dynamical system. We prove that if the given manifold is approximately invariant and approximately normally hyperbolic, then the dynamical system has a true invariant manifold nearby. We apply this result to reveal the global dynamics of boundary spike states for the generalized Allen–Cahn equation.

### 1. Introduction

The motivation for this work derives from many results associated with the following generalization of the Allen–Cahn equation with small diffusion parameter  $0 < \varepsilon \ll 1$

$$(1.1) \quad \begin{cases} u_t = \varepsilon^2 \Delta u - u + f(u), & x \in \Omega \\ \frac{\partial u}{\partial N} = 0, & x \in \partial\Omega. \end{cases}$$

We assume that  $\Omega$  is a smoothly bounded domain in  $\mathbb{R}^n$ , and take  $N$  to be the outward unit normal vector to  $\partial\Omega$ . The nonlinearity  $f \in C^1$  and is such

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that there is a non-degenerate positive radially symmetric ground state of the corresponding rescaled elliptic problem on  $\mathbb{R}^n$ .

Stationary solutions were investigated by Lin, Ni and Takagi in [LNT] and Ni and Takagi in [NT1], [NT2], and [NT3], where the authors proved the existence of solutions that are almost zero on most of the domain but have a single sharp peak (spike) on the boundary. The approach was variational, using constrained optimization, then giving a refined analysis of the critical point.

The profile of a peak solution was shown to be roughly given by a translation of the rescaled ground state  $w$  of the elliptic equation

$$(1.2) \quad \begin{cases} \Delta w - w + f(w) = 0, & y \in \mathbb{R}^n, \\ w(0) = \max w(y), & w > 0, \\ w(y) \rightarrow 0, & y \rightarrow \infty. \end{cases}$$

In the third paper [NT3], the peak was shown to have its maximum exactly on  $\partial\Omega$  and located close to where  $\partial\Omega$  had greatest mean curvature. In heuristic terms, this is because the energy associated with this equation has two parts, bulk and interfacial, both of which are minimized by having the spike located at such a point. To explain further, in minimizing among non-trivial states the bulk energy is almost zero due to the profile being almost zero except for an  $\varepsilon$ -small region where the spike occurs, and the interfacial energy being roughly proportional to the surface area of the region in  $\Omega$  on which the solution makes its excursion. Both of these remain essentially unchanged as the center of the peak moves through the interior of the domain with distance from the boundary  $O(1)$  but decrease to approximately one half their former values as the center of the peak moves to the boundary. They are further reduced by moving the center to a point on the boundary to minimize the volume/area of that part of the  $\varepsilon$ -sphere which lies in  $\Omega$ , that is, the point on the boundary at the point where the mean curvature is greatest.

The role of the mean curvature in localizing and determining the Morse index of stationary boundary spike solutions was investigated further in [BDS], [BS], and [We2].

In this paper, we go beyond that analysis, first building approximate solutions out of modifications of rescaled and translated ground states centered at each point of the boundary,  $\partial\Omega$ . This family of spikes then forms a global manifold in function space that is diffeomorphic to  $\partial\Omega$  and which we show is *approximately invariant* and *approximately normally hyperbolic* relative to (1.1) in a sense made precise later. We then prove that, in a small neighborhood of this approximately invariant manifold of spike states, there is a true invariant manifold, being a smooth graph over the former manifold. Furthermore, the space in which we work is continuously imbedded in the space of continuous functions and the neighborhood so small that the true invariant manifold consists of spike-like functions. Finally, we prove that the dynamics of one of these spike states is governed by a vector field pro-

portional to the gradient of the mean curvature of  $\partial\Omega$  and that the maximum value of the solution is always on  $\partial\Omega$  during the flow.

The approach, involving the construction of an approximate invariant manifold of states having a certain spatial structure, was pioneered more than twenty years ago in papers of G. Fusco and J. Hale in [FH] and by J. Carr and R. Pego in [CP1]. In those papers the authors were interested in the slow dynamics of interfaces in solutions to the one-dimensional Allen–Cahn equation. The same approach was also taken to obtain similar results for the one-dimensional Cahn–Hilliard equation in [ABF] and [BX1], [BX2], and to rigorously establish the slow motion of “bubble”-like solutions [AF] and multi-peaked stationary solutions to the Cahn–Hilliard equation [BFu] in multi-dimensional domains. The approach was also used to produce spike-like stationary solutions to the shadow Gierer–Meinhardt system of biological pattern formation [Ko]. In most of these papers, the qualitative behavior of solutions was the point of interest and so a true invariant manifold was not shown to exist, although that was done in a subsequent paper by Carr and Pego in [CP2] and also in [BX2]. Recently, Zelik and Mielke in [ZM] studied the dynamics of multi-pulse solutions for parabolic dissipative systems in  $\mathbb{R}^n$  by using an invariant manifold approach with a nonautonomous perturbation. We have also learned of results by Ackermann, Bartsch, and Kaplicky [ABK] in which an invariant set of spike states is found for (1.1), this set being topologically equivalent to  $\partial\Omega$ . The approach is quite different from ours, but starts by considering the boundary of the basin of attraction of the zero solution of (1.1).

What has been lacking is a systematic way to deduce the existence of a true invariant manifold in a small neighborhood of the approximately invariant manifold constructed by hand, as described above. Here we give a general result, simplifying some of the analysis needed in the applications mentioned previously, and at the same time giving stronger conclusions. In our example we give, to first order, the dynamical system that describes the motion of spikes globally, not just in a neighborhood of stationary points and more precisely than only giving the order of magnitude of the speed, as is usually the case.

To state our result on the existence of a global invariant manifold of boundary spike states and to give the dynamics of the spikes along the boundary of  $\Omega$ , we need to use a scaled norm and metric.

Define, for any  $q \in [1, \infty)$ , positive integer  $k$ , and smooth function  $u : \Omega \rightarrow \mathbb{R}$ ,

$$|u|_{W_{\varepsilon}^{k,q}(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \varepsilon^{|\alpha| - \frac{n}{q}} |\partial^{\alpha} u|_{L^q(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \varepsilon^{|\alpha|} |\partial^{\alpha} u|_{L^q(\Omega, \varepsilon^{-n} d\mu)}.$$

Note that  $|\cdot|_{W_{\varepsilon}^{k,q}(\Omega)} = |\cdot|_{W^{k,q}(\Omega_{\varepsilon, x_0})}$ , where  $x_0 \in \mathbb{R}^n$  and  $\Omega_{\varepsilon, x_0} = \{\frac{x-x_0}{\varepsilon} : x \in \Omega\}$ .

Let  $(\partial\Omega, \frac{1}{\varepsilon} \langle \cdot, \cdot \rangle)$  denote the Riemannian manifold  $\partial\Omega$  with the metric scaled by  $\frac{1}{\varepsilon}$ . We have the following theorem on existence of dynamic spike solutions.

**Theorem 1.1.** *Under assumptions (F1)–(F3) given in Sect. 7 with  $f \in C^m$ ,  $m \geq 1$ , for any sufficiently small  $\varepsilon > 0$ , there exists a mapping  $\Psi_\varepsilon \in C^m((\partial\Omega, \frac{1}{\varepsilon^2}(\cdot, \cdot)), W_\varepsilon^{2,2}(\Omega))$  such that*

- (1) *For any  $q \in [2, \infty)$ , there exists  $C > 0$  independent of  $p \in \partial\Omega$  and sufficiently small  $\varepsilon > 0$  such that*

$$\begin{aligned} |\Psi_\varepsilon - W_\varepsilon|_{C^0((\partial\Omega, \frac{1}{\varepsilon^2}(\cdot, \cdot)), W_\varepsilon^{2,2}(\Omega) \cap W_\varepsilon^{2,q}(\Omega))} &\leq C\varepsilon \\ |\Psi_\varepsilon - W_\varepsilon|_{C^1((\partial\Omega, \frac{1}{\varepsilon^2}(\cdot, \cdot)), W_\varepsilon^{2,2}(\Omega) \cap W_\varepsilon^{2,q}(\Omega))} &\rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

where  $W_\varepsilon(p) = w(\frac{\cdot - p}{\varepsilon})$ .

If  $f \in C^2$ , then  $|\Psi_\varepsilon - W_\varepsilon|_{C^1((\partial\Omega, \frac{1}{\varepsilon^2}(\cdot, \cdot)), W_\varepsilon^{2,2}(\Omega) \cap W_\varepsilon^{2,q}(\Omega))} \leq C\varepsilon$ .

- (2)  $M_\varepsilon^* \equiv \Psi_\varepsilon(\partial\Omega)$  is a normally hyperbolic invariant manifold of the semi-flow generated by (1.1).  
 (3) On the invariant manifold  $M_\varepsilon^*$ , (1.1) is conjugate through  $\Psi_\varepsilon$  to the ODE given by a vector field  $Y_\varepsilon(p)$  on  $\partial\Omega$  such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{p \in \partial\Omega} \left\{ \frac{1}{\varepsilon^3} |Y_\varepsilon(p) - c\varepsilon^3 \nabla \kappa(p)| \right\} = 0$$

for some  $c > 0$  determined only by  $w$ , where  $\kappa(p) = H(p) \cdot N(p)$  and  $H(p)$  is the mean curvature vector of  $\partial\Omega$ . Moreover, if  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ , then there exists  $C > 0$  independent of  $p \in \partial\Omega$  and sufficiently small  $\varepsilon > 0$  such that

$$|Y_\varepsilon(p) - c\varepsilon^3 \nabla \kappa(p)| \leq C\varepsilon^{3+\beta}.$$

- (4) If  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ , then for each  $p \in \partial\Omega$ , there exists a unique  $\tilde{p} \in \partial\Omega$  such that  $\max_{x \in \tilde{\Omega}} \Psi_\varepsilon(p)(x) = \Psi_\varepsilon(p)(\tilde{p})$ . Moreover  $|p - \tilde{p}| < C\varepsilon^2$  for some  $C > 0$  independent of  $0 < \varepsilon \ll 1$ .

*Remarks.* As might be expected, since the flow is gradient and the energy derives mainly from the interface, the spike solutions move along the boundary of the domain in a way that decreases the interface within  $\Omega$ . Thus, they are driven by the mean curvature, and are stationary at a point in an  $\varepsilon$ -neighborhood of a non-degenerate critical point of the mean curvature. This recovers results of others to which we alluded earlier, but we give the global flow and the location of stationary points is just a corollary.

The main abstract results in this paper, used in establishing the above theorem, provide for the existence of invariant manifolds and their invariant foliations for infinite dimensional dynamical systems when such systems have approximately invariant manifolds that are approximately normally hyperbolic.

The theory of invariant manifolds and foliations provides indispensable tools for the study of dynamics of nonlinear systems in finite or infinite dimensional space. As is the case here, invariant manifolds can be used to capture complex dynamics and the long term behavior of solutions and to reduce high dimensional problems to the analysis of lower dimensional structures. Invariant manifolds with invariant foliations provide a coordinate system in which systems of differential equations may be decoupled and normal forms derived. These play an important role in the study of structural stability of dynamical systems or, when a degeneracy occurs, in understanding the nature of bifurcations.

The rich history of developments in the field dates back to work of Hadamard, Perron, and Lyapunov, and includes notable advances due to Androsov, Bogoliubov, Fenichel, Hale, Hartman, Henry, Hirsch–Pugh–Shub, Krylov, Kurzweil, Levinson, Mañé, Marsden, Pliss, Ruelle, Sacker, Sell, and many others, too numerous to list here. Our abstract results provide another step in the development by giving weaker conditions under which invariant manifolds and foliations can be shown to exist, but conditions that arise in the study of nonlinear partial differential equations.

When the invariant manifold is an isolated equilibrium point, this problem was studied by Newton and others. The problem can be formulated as: *Given that a differentiable function has an approximate zero, does it have a true zero?* Newton’s theorem says that if the approximation is a “good” one and if a nondegeneracy condition holds, then there is a true zero nearby. Our result can be seen as a generalization of this, giving a nondegeneracy condition (approximate normal hyperbolicity) under which an approximately invariant manifold gives rise to a true invariant manifold nearby. Just as Newton’s theorem is related to the Implicit Function Theorem, which gives persistence of zeros under perturbation, our result is related to the persistence problem for invariant manifolds.

When the approximately invariant manifold of a dynamical system is a true invariant manifold of another nearby dynamical system, one may naturally think that existence of an invariant manifold for the original system is a question of persistence under perturbation. That problem can be stated as: *Assuming that a dynamical system has an invariant manifold, does a perturbation of this system also have an invariant manifold?* Both local theory (theory of stable, unstable, and center manifolds) and global theory (theory of normally hyperbolic invariant manifolds) have been developed to address this question.

But there is a subtle difference between the question of persistence and the one we address here. We are not asking how close another dynamical system must be to one having a true invariant manifold in order to also enjoy that property, and then hoping that our dynamical system is that close. Instead, we look for a true invariant manifold for the given system as a perturbation of the approximately invariant manifold we have at hand. However, the persistence result is a special case of the results in this paper, since an invariant manifold of a dynamical system is actu-

ally an approximately invariant manifold of the perturbed dynamical system.

Furthermore, in applications and numerical computations, the approximately invariant manifold one has may not be a true invariant manifold of another system. In fact, the approximately invariant manifold that we construct for (1.1) is not a true invariant manifold of any nearby system. The abstract problem we consider here, therefore, is more general and can be regarded as an extension of both Newton's theorem and the classical theory of perturbation of invariant manifolds.

We now give the general formulation of the problem we address. Before we discuss results for continuous dynamics, we present the theory for maps.

Let  $X$  be a Banach space and let  $T$  be a  $C^1$  map from  $X$  into  $X$ . We do not assume invertibility in general and so the results will apply to semi-dynamical systems. A typical example is the time- $t$  map of the solution operator for a nonlinear parabolic partial differential equation.

Suppose that there exists a smooth manifold,  $\tilde{M}$ , embedded in  $X$ , which is approximately invariant with respect to  $T$ , that is, for some small  $\delta > 0$

$$T(\tilde{M}) \subset B(\tilde{M}, \delta)$$

and

$$\tilde{M} \subset B(T(\tilde{M}), \delta),$$

where  $B(\tilde{M}, \delta) = \{x \in X : \text{dist}(x, \tilde{M}) < \delta\}$  is a  $\delta$  neighborhood of  $\tilde{M}$ .

Our general results include the cases where the manifold is immersed, rather than embedded in  $X$  but it is better to keep in mind the most straightforward situation at first.

The questions which are addressed here concern the existence of a true invariant manifold for  $T$  and the qualitative behavior of the orbits near this invariant manifold. In general there will be no true invariant manifold for  $T$  even in finite dimensional space. One can easily construct examples that violate the conditions of Newton's theorem and have an approximate zero but no true zero. In order to guarantee the existence of a true invariant manifold, a nondegeneracy condition on the approximately invariant manifold is necessary. This condition is *approximate normal hyperbolicity*. The condition gives, for each  $m \in \tilde{M}$ , a decomposition  $X = X_m^c \oplus X_m^u \oplus X_m^s$ , with  $X_m^c$  an approximation of the tangent space to  $\tilde{M}$  at  $m$  and such that

- (a) This splitting is approximately invariant under the linearized map,  $DT$ ,
- (b)  $DT(m)|_{X_m^u}$  expands and does so to a greater degree than does  $DT(m)|_{X_m^c}$  while  $DT(m)|_{X_m^s}$  contracts and does so to a greater degree than does  $DT(m)|_{X_m^c}$ .

The superscripts  $c$ ,  $u$  and  $s$  stand for "center", "unstable", and "stable", respectively. The precise definition of approximate normal hyperbolicity is given in Sect. 2, where we give notation and state the main results.

Heuristically, our main results may be summarized by

**Theorem 1.2.** *Suppose that  $\tilde{M}$  is a  $C^1$  manifold which is approximately invariant and approximately normally hyperbolic with respect to  $T$ , the approximation being sufficiently good and the “twisting” of  $\tilde{M}$  being uniformly bounded, then*

- (1) *Existence:  $T$  has a true  $C^1$  normally hyperbolic invariant manifold  $M$  near  $\tilde{M}$ .*
- (2) *Smoothness: If  $T$  is  $C^k$  and a “spectral gap” condition holds, then  $M$  is  $C^k$ .*
- (3) *Stable and unstable manifolds: There is a stable manifold  $W^s(M)$  and an unstable manifold  $W^u(M)$  of  $T$  at  $M$ .*
- (4) *Invariant foliations: Both  $W^s(M)$  and  $W^u(M)$  are foliated by invariant foliations:*

$$W^s(M) = \bigcup_{m \in M} W_m^{ss} \quad \text{and} \quad W^u(M) = \bigcup_{m \in M} W_m^{uu},$$

where leaves  $W_m^{ss}$  and  $W_m^{uu}$  are  $C^k$  submanifolds and are Hölder continuous in  $m$ .

- (5) *Characterization of foliations: For any  $x, \tilde{x} \in W_m^{ss}$ ,  $|T^n(\tilde{x}) - T^n(x)| \rightarrow 0$  exponentially, as  $n \rightarrow +\infty$ ; For any  $y, \tilde{y} \in W_m^{uu}$ ,  $|T^n(\tilde{y}) - T^n(y)| \rightarrow 0$  exponentially, as  $n \rightarrow -\infty$ .*
- (6) *Semiflow: If  $\tilde{M}$  is an approximately invariant manifold of time- $t_0$  map  $T^{t_0}$  of a semiflow at  $t_0 > 0$ , then the semiflow  $T^t$  has a normally hyperbolic invariant manifold.*

*Remarks.* We do not assume that  $\tilde{M}$  is compact or finite dimensional. Also,  $\tilde{M}$  is not necessarily an embedded manifold, but may be an immersed manifold. We assume that the immersed manifold  $\tilde{M}$  does not twist very much locally, and  $DT$  has a certain uniform continuity in a neighborhood of  $\tilde{M}$ .

The above result can be viewed as an extension of [BLZ1] and [BLZ2] where perturbations of semiflows are considered. Note that in Item 5, above, it is part of the result that  $T^{-1}$  exists on the unstable manifold.

In the present paper, we also consider the more general case where the manifold  $\tilde{M}$  has boundary and is approximately *overflowing* (intuitively, “approximately negatively invariant and the semiflow crosses the boundary transversally”) or approximately *inflowing* (intuitively, “approximately positively invariant and the semiflow crosses the boundary transversally”). In fact the theorem above is obtained by first finding overflowing and inflowing invariant manifolds and taking their intersection.

As an example, the local unstable manifold of a periodic orbit is an overflowing invariant manifold and the local stable manifold is an inflowing invariant manifold.



For approximately overflowing manifolds and inflowing manifolds, our results may be summarized as

**Theorem 1.3.** *Given that the approximately overflowing invariant manifold  $\tilde{M}$  is approximately normally hyperbolic, then*

- (i) *Existence:  $T$  has a true  $C^1$  center-unstable manifold  $W^{cu}$ , which is an overflowing invariant manifold.*
- (ii) *Smoothness: If  $T$  is  $C^k$  and a “spectral gap” condition holds, then  $W^{cu}$  is  $C^k$ .*
- (iii) *Invariant foliation:  $W^{cu}$  is foliated by an invariant foliation:*

$$W^{cu} = \bigcup_{m \in M} W_m^{uu},$$

*where each leaf  $W_m^{uu}$  is a  $C^k$  submanifold and the family is Hölder continuous in  $m$ .*

- (iv) *Characterization of foliation: For any  $y, \tilde{y} \in W_m^{uu}$ ,  $|T^{(n)}(\tilde{y}) - T^{(n)}(y)| \rightarrow 0$  exponentially, as  $n \rightarrow -\infty$ ;*
- (v) *Semiflow: If  $\tilde{M}$  is an approximately overflowing invariant manifold that is approximately normally hyperbolic for the time- $t_0$  map  $T^{t_0}$  of a semiflow at  $t_0 > 0$ , then the semiflow  $T^t$  has a true center-unstable manifold  $W^{cu}$  that has the same properties as maps.*
- (vi) *Inflowing manifolds: Similar results hold for approximately inflowing manifolds.*

*Remark.* The perturbation theory for overflowing/inflowing invariant manifolds and their invariant foliations was developed in [BLZ2] and [BLZ3]. We reported on the basic results of this paper in [BLZ4] as well as at several conferences in the intervening years.

We develop the abstract results first, giving a precise formulation for immersed approximately inflowing manifolds in the next section. Coordinate systems in a tubular neighborhood of the manifold are constructed in Sect. 3. In Sect. 4 we prove the existence of a center-stable manifold, as a graph over the stable bundle of the given approximately invariant inflowing manifold. Section 5 provides an invariant foliation of the center-stable manifold with stable fibers. Section 6 includes a discussion of the modifications needed for the case of approximately overflowing invariant manifolds, smooth dependence on parameters, results for semiflows as a consequence of the previously obtained results for maps, and how perturbation theorems follow from the present work. The application of our abstract results to the manifold of spike states and their dynamics governed by (1.1) is provided in Sect. 7.

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## 2. Main results for inflowing invariant manifolds

In this section, we first introduce basic notations, hypotheses, and results for approximate inflowing invariant manifolds. Overflowing manifolds and several other issues will be considered in Sect. 6. As mentioned in the introduction, our results are for the general case of immersed manifolds.

Let  $X$  be a Banach space and  $T \in C^J(X, X)$ ,  $J \geq 1$ . Suppose that  $M$  is a connected  $C^1$  Banach manifold (with boundary removed for convenience) and that  $\psi : M \rightarrow X$  is an immersion.

For a subset  $A \subset X$ , and  $a > 0$ , let

$$B(A, a) = \{x \in X : d(x, A) < a\}.$$

For  $m_0 \in M$ , let  $B_c(m_0, a)$  denote the connected component of the set  $\psi^{-1}(B(\psi(m_0), a))$  containing  $m_0$ , this being the natural neighborhood of  $m_0$  in the model manifold,  $M$ .

**Definition 2.1.**  $\psi(M)$  is said to be approximately inflowing invariant if the following conditions hold

- (1) There exist  $\eta > 0$  and  $u \in C^0(M, M)$ , such that

$$|T(\psi(m)) - \psi(u(m))| < \eta$$

for all  $m \in M$ ;

- (2) There exists  $r_0 \in (0, 1)$  such that  $\psi(\overline{B_c(m_0, r_0)})$  is closed in  $X$  for any  $m_0 \in u(M)$ .

Condition (1) means that  $\psi(M)$  is approximately invariant under  $T$  and  $u$  is an approximation of  $T$  on  $\psi(M)$ . Condition (2) essentially states that the ‘distance’ between the projection of  $T(\psi(M))$  into  $\psi(M)$  and the boundary of  $\psi(M)$  is bounded from below. Lemma 3.7 makes this more precise.

For an example of an immersed, but not embedded invariant manifold, see [BLZ2].

The inflowing manifolds we consider here have a certain “normal hyperbolicity” property, i.e., the linearization  $DT$  has different growth rates in different directions. More precisely, conditions (H1)–(H3) hold:

- (H1) For each  $m \in M$  there is a decomposition

$$X = X_m^c \oplus X_m^s \oplus X_m^u$$

of closed subspaces with associated projections  $\Pi_m^c$ ,  $\Pi_m^u$ , and  $\Pi_m^s$ .

- (H2) For any  $m \in M$ ,  $\Pi_m^c$  is an isomorphism from  $D\psi(m)T_m M$  to  $X_m^c$ . Furthermore there exist constants  $B, L \geq 1$ ,  $\chi \in (0, \frac{1}{2})$ , such that, for any  $m_0 \in M$ ,  $m_1, m_2 \in B_c(m_0, r_0)$ ,  $m_1 \neq m_2$ ,  $\alpha = c, u, s$ ,

$$(2.1) \quad \begin{cases} \|\Pi_{m_0}^\alpha\| \leq B, & \|\Pi_{m_1}^\alpha - \Pi_{m_2}^\alpha\| \leq L|\psi(m_1) - \psi(m_2)| \\ \frac{|\psi(m_1) - \psi(m_2) - \Pi_{m_0}^c(\psi(m_1) - \psi(m_2))|}{|\psi(m_1) - \psi(m_2)|} \leq \chi. \end{cases}$$

Hypothesis (H2) means that  $\Pi_m^c$  is approximately the tangent space  $T_m \psi(M)$  and  $\psi(M)$  does not twist too much, allowing us to construct a local tubular neighborhood with a uniform diameter. We notice that if (H2) holds for  $r_0$  then it also holds for any  $0 < r \leq r_0$ .

The following hypothesis is a weak normal hyperbolicity condition.

(H3) There exist  $\sigma, \lambda \in (0, 1)$  such that, for any  $m_0 \in M$ , if  $m_1 \equiv u(m_0)$ , and  $\alpha \in \{c, s\}$ ,  $\beta \in \{c, s, u\}$ , with  $\alpha \neq \beta$ , then

$$(2.2) \quad \left\| \Pi_{m_1}^\beta DT(\psi(m_0)) \big|_{X_{m_0}^\alpha} \right\| \leq \sigma$$

$$(2.3) \quad \left\| \Pi_{m_1}^s DT(\psi(m_0)) \big|_{X_{m_0}^s} \right\| < \lambda,$$

$$(2.4) \quad \lambda \left\| \left( \Pi_{m_1}^u DT(\psi(m_0)) \big|_{X_{m_0}^u} \right)^{-1} \right\|^{-1} > \max \left\{ 1, \left\| \Pi_{m_1}^c DT(\psi(m_0)) \big|_{X_{m_0}^c} \right\|^J \right\}.$$

Condition (2.2) represents the approximate invariance of  $X^c$  and  $X^s$  under the linearization of the map  $T$ . Note that we do not require  $X^u$  to be approximately invariant. Different growth or decay properties of  $DT$  in the unstable and center-stable directions are assumed in (2.3) and (2.4). Here, (H3) implicitly assumes that, for any  $m_0 \in M$ ,

$$\Pi_{m_1}^u DT(\psi(m_0)) : X_{m_0}^u \rightarrow X_{m_1}^u$$

is an isomorphism. Though we do not assume  $DT$  contracts more strongly in the direction of  $X^s$  than it does in the direction of  $X^c$  at this moment, (H3) is sufficient in the proof of the existence of a unique inflowing invariant manifold in the direction of  $X^c \oplus X^s$ . Finally, we have the following technical assumption on  $T$ .

(H4) There exists  $B_1 \geq 1$  such that, for any  $1 \leq j \leq J$

$$\left\| D^j T|_{B(\psi(M), r_0)} \right\| \leq B_1.$$

When  $j > 1$ ,  $D^j T$  is understood as a multilinear map. Hypothesis (H4) holds automatically if  $\psi(M)$  is precompact. When  $J = 1$ , we need the following function

$$(2.5) \quad \mathcal{A}(\delta) = \sup \{ \|DT(x_1) - DT(x_2)\| : x_1, x_2 \in B(\psi(M), \delta), |x_1 - x_2| < \delta \}.$$

Note that if  $DT$  is uniformly continuous, then  $\inf \mathcal{A}(\delta) = 0$ . In fact, we will only require  $\inf \mathcal{A}(\delta)$  to be small, which is a weaker version of uniform continuity. When  $J > 1$ , it is obvious that  $\mathcal{A}(\delta) \leq B_1 \delta$ . For  $\alpha = c, u, s$ , let

$$X_m^\alpha(\varepsilon) = \{x \in X_m^\alpha : |x| < \varepsilon\} \text{ and } X^\alpha(\varepsilon) = \{(m, x) : m \in M, x \in X_m^\alpha(\varepsilon)\}.$$

**Theorem 2.2.** *Assume that (H1)–(H4) hold. Depending on  $r_0, B, B_1, \lambda, L$ , when  $\eta, \chi, \sigma$ , and  $\inf \mathcal{A}(\delta)$  are sufficiently small, there exists a  $C^J$  positively invariant manifold  $W^{cs}$ , which is given as the image of a map*

$$h : \{(m, x^s) : m \in M, x^s \in X_m^s(\delta_0)\} \rightarrow X,$$

for some  $\delta_0 > 0$ . The mapping  $h$  also satisfies

$$h(m, x^s) - (\psi(m) + x^s) \in X_m^u(\delta_0).$$

*Remark 2.3.* a.) The important point is that the smallness of the constants  $\eta, \chi, \sigma$ , and  $\inf \mathcal{A}(\delta)$  depend on  $M$  and  $T$  only through the parameters  $r_0, B, B_1, \lambda, L$ . A more precise statement can be found in Theorem 4.2.  
b.) From the proof, it holds that, for any  $m_0 \in M$ , there exists  $\tilde{h} : X_{m_0}^c(\delta_0) \oplus X_{m_0}^s(\delta_0) \rightarrow X_{m_0}^u$  so that

$$\begin{aligned} & \left\{ h(m, x^s) : m \in B_c(m_0, r_0) \cap \psi^{-1} \left( B \left( \psi(m_0), \frac{\delta_0}{4} \right) \right), x^s \in X_m^s \left( \frac{\delta_0}{4} \right) \right\} \\ & \subset \psi(m_0) + \text{graph} \left( \tilde{h}|_{X_{m_0}^c(\frac{\delta_0}{2}) \oplus X_{m_0}^s(\frac{\delta_0}{2})} \right) \\ & \subset \left\{ h(m, x^s) : m \in B_c(m_0, r_0) \cap \psi^{-1}(B(\psi(m_0), \delta_0)), x^s \in X_m^s(\delta_0) \right\}. \end{aligned}$$

Moreover, there exists  $C > 0$  depending on  $r_0, B, B_1, \lambda, L$  such that

$$|\tilde{h}|_{C^J(X_{m_0}^c(\delta_0) \oplus X_{m_0}^s(\delta_0))} \leq C.$$

In fact,  $\text{Lip } \tilde{h}$  can be rather small (see Theorem 4.2 for details).

This positively invariant manifold is called the center stable manifold associated with  $\psi(M)$  as it stretches in approximately the tangent direction and the stable direction of  $\psi(M)$ . In some cases, an embedded inflowing invariant manifold, instead of an immersed manifold, is desired. To obtain the existence of an embedded inflowing invariant manifold, we need not only the original manifold to be embedded, but also the following condition

(H2') For  $m_1, m_2 \in \psi^{-1}(B(\psi(m_0), r_0))$ , (2.1) holds.

A natural question after the existence and uniqueness of the positively invariant center-stable manifold  $W^{cs}$  is if there exists a unique positively invariant manifold close and homeomorphic to  $\psi(M)$ . The answer is that there generally exist such manifolds, but not uniquely, and its smoothness is more delicate. To understand this, think of an ODE which has a hyperbolic equilibrium with exactly two eigenvalues with negative real parts. If they are both real but not equal, then the stable manifold, which corresponds to our center-stable manifold here, and the strong stable manifold, which corresponds to one of the stable fibers to be constructed in Sect. 5 both uniquely exist. However, the weak stable manifold (in the direction of the eigenvector of the weak negative eigenvalue), corresponding to the center manifold here, exists, but not uniquely. The construction of such positively invariant center manifolds or weakly stable manifolds is usually done by first modifying the system outside a bounded set in the phase space and then applying general theorems like the ones obtained in this paper.

If  $M$  is both inflowing and overflowing as described in Subsect. 6.2, which usually implies that  $M$  is a closed manifold without boundary, then there also exists an negatively invariant center unstable manifold. The intersection of the center stable and center unstable manifold is a normally hyperbolic invariant manifold close to  $\psi(M)$ .

The following result concerns a stable foliation of the center-stable manifold. In the construction we will need a further condition requiring nondegeneracy in the ‘tangential’ direction:

(H3’) There exists  $a > 0$  such that, for any  $m_0 \in M$ , writing  $m_1 = u(m_0)$ ,

$$\Pi_{m_1}^c DT(\psi(m_0)) : X_{m_0}^c \rightarrow X_{m_1}^c$$

is an isomorphism with

$$(2.6) \quad \left\| \left( \Pi_{m_1}^c DT(\psi(m_0)) \big|_{X_{m_0}^c} \right)^{-1} \right\|^{-1} > a$$

$$(2.7) \quad \left\| \Pi_{m_1}^s DT(\psi(m_0)) \big|_{X_{m_0}^s} \right\| < \lambda \left\| \left( \Pi_{m_1}^c DT(\psi(m_0)) \big|_{X_{m_0}^c} \right)^{-1} \right\|^{-1}.$$

As a convention, throughout the paper,  $a$  is taken to be 1 if (H3’) is not assumed.

In order to avoid technical complications near the boundary of  $W^{cs}$ , we will construct stable fibers with base points in an open subset,  $\tilde{W}^{cs}$ , which is away from the boundary of  $W^{cs}$ . Let

$$\tilde{W}^{cs} = \left\{ h(m, x^s) : \psi(\overline{B_c(m, \delta_0)}) \text{ is closed in } X, x^s \in X_m^s \left( \frac{\delta_0}{5} \right) \right\},$$

where  $h$  is the map in Theorem 2.2 representing  $W^{cs}$ .

**Theorem 2.4.** *Assume that (H1)–(H4) and (H3’) hold. Depending on  $r_0, B, B_1, \lambda, L, a$ , when  $\eta, \chi, \sigma$ , and  $\inf \mathcal{A}(\delta)$  are sufficiently small, for any  $y \in \tilde{W}^{cs}$ , there exists a unique  $C^J$  submanifold  $y \in W_y^{ss} \subset W^{cs}$ , such that*

- (1)  $T(W_y^{ss}) \subset W_{T(y)}^{ss}$ ;
- (2) For  $y, \tilde{y} \in \tilde{W}^{cs}$ , “ $\tilde{y} \in W_y^{ss}$ ” is an equivalence relation;
- (3) for any  $\tilde{y} \in W_y^{ss}$ ,  $|T^{(n)}(\tilde{y}) - T^{(n)}(y)| \rightarrow 0$  exponentially, as  $n \rightarrow +\infty$ ;
- (4)  $W_y^{ss}$  is Hölder continuous in  $y$  and  $T_y W_y^{ss}$  is continuous in  $y$ .

Again the precise statement on the smallness of the quantities  $\eta, \chi, \sigma$ , and  $\inf \mathcal{A}(\delta)$  is same as in the existence of the center stable manifold and can be found in Theorem 4.2.

In fact, for any  $y = \psi(m) + x^s + x^u \in W^{cs}$ , the stable fiber  $W_y^{ss}$  through  $y$  can be written as the graph of a  $C^J$  map  $g_y$  from an open set of  $X_m^s$  to  $X_m^c \oplus X_m^u$  and  $g_y$  has small Lipschitz constant. Therefore, the stable fibers are roughly in the stable directions. We also want to point out that the stable foliation is constructed without using the existence of an invariant center manifold

in  $W^{cs}$ . Precise statements of the properties of the stable foliation can be found in Sect. 5.

For applications, it is important to have some estimate on the quantities such as  $\delta_0$ , the width of the center stable manifold in the stable direction, etc. In general, when we are interested in the dynamics near  $\psi(M)$ , we would like  $\delta_0$  be as large as possible to provide some characterization of the dynamics in a large neighborhood of  $\psi(M)$ . Sometimes, it is possible that the center stable manifold contains a center manifold and we may be interested in the location of the center manifold. In this case, we would like  $\delta_0$  be as small as possible. Therefore, throughout this paper, we will track these constants and their dependence on the given quantities. A precise statement about these parameters can be found in Theorem 4.2. In the proofs, the generic constants  $C = C(\frac{1}{r_0}, B, B_1, \frac{1}{\lambda}, \frac{1}{a}, L)$  and  $C_0 = C_0(B, B_1, \frac{1}{\lambda}, \frac{1}{a})$  are increasing in their arguments but may change from line to line. In the construction, the upper bound  $\sigma_0$  of  $\sigma$  is always taken independent of  $L$ . The independence of  $\sigma_0$  on  $L$  will be needed in one of the applications (see Subsect. 6.4).

### 3. Preliminaries

In this section, we establish various local coordinate systems in neighborhoods of  $\psi(M)$  and study their relationships. We assume (H1) and (H2) throughout this section, if not otherwise specified.

Since  $\psi$  is not an embedding, care must be taken to construct the coordinate systems in a tubular neighborhood of  $\psi(M)$ . We proceed as in [BLZ2]. Since  $\psi$  does not locally twist the manifold  $M$  very much (see (H2)), we are able to establish local tubular neighborhoods around  $\psi(B_c(m, r))$ , for some  $r > 0$  and every  $m \in M$ , and to obtain basic estimates. It may happen that some points in these tubular neighborhoods do not have globally unique representations, but this difficulty will be overcome.

**Lemma 3.1.** *If two projections  $P_1, P_2$  on  $X$  satisfy*

$$\|P_1 - P_2\| < 1,$$

*then  $P_2|_{P_1(X)}$  is an isomorphism from  $P_1(X)$  to  $P_2(X)$ .*

*Proof.* Note that

$$(P_1 - P_2)^2 = P_1 + P_2 - P_1 P_2 - P_2 P_1.$$

For any  $x \in P_1(X)$ , we have

$$x - P_1 P_2 x = (P_1 - P_2)^2 x,$$

which implies

$$\|(I - P_1 P_2)|_{P_1(X)}\| < 1.$$

Therefore,  $(P_1 P_2)|_{P_1(X)}$  is an isomorphism. Similarly,  $(P_2 P_1)|_{P_2(X)}$  is also an isomorphism, which implies  $P_2|_{P_1(X)}$  is an isomorphism.  $\square$

By the lemma and according to (H2), for any  $0 < r < \min\{r_0, \frac{1}{L}\}$ ,  $m_0 \in M$  and  $m_1 \in B_c(m_0, r)$ ,  $\Pi_{m_1}^\alpha|_{X_{m_0}^\alpha}$  is an isomorphism from  $X_{m_0}^\alpha$  to  $X_{m_1}^\alpha$  for  $\alpha = c, s, u$ . For any  $m_0 \in M$  and  $\varepsilon > 0$ , define

$$N(m_0, r, \varepsilon) = \{\psi(m) + x^s + x^u : m \in B_c(m_0, r), x^\alpha \in X_m^\alpha, |x^\alpha| < \varepsilon\}$$

which is the union of  $X_m^s(\varepsilon) \oplus X_m^u(\varepsilon)$  attached to  $\psi(m)$  for all  $m \in B_c(m_0, r)$ . We shall prove that when  $r$  and  $\varepsilon$  are small,  $N(m_0, r, \varepsilon)$  is a neighborhood of  $\psi(B_c(m_0, r))$  in which every point has a unique representation as given in the definition of  $N(m_0, r, \varepsilon)$ . First, we write points in  $N(m_0, r, \varepsilon)$  in different coordinate systems and study how they are related to each other.

Let  $m_i \in B_c(m_0, r_0)$  and  $x_i^\alpha \in X_{m_i}^\alpha$  for  $i = 1, 2$  and  $\alpha = u, s$ . We may write  $\psi(m_i) + x_i^s + x_i^u$  for  $i = 1, 2$  as

$$(3.1) \quad \begin{aligned} \psi(m_i) + x_i^s + x_i^u &= \psi(m_0) + \bar{x}_i^c + \bar{x}_i^s + \bar{x}_i^u \\ &= \psi(m_i) + \Pi_{m_i}^s \tilde{x}_i^s + \Pi_{m_i}^u \tilde{x}_i^u, \end{aligned}$$

where  $\tilde{x}_i^\alpha, \bar{x}_i^\alpha \in X_{m_0}^\alpha$ . A basic comparison between  $\tilde{x}_i^\alpha$  and  $\bar{x}_i^\alpha$  is given by

**Lemma 3.2.** *If  $|\tilde{x}_i^s| < \varepsilon$  for  $i = 1, 2$ , then*

$$(3.2) \quad \begin{aligned} &|\bar{x}_1^c - \bar{x}_2^c - (\psi(m_1) - \psi(m_2))| \\ &\leq (\chi + 2BL\varepsilon)|\psi(m_1) - \psi(m_2)| \\ &\quad + BL|\psi(m_i) - \psi(m_0)|(|\tilde{x}_1^s - \tilde{x}_2^s| + |\tilde{x}_1^u - \tilde{x}_2^u|) \end{aligned}$$

and, for  $\alpha = u, s$ ,

$$(3.3) \quad \begin{aligned} &|\bar{x}_1^\alpha - \bar{x}_2^\alpha - (\tilde{x}_1^\alpha - \tilde{x}_2^\alpha)| \\ &\leq (B\chi + 2BL\varepsilon)|\psi(m_1) - \psi(m_2)| \\ &\quad + BL|\psi(m_i) - \psi(m_0)|(|\tilde{x}_2^s - \tilde{x}_2^s| + |\tilde{x}_2^u - \tilde{x}_2^u|). \end{aligned}$$

*Proof.* From (3.1), we have

$$(3.4) \quad \begin{aligned} &\bar{x}_1^c - \bar{x}_2^c + \bar{x}_1^s - \bar{x}_2^s + \bar{x}_1^u - \bar{x}_2^u \\ &= \psi(m_1) - \psi(m_2) + (\Pi_{m_1}^s \tilde{x}_1^s - \Pi_{m_2}^s \tilde{x}_2^s) + (\Pi_{m_1}^u \tilde{x}_1^u - \Pi_{m_2}^u \tilde{x}_2^u). \end{aligned}$$

Applying the projection  $\Pi_{m_0}^c$  to (3.4), we obtain

$$\begin{aligned} \bar{x}_1^c - \bar{x}_2^c &= \Pi_{m_0}^c(\psi(m_1) - \psi(m_2)) + (\Pi_{m_0}^c - \Pi_{m_1}^c)\Pi_{m_1}^s(\tilde{x}_1^s - \tilde{x}_2^s) \\ &\quad + \Pi_{m_0}^c(\Pi_{m_1}^s - \Pi_{m_2}^s)\tilde{x}_2^s + (\Pi_{m_0}^c - \Pi_{m_1}^c)\Pi_{m_1}^u(\tilde{x}_1^u - \tilde{x}_2^u) \\ &\quad + \Pi_{m_0}^c(\Pi_{m_1}^u - \Pi_{m_2}^u)\tilde{x}_2^u. \end{aligned}$$

It follows from (H2) that

$$\begin{aligned}
& |\bar{x}_1^c - \bar{x}_2^c - (\psi(m_1) - \psi(m_2))| \\
& \leq \chi |\psi(m_1) - \psi(m_2)| + BL |\psi(m_1) - \psi(m_0)| (|\tilde{x}_1^s - \tilde{x}_2^s| + |\tilde{x}_1^u - \tilde{x}_2^u|) \\
& \quad + 2BL\varepsilon |\psi(m_1) - \psi(m_2)| \\
& \leq (\sigma + 2BL\varepsilon) |\psi(m_1) - \psi(m_2)| \\
& \quad + BL |\psi(m_1) - \psi(m_0)| (|\tilde{x}_1^s - \tilde{x}_2^s| + |\tilde{x}_1^u - \tilde{x}_2^u|),
\end{aligned}$$

which is (3.2) for  $i = 1$ . The case for  $i = 2$  can be obtained in the same fashion.

Finally, applying  $\Pi_{m_0}^s$  to (3.4), we have

$$\begin{aligned}
\bar{x}_1^s - \bar{x}_2^s &= \Pi_{m_0}^s (\psi(m_1) - \psi(m_2)) + \Pi_{m_0}^s \Pi_{m_1}^s (\tilde{x}_1^s - \tilde{x}_2^s) \\
&\quad + \Pi_{m_0}^s (\Pi_{m_1}^s - \Pi_{m_2}^s) \tilde{x}_2^s + \Pi_{m_0}^s (\Pi_{m_1}^u - \Pi_{m_0}^u) (\tilde{x}_1^u - \tilde{x}_2^u) \\
&\quad + \Pi_{m_0}^s (\Pi_{m_1}^u - \Pi_{m_2}^u) \tilde{x}_2^u.
\end{aligned}$$

Hence, from (H2), we obtain (3.3) for the stable direction. Applying  $\Pi_{m_0}^u$  yields the same estimate for the unstable direction.  $\square$

Note that we did not require  $\varepsilon$  to be a small number. As a consequence, we have the uniqueness of the local representation of points in  $N(m_0, r, \varepsilon)$  following a similar proof as in the paper [BLZ2] for overflowing invariant manifolds.

**Lemma 3.3.** *If  $m_i \in B_c(m_0, r)$ ,  $x_i^\alpha \in X_{m_i}^\alpha$ ,  $i = 1, 2$ , and*

$$(3.5) \quad r < \min \left\{ \frac{1}{4L}, \frac{r_0}{2} \right\}, \quad |x_i^\alpha| < \varepsilon \leq \frac{1}{8BL},$$

for  $\alpha = u, s$ , then

$$\psi(m_1) + x_1^u + x_1^s = \psi(m_2) + x_2^u + x_2^s$$

if and only if  $m_1 = m_2$ ,  $x_1^s = x_2^s$  and  $x_1^u = x_2^u$ .

*Proof.* Since  $|\psi(m_1) - \psi(m_2)| < 2r$ , from (H2), we have

$$|(\Pi_{m_2}^\alpha|_{X_{m_1}^\alpha})^{-1} x_2^\alpha| \leq (1 - 2Lr)^{-1} |x_2^\alpha| \leq 2|x_2^\alpha|,$$

for  $\alpha = u, s$ . Since  $m_2 \in B_c(m_0, r) \subset B_c(m_1, r_0)$ , one can apply Lemma 3.2 to the points  $\psi(m_1)$  and  $\psi(m_2) + x_2^s + x_2^u$ , with  $m_0$  replaced by  $m_1$ . Thus, we have

$$|\psi(m_1) - \psi(m_2)| \leq (\chi + 4BL\varepsilon) |\psi(m_1) - \psi(m_2)|,$$

where the assumption  $\psi(m_1) + x_1^s + x_1^u = \psi(m_2) + x_2^s + x_2^u$  is used. This implies that  $m_1 = m_2$ , and completes the proof.  $\square$



**Remark 3.4.** If (H2') is satisfied, one can obtain global uniqueness of representation with this coordinate system in the tubular neighborhoods of  $\psi(M)$ .

In the rest of the paper, we always assume that  $r$  and  $\varepsilon$  satisfy the condition (3.5) in Lemma 3.3. Since each point in  $N(m_0, r, \varepsilon)$  has a unique representation, we call  $m$  the base point of  $\psi(m) + x^s + x^u \in N(m_0, r, \varepsilon)$ . Our next step is to prove that  $N(m_0, r, \varepsilon)$  is a neighborhood of  $\psi(B_c(m_0, r))$ . Let  $P_{m_0}$  denote the map from  $X$  to  $X_{m_0}^c$  which is given by  $P_{m_0}(x) = \Pi_{m_0}^c(x - \psi(m_0))$ . Then, we have

**Lemma 3.5.**  $P_{m_0} \circ \psi$  is a diffeomorphism from  $B_c(m_0, r_0)$  to its image which is an open subset of  $X_{m_0}^c$ .

The proof is exactly the same as that of Lemma 3.4 in [BLZ2] and so we omit it. The next result states that any point in  $X$  close to  $\psi(m_0) + x_0^s + x_0^u$  can be written as  $\psi(m) + x^s + x^u$ .

**Lemma 3.6.** Let  $r$  and  $\varepsilon$  satisfy (3.5) and  $r_1 \in (0, r]$  be a number such that

$$X_{m_0}^c(r_1) \subset P_{m_0}\psi(B_c(m_0, r)).$$

Then for any  $x_0^\alpha \in X_{m_0}^\alpha(\varepsilon)$ ,  $\alpha = u, s$ , there exists  $\theta > 0$  such that for any  $x \in X$ , satisfying

$$|x - (\psi(m_0) + x_0^s + x_0^u)| < \theta,$$

there exist  $m \in (P_{m_0}\psi)^{-1}X_{m_0}^c(r_1)$  and  $x^\alpha \in X_m^\alpha(\varepsilon)$ ,  $\alpha = u, s$ , such that

$$x = \psi(m) + x^s + x^u.$$

Here  $\theta$  depends only on  $B, L, \varepsilon, r_1$ , and the norms  $|x_0^u|, |x_0^s|$ .

*Proof.* First note that, from Lemma 3.5,  $(P_{m_0}\psi)^{-1}$  is  $C^1$  from  $X_{m_0}^c(r_1)$  to  $B_c(m_0, r)$ . To show that  $x$  can be written as  $x = \psi(m) + x^s + x^u$ , we consider a map  $f$  from  $\overline{X_{m_0}^c(r_2)}$  to  $X_{m_0}^c$ , for some  $r_2 < r_1$ , which is defined as follows. For any  $x^c \in \overline{X_{m_0}^c(r_2)}$ , let  $m = (P_{m_0}\psi)^{-1}x^c$ . Notice that  $m \in B_c(m_0, r)$ . We define

$$f(x^c) = \Pi_{m_0}^c \Pi_m^c(x - \psi(m)) + x^c.$$

For  $x_i^c \in \overline{X_{m_0}^c(r_2)}$ ,  $i = 1, 2$ , letting  $m_i = (P_{m_0}\psi)^{-1}x_i^c$ , we compute

$$\begin{aligned} |f(x_2^c) - f(x_1^c)| &= |\Pi_{m_0}^c \Pi_{m_2}^c(\psi(m_1) - \psi(m_2)) + x_2^c - x_1^c \\ &\quad + \Pi_{m_0}^c(\Pi_{m_2}^c - \Pi_{m_1}^c)(x - \psi(m_1))| \\ &= |\Pi_{m_0}^c(\Pi_{m_2}^c - \Pi_{m_0}^c)(\psi(m_1) - \psi(m_2)) \\ &\quad + \Pi_{m_0}^c(\Pi_{m_2}^c - \Pi_{m_1}^c)(x - \psi(m_1))| \end{aligned}$$

$$\begin{aligned}
&\leq BL|\psi(m_2) - \psi(m_1)|(|\psi(m_2) - \psi(m_0)| + \theta + |x_0^s| \\
&\quad + |x_0^u| + |\psi(m_1) - \psi(m_0)|) \\
&\leq 2BL(\theta + |x_0^s| + |x_0^u| + 4r_2)|x_2^c - x_1^c|.
\end{aligned}$$

Therefore, by choosing  $\theta$  and  $r_2$  small enough such that

$$(3.6) \quad 2BL(\theta + |x_0^s| + |x_0^u| + 4r_2) < 1,$$

we have that  $f$  is a contraction. When  $x^c = 0$ , we have  $m = m_0$  and

$$|f(0)| = |\Pi_{m_0}^c \Pi_{m_0}^c(x - \psi(m_0))| \leq B\theta.$$

This together with (3.6) implies that for a smaller  $\theta$  such that

$$B\theta + 2BL(\theta + |x_0^s| + |x_0^u| + 4r_2)r_2 \leq r_2,$$

we have  $f$  maps  $\overline{X_{m_0}^c(r_2)}$  into itself. Therefore, by the contraction mapping theorem, there exists a unique  $x^c \in \overline{X_{m_0}^c(r_2)}$  satisfying

$$f(x^c) = x^c.$$

This implies that for the fixed point  $x^c$  and  $m = (P_{m_0}\psi)^{-1}x^c$ , we have that  $\Pi_m^c(x - \psi(m)) = 0$ . Denote  $x - \psi(m) = x^s + x^u$ . We want to show that  $|x^s|, |x^u| < \varepsilon$ . We note that

$$\begin{aligned}
|x^s| &= |\Pi_m^s(x - \psi(m))| \\
&\leq B\theta + |\Pi_m^s x_0^s| + |\Pi_m^s x_0^u| + |\Pi_m^s(\psi(m) - \psi(m_0))| \\
&\leq B\theta + |x_0^s| + \|\Pi_m^s - \Pi_{m_0}^s\|(|x_0^s| + |x_0^u| + |\psi(m) - \psi(m_0)|) \\
&\quad + |\Pi_{m_0}^s(\psi(m) - \psi(m_0) - x^c)| \\
&\leq B\theta + |x_0^s| + 2Lr_2(|x_0^s| + |x_0^u| + 2r_2) + 2B\chi r_2.
\end{aligned}$$

Thus, by choosing  $\theta$  and  $r_2$  sufficiently small, we obtain  $|x^s| < \varepsilon$ . The estimate for  $x^u$  is similar. This completes the proof of the lemma.  $\square$

As a consequence of this lemma,  $N(m_0, r, \varepsilon)$  is an open set containing  $\psi(B_c(m_0, r))$ .

In our hypothesis,  $M$  is a manifold without boundary. However,  $M$  is only a model of  $\psi(M)$  which is our real interest and it is possible that  $\psi(M)$  has boundary in  $X$ . To see this, we endow  $M$  with the Finsler structure induced by the immersion  $\psi$  and derive a metric on  $M$ . However,  $M$  may not be a complete metric space. Next, we state a lemma that can be used to show that points in  $u(M)$  are away from the boundary. The proof is exactly same as that of Lemma 3.6 in [BLZ2] and so we omit it.

**Lemma 3.7.** *For  $r_1 \leq r$  and  $m_0 \in M$ , if  $\psi(\overline{B_c(m_0, r_1)})$  is closed in  $X$ , then  $X_{m_0}^c((1 - \chi)r_1) \subset P_{m_0}(\psi(B_c(m_0, r_1)))$ .*

#### 4. Center-stable manifolds

The main result of this section is that, under assumptions (H1)–(H4) with  $\eta$ ,  $\chi$ ,  $\sigma$ , and  $\inf \mathcal{A}(\delta)$  sufficiently small,  $T$  has a positively invariant manifold  $W^{cs}$  close to  $\psi(M)$  and roughly in the direction of  $X^c$  and  $X^s$ . Our main tool is the graph transform.

Throughout this section, we fix  $\mu > 0$  such that

$$(4.1) \quad 0 < \mu < \frac{(1 - \lambda) \min\{1, a\}}{500BB_1},$$

where  $a$  is taken to be 1 if (H3') is not assumed. How small  $\eta$ ,  $\chi$ , and  $\sigma$  have to be is determined by  $\mu$ .

For  $\delta > 0$  and  $\alpha = u, s, c$  let

$$X^\alpha(\delta) = \{(m, x) : m \in M, x \in X_m^\alpha(\delta)\}.$$

Fixing  $r$  satisfying (3.5) and  $r > \delta_0 \geq \varepsilon > 0$ , consider a map  $h : X^s(\delta_0) \rightarrow X$ , satisfying

$$(4.2) \quad h(m, x^s) - (\psi(m) + x^s) \in \overline{X_m^u(\varepsilon)},$$

for all  $(m, x^s) \in X^s(\delta_0)$ . We say  $h$  has Lipschitz constant  $\mu$  if for any  $m_0 \in M$ ,  $m_i \in B_c(m_0, \delta_0)$ , and  $x_i^s \in X_{m_i}^s(\delta_0)$ ,  $i = 1, 2$ , one has

$$(4.3) \quad \begin{aligned} |\Pi_{m_0}^u(h(m_1, x_1^s) - h(m_2, x_2^s))| &\leq \mu (|\Pi_{m_0}^s(h(m_1, x_1^s) - h(m_2, x_2^s))| \\ &\quad + |\Pi_{m_0}^c(h(m_1, x_1^s) - h(m_2, x_2^s))|). \end{aligned}$$

**Definition 4.1.** Define  $\Gamma = \Gamma(\varepsilon, \mu, \delta_0)$  to be the collection of all maps  $h$  from  $X^s(\delta_0)$  to  $X$  with Lipschitz constant  $\mu$ . That is,  $h \in \Gamma$  if and only if (4.2) and (4.3) hold.

**Theorem 4.2.** Fix  $\mu$  satisfying (4.1). There exist constants  $\chi_0, \sigma_0, \mathcal{A}_0 > 0$ , determined (decreasingly) only by  $B, B_1, \frac{1}{a}, \lambda, \frac{1}{\mu}$ , such that, if  $\inf \mathcal{A}(\delta) < \mathcal{A}_0$  and  $\chi \in (0, \chi_0)$ , then there exists a constant  $\delta_0^* > 0$ , determined (decreasingly) by  $\frac{1}{r_0}, B, B_1, a, \lambda, L, \frac{1}{\sup\{\mathcal{A} < \mathcal{A}_0\}}, \frac{1}{\mu}, \frac{1}{\chi}$ , such that, when

$$\sigma < \sigma_0, \quad \delta_0 \leq \delta_0^*, \quad \frac{\eta}{\varepsilon} < C_0, \quad \frac{\varepsilon}{\delta_0} \in \left( \frac{8\sigma_0\lambda}{1-\lambda}, \frac{\mu(1-\lambda)}{10BB_1} \right)$$

for some  $C_0$  depending only on  $B, B_1, a, \lambda$ , there exists a unique  $h \in \Gamma$  so that  $T(h(X^s(\delta_0))) \subset h(X^s(\delta_0))$ . Moreover, the image of  $h$  is a  $C^J$  manifold.

The proof of this theorem is accomplished through a sequence of lemmas.

Define  $\psi_0 : X^s(\delta_0) \rightarrow X$  by

$$\psi_0(m, x^s) = \psi(m) + x^s.$$

Using (3.2) and (3.3), one may verify that  $\psi_0$  has Lipschitz constant

$$\frac{B\chi + BL\delta_0 + 4BL\varepsilon}{1 - 2B\chi - 2BL\delta_0 - 8BL\varepsilon}.$$

Therefore, if  $\chi$ ,  $\mu$ ,  $\delta_0$ , and  $\varepsilon$  satisfy

$$\chi \leq \frac{\mu}{8B} \quad \delta_0, \varepsilon \leq \frac{\mu}{20BL}$$

we have  $\psi_0 \in \Gamma$ . Therefore,  $\Gamma$  is not empty. Define a metric on  $\Gamma$  by

$$\|h_1 - h_2\| = \sup \{|h_1(m, x^s) - h_2(m, x^s)| : (m, x^s) \in X^s(\delta_0)\},$$

which makes  $\Gamma$  a complete metric space.

We first give local representations of Lipschitz maps  $h \in \Gamma$ . For any  $(m_0, x_0^s) \in X^s(\delta_0)$ , let  $\delta > 0$  be such that

$$(4.4) \quad \delta < \delta_0 - |x_0^s| \quad \text{and} \quad \psi(\overline{B_c(m_0, \delta)}) \text{ is closed in } X,$$

which roughly means that  $\delta$  is less than the distance from  $(m_0, x_0^s)$  to the boundary of  $X^s(\delta_0)$ . For any  $h \in \Gamma$ , a local representation will be constructed in a  $\rho\delta$ -neighborhood of  $(m_0, x_0^s)$ , where  $\rho \in (0, 1)$  is needed in the coordinate transformations.

**Lemma 4.3.** *There exist a constant  $C_0 > 0$  depending only on  $B$  and a constant  $C > 0$  depending only on  $B, L, r_0$  such that, for any  $\rho \in [\frac{3}{5}, \frac{9}{10}]$  and  $\chi \in (0, \frac{1}{2})$  satisfying  $C\delta_0 + C_0\chi < 1$ , any  $\delta_0 \geq 2\varepsilon$  and  $\delta > 0$  satisfying (4.4), and any  $h \in \Gamma$ , there exists a map*

$$f : \overline{X_{m_0}^c(\rho\delta)} \oplus \overline{X_{m_0}^s(\rho\delta)} \rightarrow X_{m_0}^u,$$

such that

$$\begin{aligned} & \{\psi(m_0) + x_0^s + x^c + x^s + f(x^c, x^s) : x^c \in \overline{X_{m_0}^c(\rho\delta)}, x^s \in \overline{X_{m_0}^s(\rho\delta)}\} \\ & \subset \{h(m, x^s) : m \in B_c(\psi(m_0), \delta), x^s \in X_{m_0}^s(\delta_0)\}. \end{aligned}$$

Furthermore, for any  $m \in B_c(m_0, \rho^2\delta)$ , and  $x^s \in X_{m_0}^s(\rho^2\delta)$ , we have

$$h(m, \Pi_m^s(x_0^s + x^s)) = \psi(m_0) + x_0^s + \bar{x}^c + \bar{x}^s + f(\bar{x}^c, \bar{x}^s),$$

for some  $\bar{x}^\alpha \in X_{m_0}^\alpha(\rho\delta)$ ,  $\alpha = c, s$ .

From the definition of  $\Gamma$ ,  $f$  has Lipschitz constant  $\mu$ . That is, for any  $x_i^\alpha \in X_{m_0}^\alpha(\rho\delta)$ ,  $|x_i^\alpha| \leq \rho\delta$ , for  $i = 1, 2$ , and  $\alpha = s, c$ , one has

$$|f(x_2^c, x_2^s) - f(x_1^c, x_1^s)| \leq \mu(|x_2^s - x_1^s| + |x_2^c - x_1^c|).$$

*Proof.* Let

$$h(m_0, x_0^s) = \psi(m_0) + x_0^s + x_0^u.$$

For  $m \in B_c(m_0, \rho^2\delta)$ ,  $x_0^s \in X_m^s(\rho^2\delta)$ , since

$$|\Pi_m^s(x_0^s + x^s)| \leq (1 + L\delta)(\delta_0 - \delta + \rho^2\delta) < \delta_0$$

we write

$$\begin{aligned} h(m, \Pi_m^s(x_0^s + x^s)) &= \psi(m) + \Pi_m^s(x_0^s + x^s) + \Pi_m^u(x_0^u + x^u) \\ &= \psi(m_0) + \bar{x}^c + \bar{x}^s + \bar{x}^u, \end{aligned}$$

where  $x^u \in X_{m_0}^u$  and  $\bar{x}^\alpha \in X_{m_0}^\alpha$ . From (3.2) and (3.3), we obtain

$$\begin{aligned} |\bar{x}^c| &\leq (1 + \chi + C\delta_0)|\psi(m) - \psi(m_0)| \leq \rho\delta, \\ \text{and } |\bar{x}^s - x_0^s| &\leq (B\chi + C\delta_0)|\psi(m) - \psi(m_0)| + |x^s| \leq \rho\delta. \end{aligned}$$

In order to finish the proof, we first show that for any  $x^c \in \overline{X_{m_0}^c(\rho\delta)}$  and  $x^s \in \overline{X_{m_0}^s(\rho\delta)}$ , there exists a unique  $x^u \in \overline{X_{m_0}^u(\delta_0)}$ , such that

$$\psi(m_0) + x_0^s + x^c + x^s + x^u$$

is in the image of  $h$ .

For any point in the above form, we would like to find  $m \in M$  such that it is in  $\psi(m) + X_m^u \oplus X_m^s$ . Lemma 3.6 does not apply directly. However, for small  $\tau \in [0, 1]$ , it follows from Lemma 3.6 that there exist unique  $m_\tau \in B_c(m_0, \delta)$  and  $\hat{x}_\tau^\alpha \in X_{m_0}^\alpha(\tilde{\varepsilon})$  such that

$$\psi(m_0) + x_0^s + \tau(x^s + x^u + x^c) = \psi(m_\tau) + \Pi_{m_\tau}^s \hat{x}_\tau^s + \Pi_{m_\tau}^u \hat{x}_\tau^u$$

where, from Lemma 3.3,

$$\tilde{\varepsilon} = \frac{1}{8BL}.$$

From Lemma 3.2, we have

$$\begin{aligned} |\psi(m_\tau) - \psi(m_0)| &\leq (1 - (\chi + 2BL\tilde{\varepsilon}))^{-1} |\tau x^c| \leq C_0\delta \\ \text{and } |\hat{x}_\tau^\alpha| &\leq \delta_0 + C_0\delta \end{aligned}$$

for  $\alpha = u, s$ . Therefore, using (3.2) and (3.3) again, we obtain

$$(4.5) \quad |\psi(m_\tau) - \psi(m_0)| \leq (1 - (\chi + C\delta_0))^{-1} |\tau x^c| < \delta.$$

Furthermore, we have,

$$\begin{aligned} |\hat{x}_\tau^s - x_0^s - \tau x^s| &\leq (B\chi + C\delta_0)|\psi(m_\tau) - \psi(m_0)|, \\ \text{and } |\hat{x}_\tau^u - \tau x^u| &\leq (B\chi + C\delta_0)|\psi(m_\tau) - \psi(m_0)|. \end{aligned}$$

Thus,

$$(4.6) \quad |\Pi_{m_\tau}^s \hat{x}_\tau^s|, |\hat{x}_\tau^s| < \delta_0, \quad \text{and} \quad |\hat{x}_\tau^u| \leq \delta_0 + (B\chi + C\delta_0)\delta.$$

From a continuation argument, there exist unique  $m \in B_c(m_0, \delta)$  and  $\hat{x}^\alpha \in X_{m_0}^\alpha$  such that

$$\psi(m_0) + x_0^s + x^s + x^u + x^c = \psi(m) + \Pi_m^s \hat{x}^s + \Pi_m^u \hat{x}^u$$

and  $m, \hat{x}^\alpha$  satisfy (4.5) and (4.6).

Thus, we can define the map from  $\overline{X_{m_0}^u(\delta_0)}$  to  $X_{m_0}^u$ :

$$E(x^u) \equiv \Pi_{m_0}^u(h(m, \Pi_m^s \hat{x}^s) - \psi(m_0)).$$

We shall show that  $E$  is a contraction on  $\overline{X_{m_0}^u(\delta_0)}$ . For  $x_i^u \in \overline{X_{m_0}^u(\delta)}$ , let  $\hat{x}_i^\alpha$  be defined as above for  $i = 1, 2$ , and  $\alpha = u, s$ . From (4.5), (3.2), and (3.3), it follows that

$$\begin{aligned} |\psi(m_2) - \psi(m_1)| &\leq (\chi + C\delta_0)|\psi(m_2) - \psi(m_1)| \\ &\quad + C\delta(|\hat{x}_2^u - \hat{x}_1^u| + |\hat{x}_2^s - \hat{x}_1^s|), \\ |\hat{x}_2^s - \hat{x}_1^s| &\leq (B\chi + C\delta_0)|\psi(m_2) - \psi(m_1)| \\ &\quad + C\delta(|\hat{x}_2^u - \hat{x}_1^u| + |\hat{x}_2^s - \hat{x}_1^s|), \\ \text{and } |\hat{x}_2^u - \hat{x}_1^u| &\leq (B\chi + C\delta_0)|\psi(m_2) - \psi(m_1)| \\ &\quad + C\delta(|\hat{x}_2^u - \hat{x}_1^u| + |\hat{x}_2^s - \hat{x}_1^s|) + |x_2^u - x_1^u|, \end{aligned}$$

which imply that

$$\begin{aligned} |\psi(m_2) - \psi(m_1)| + |\hat{x}_2^s - \hat{x}_1^s| &\leq C\delta|x_2^u - x_1^u| \\ \text{and } |\hat{x}_2^u - \hat{x}_1^u| &\leq (1 + C\delta)|x_2^u - x_1^u|. \end{aligned}$$

Write  $h(m_i, \Pi_{m_i}^s \hat{x}_i^s)$  for  $i = 1, 2$ , as

$$h(m_i, \Pi_{m_i}^s \hat{x}_i^s) = \psi(m_i) + \Pi_{m_i}^s \hat{x}_i^s + \Pi_{m_i}^u \tilde{x}_i^u = \psi(m_0) + \bar{x}_i^c + \bar{x}_i^s + \bar{x}_i^u,$$

where  $\tilde{x}_i^\alpha, \bar{x}_i^\alpha \in X_{m_0}^\alpha$ . From (3.2) and (3.3), we obtain

$$|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s| \leq C\delta(|\tilde{x}_2^u - \tilde{x}_1^u| + |x_2^u - x_1^u|).$$

Therefore,

$$|\bar{x}_2^u - \bar{x}_1^u| \leq (1 + C\delta)|\bar{x}_2^u - \bar{x}_1^u| + C\delta|x_2^u - x_1^u|,$$

which implies

$$|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s| \leq C\delta(|\bar{x}_2^u - \bar{x}_1^u| + |x_2^u - x_1^u|).$$

Since  $h$  is Lipschitz, we have,

$$|\bar{x}_2^u - \bar{x}_1^u| \leq \mu(|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s|) \leq C\mu\delta|x_2^u - x_1^u|.$$

This shows that  $E$  is a contraction with Lipschitz constant  $C\mu\delta$ . On the other hand, from (3.3), (4.5), and the definition of  $\Gamma$ ,

$$|\bar{x}_i^u| \leq |\tilde{x}_i^u| + (B\chi + C\delta_0)\delta \leq (1 - L\delta)^{-1}\varepsilon + (B\chi + C\delta_0)\delta < \delta_0.$$

Therefore,  $E$  maps  $\overline{X_{m_0}^u(\delta_0)}$  to itself. The contraction mapping principle implies that there exists a unique fixed point  $x^u$ .

By the construction of  $E$  and (4.5), there exists a unique  $m \in B_c(m_0, \delta)$  and  $\bar{x}^c \in X_{m_0}^c$ ,  $\hat{x}^s, \bar{x}^s \in X_{m_0}^s$ , and  $\hat{x}^u, \tilde{x}^u \in X_{m_0}^u$ , such that  $|\Pi_m^s \hat{x}^s| \leq \delta_0$ ,

$$\psi(m_0) + x_0^s + x^s + x^u + x^c = \psi(m) + \Pi_m^s \hat{x}^s + \Pi_m^u \hat{x}^u,$$

$$\text{and } h(m, \Pi_m^s \hat{x}^s) = \psi(m) + \Pi_m^s \hat{x}^s + \Pi_m^u \tilde{x}^u = \psi(m_0) + \bar{x}^c + \bar{x}^s + x^u.$$

Inequalities (3.2), (3.3), and (4.5) immediately imply that  $\hat{x}^u = \tilde{x}^u$ . By defining  $f(x^c, x^s) = x^u$ , we complete the proof.  $\square$

Our next step is to construct a transformation  $\mathcal{F}$  on the space  $\Gamma$  of  $\mu$ -Lipschitz maps. The basic idea is that, for any  $h \in \Gamma$ ,

$$\mathcal{F}(h)(X^s(\delta_0)) \subset T^{-1}(h(X^s(\delta_0))),$$

i.e., the image of  $\mathcal{F}(h)$  is contained in the preimage of the image of  $h$  under the map  $T$ . As  $T$  is not assumed to be a homeomorphism, the preimage of this Lipschitz section might be rather wild. However, the approximate hyperbolicity condition (H3) guarantees that such a transformation  $\mathcal{F}$  can be defined.

Recall that the parameter  $\eta > 0$  is from Definition 2.1 of approximate inflowing manifolds. In this and the following sections, several lemmas will have the same preamble as that of Theorem 4.2 and so we single it out and refer to it as

- (S) “Fix  $\mu$  satisfying (4.1). There exist constants  $\chi_0, \sigma_0, \mathcal{A}_0 > 0$ , determined (decreasingly) only by  $B, B_1, \frac{1}{a}, \lambda, \frac{1}{\mu}$ , such that, if  $\inf \mathcal{A}(\delta) < \mathcal{A}_0$  and  $\chi \in (0, \chi_0)$ , then there exists a constant  $\delta_0^* > 0$ , determined (decreasingly) by  $\frac{1}{r_0}, B, B_1, a, \lambda, L, \frac{1}{\sup\{\mathcal{A} < \mathcal{A}_0\}}, \frac{1}{\mu}, \frac{1}{\chi}$ , such that, when

$$(4.7) \quad \sigma < \sigma_0, \quad \delta_0 \leq \delta_0^*, \quad \frac{\eta}{\varepsilon} < C_0, \quad \frac{\varepsilon}{\delta_0} \in \left( \frac{8\sigma_0\lambda}{1-\lambda}, \frac{\mu(1-\lambda)}{10BB_1} \right)$$

for some  $C_0$  depending only on  $B, B_1, a, \lambda$ ”.

When (H3') is not assumed,  $a$  is understood to be 1.

**Lemma 4.4.** *With preamble (S), for any  $(m, x^s) \in X^s(\delta_0)$ , and  $x^u \in \overline{X_m^u(\varepsilon)}$ , there exist unique  $m_1 \in B_c(u(m), r)$ ,  $x_1^s \in X_{m_1}^s(\frac{1+\lambda}{2}\delta_0)$ , and  $x_1^u \in X_{m_1}^u(\frac{\delta_0}{2})$  such that*

$$T(\psi(m) + x^s + x^u) = \psi(m_1) + x_1^s + x_1^u \quad \text{and} \quad |\psi(u(m)) - \psi(m_1)| < \frac{\delta_0}{2}.$$



*Proof.* Let  $\bar{m} = u(m)$ . We have

$$\begin{aligned} & |\psi(\bar{m}) - T(\psi(m) + x^s + x^u)| \\ & \leq |\psi(\bar{m}) - T(\psi(m))| + |T(\psi(m) + x^s + x^u) - T(\psi(m))| \\ & \leq \eta + B_1(\delta_0 + \varepsilon). \end{aligned}$$

According to Lemmas 3.6 and 3.7, there exist  $m_1 \in B_c(\bar{m}, r)$ ,  $x_1^s \in X_{m_1}^s$ , and  $x_1^u \in X_{m_1}^u$ , such that

$$T(\psi(m) + x^s + x^u) = \psi(m_1) + x_1^s + x_1^u \quad \text{with } |x_1^\alpha| \leq \frac{1 - \chi}{20BL}.$$

In order to complete the proof, we only need to estimate  $|x_1^s|$ ,  $|x_1^u|$ , and  $|\psi(\bar{m}) - \psi(m_1)|$ . Write

$$\psi(m_1) + x_1^s + x_1^u = \psi(m_1) + \Pi_{m_1}^s \tilde{x}^s + \Pi_{m_1}^u \tilde{x}^u = \psi(\bar{m}) + \bar{x}^c + \bar{x}^s + \bar{x}^u,$$

where  $\tilde{x}^\alpha, \bar{x}^\alpha \in X_{\bar{m}}^\alpha$ . For  $\tau \in [0, 1]$ , let

$$x_\tau = \psi(m) + \tau(x^s + x^u).$$

We have

$$\bar{x}^\alpha = \Pi_{\bar{m}}^\alpha \int_0^1 DT(x_\tau)(x^s + x^u) d\tau + \Pi_{\bar{m}}^\alpha (T(\psi(m)) - \psi(\bar{m})).$$

Therefore,

$$|\bar{x}^\alpha - \Pi_{\bar{m}}^\alpha DT(\psi(m))(x^s + x^u)| \leq B\eta + B\mathcal{A}(2\delta_0)(|x^u| + |x^s|).$$

From condition (H3), we have

$$\begin{aligned} (4.8) \quad & |\bar{x}^u|, |\bar{x}^c| \leq B\eta + (BB_1 + B\mathcal{A}(2\delta_0))|x^u| + (\sigma + B\mathcal{A}(2\delta_0))|x^s|, \\ & |\bar{x}^s| \leq B\eta + (BB_1 + B\mathcal{A}(2\delta_0))|x^u| + (\lambda + B\mathcal{A}(2\delta_0))|x^s|. \end{aligned}$$

Therefore, the desired estimate follows immediately from (3.2) and (3.3) and the uniqueness follows from Lemma 3.3.  $\square$

In the following lemma, for  $m_0 \in M$ , we denote  $m_1 = u(m_0)$ . Since  $M$  is approximately invariant,  $\psi(m_1)$  is an approximation of  $T(\psi(m))$ .

**Lemma 4.5.** *With preamble (S), for any  $h \in \Gamma$  and  $(m_0, x_0^s) \in X^s(\delta_0)$ , there exists a unique  $x_0^u \in \overline{X}_{m_0}^u(\varepsilon)$ , such that  $T(\psi(m_0) + x_0^s + x_0^u)$  lies in the image of  $h$ .*

*Proof.* From Lemma 4.4, we can write

$$T(\psi(m_0) + x_0^s) = \psi(m_*) + x_*^s + x_*^u = \psi(m_1) + \bar{x}_*^s + \bar{x}_*^u + \bar{x}_*^c,$$

where  $m_* \in B_c(m_1, r)$ ,  $x_*^\alpha \in X_{m_*}^\alpha$ , and  $\bar{x}_*^\alpha \in X_{m_1}^\alpha$  with

$$|\psi(m_*) - \psi(m_1)| < \frac{1}{2}\delta_0, \quad |x_*^u| < \frac{1}{2}\delta_0, \quad |x_*^s| < \frac{1+\lambda}{2}\delta_0.$$

From (3.2), (3.3), (4.7), and (4.8), we have

$$(4.9) \quad |x_*^u| \leq (1 + C\delta_0)(|\bar{x}_*^u| + (B\chi + C\delta_0)(1 - \chi - C\delta_0)^{-1}|\bar{x}_*^c|) \\ \leq 2B\eta + (2\sigma + 2B\mathcal{A}(\delta_0))\frac{\delta_0}{\varepsilon} \leq \frac{1-\lambda}{2\lambda}\varepsilon.$$

Let  $\delta = \frac{1-\lambda}{2}\delta_0$ . For  $h \in \Gamma$ , Definition 2.1 and Lemma 4.3 imply that there exists a Lipschitz representation  $f : \overline{X_{m_*}^c(\frac{\delta}{2})} \times \overline{X_{m_*}^s(\frac{\delta}{2})} \rightarrow X_{m_*}^u$  of  $h$  near  $(m_*, x_*^s)$ . For any  $x^u \in \overline{X_{m_0}^u(\varepsilon)}$ , let

$$T(\psi(m_0) + x_0^s + x^u) = m_* + x_*^s + x_*^u + \bar{x}^c + \bar{x}^u + \bar{x}^s,$$

with  $\bar{x}^\alpha \in X_{m_*}^\alpha$ . Then we have, from (H3) and (4.7), for  $\alpha = s, c$ ,

$$|\bar{x}^\alpha| = |\Pi_{m_*}^\alpha(T(\psi(m_0) + x_0^s + x^u) - T(\psi(m_0) + x_0^s))| \leq BB_1\varepsilon \leq \frac{\delta}{2}.$$

Define

$$E(x^u) = x^u + (\Pi_{m_1}^u DT(\psi(m_0))|_{X_{m_0}^u})^{-1} \Pi_{m_1}^u (f(\bar{x}^c, \bar{x}^s) - \bar{x}^u - x_*^u).$$

We shall prove that  $E$  is a contraction on  $\overline{X_{m_0}^u(\varepsilon)}$ . In fact, for any  $x_i^u \in \overline{X_{m_0}^u(\varepsilon)}$ ,  $i = 1, 2$ , let  $\bar{x}_i^\alpha \in X_{m_*}^\alpha$  be defined as above. Then

$$(4.10) \quad |\bar{x}_2^\alpha - \bar{x}_1^\alpha - \Pi_{m_1}^\alpha DT(\psi(m_0))(x_2^u - x_1^u)| \\ \leq |\Pi_{m_*}^\alpha(T(\psi(m_0) + x_0^s + x_2^u) - T(\psi(m_0) + x_0^s + x_1^u)) \\ - \Pi_{m_1}^\alpha DT(\psi(m_0))(x_2^u - x_1^u)| \\ \leq (C\delta_0 + B\mathcal{A}(2\delta_0))|x_2^u - x_1^u|,$$

for  $\alpha = c, s, u$ . Since  $f$  has Lipschitz constant  $\mu$ , from (H3), we obtain

$$|E(x_2^u) - E(x_1^u)| = |(\Pi_{m_1}^u DT(\psi(m_0))|_{X_{m_0}^u})^{-1} \\ \times \Pi_{m_1}^u [-(\bar{x}_2^u - \bar{x}_1^u) + \Pi_{m_1}^u DT(\psi(m_0))(x_2^u - x_1^u) \\ + f(\bar{x}_2^c, \bar{x}_2^s) - f(\bar{x}_1^c, \bar{x}_1^s)]| \\ \leq (C\delta_0 + C_0\mathcal{A}(2\delta_0))|x_2^u - x_1^u| \\ + (1 + C\delta_0)\lambda\mu(|\bar{x}_2^s - \bar{x}_1^s| + |\bar{x}_2^c - \bar{x}_1^c|) \\ \leq (C\delta_0 + C_0\mathcal{A}(2\delta_0) + 3BB_1\lambda\mu)|x_2^u - x_1^u| \\ \leq \frac{1-\lambda}{10}|x_2^u - x_1^u|,$$

where (4.1), (4.10), and (4.7) were used. Recall that  $C_0$  depends only on  $B, B_1$ , and  $\lambda$ . Finally, when  $x^u = 0$ , we have  $\bar{x}^\alpha = 0$  for  $\alpha = c, s, u$ . Consequently, from (4.9)

$$\begin{aligned} |E(0)| &= \left| \left( \Pi_{m_1}^u DT(\psi(m_0)) \Big|_{x_{m_0}^u} \right)^{-1} \Pi_{m_1}^u (f(0, 0) - x_*^u) \right| \\ &\leq (1 + C\delta_0)\lambda(\varepsilon + |x_*^u|) \leq \frac{2\lambda + 1}{3}\varepsilon. \end{aligned}$$

Therefore,  $E$  is a contraction on  $\overline{X_{m_0}^u(\varepsilon)}$ . Let  $x_0^u \in \overline{X_{m_0}^u(\varepsilon)}$  be the unique fixed point, then clearly it satisfies the requirement in the lemma.  $\square$

Fixing  $\rho \in [\frac{3}{5}, \frac{9}{10}]$ , Lemma 4.5 allows us to define a map  $\tilde{h} : X^s(\delta_0) \rightarrow X$  by

$$\tilde{h}(m_0, x_0^s) = \psi(m_0) + x_0^s + x_0^u,$$

where  $x_0^u \in \overline{X_{m_0}^u(\varepsilon)}$  and it satisfies

$$T(\tilde{h}(X^s(\delta_0))) \subset h(X^s(\delta_0)).$$

The next lemma states that  $\tilde{h}$  has Lipschitz constant  $\mu$  and thus  $\tilde{h} \in \Gamma$ .

**Lemma 4.6.** *With preamble (S), for any  $h \in \Gamma$ ,  $\tilde{h}$  defined above has Lipschitz constant  $\mu$ .*

*Proof.* Taking any  $m_0 \in M$ ,  $m_i \in B_c(m_0, \delta_0)$ , and  $x_i^s \in X_{m_i}^s(\delta_0)$ ,  $i = 1, 2$ , let

$$\begin{aligned} (4.11) \quad \tilde{h}(m_i, x_i^s) &= \psi(m_i) + x_i^s + x_i^u = \psi(m_i) + \Pi_{m_i}^s x_i^{s*} + \Pi_{m_i}^u x_i^{u*} \\ &= \psi(m_0) + \bar{x}_i^c + \bar{x}_i^s + \bar{x}_i^u, \end{aligned}$$

where  $x_i^u \in \overline{X_{m_i}^u(\varepsilon)}$  and  $x_i^{\alpha*}, \bar{x}_i^\alpha \in X_{m_0}^\alpha$ . According to (4.3), we need to prove

$$(4.12) \quad |\bar{x}_2^u - \bar{x}_1^u| \leq \mu(|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s|).$$

From the definition of  $\tilde{h}$ , for  $i = 1, 2$ , there exist  $(\tilde{m}_i, \tilde{x}_i^s) \in X^s(\delta_0)$ ,  $\tilde{x}_i^u \in X_{m_i}^u(\varepsilon)$ , such that

$$(4.13) \quad T(\tilde{h}(m_i, x_i^s)) = h(\tilde{m}_i, \tilde{x}_i^s) = \psi(\tilde{m}_i) + \tilde{x}_i^s + \tilde{x}_i^u = \psi(\hat{m}) + \hat{x}_i^c + \hat{x}_i^s + \hat{x}_i^u,$$

where  $\hat{m} = u(m_0)$ ,  $\hat{x}_i^\alpha \in X_{\hat{m}}^\alpha$ ,  $i = 1, 2$ , and  $\alpha = c, s, u$ . Since

$$|x_i^{s*}| \leq (1 - L\delta_0)^{-1}\delta_0, \quad |x_i^{u*}| \leq (1 - L\delta_0)^{-1}\varepsilon,$$

(4.11) and inequalities (3.2) and (3.3) imply

$$\begin{aligned} |\bar{x}_i^c| &\leq (1 + \chi + C\delta_0)\delta_0, \quad |\bar{x}_i^s| \leq (1 + B\chi + C\delta_0)\delta_0, \\ \text{and} \quad |\bar{x}_i^u| &\leq \varepsilon + (B\chi + C\delta_0)\delta_0. \end{aligned}$$

In order to prove (4.12), we only need to consider the case when

$$(4.14) \quad \mu(|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s|) \leq 2\varepsilon + (2B\chi + C\delta_0)\delta_0.$$

Using (4.13), we have, for  $i = 1, 2$ ,

$$\begin{aligned} |\psi(\tilde{m}_i) - \psi(\hat{m})| &\leq |\tilde{x}_i^s| + |\tilde{x}_i^u| + |T(\tilde{h}(m_i, x_i^s)) - T(\psi(m_0))| + \eta \\ &\leq \eta + C_0\varepsilon + C_0\delta_0 \leq C_0\delta_0. \end{aligned}$$

From Lemma 4.4,  $m_i \in B_c(\hat{m}, r)$ . For  $\tau \in [0, 1]$ , let

$$x_\tau = \psi(m_0) + (1 - \tau)\bar{x}_1^c + \tau\bar{x}_2^c + (1 - \tau)\bar{x}_1^s + \tau\bar{x}_2^s + (1 - \tau)\bar{x}_1^u + \tau\bar{x}_2^u.$$

Write

$$\begin{aligned} h(\tilde{m}_2, \tilde{x}_2^s) - h(\tilde{m}_1, \tilde{x}_1^s) &= T(\tilde{h}(m_2, x_2^s)) - T(\tilde{h}(m_1, x_1^s)) \\ &= \int_0^1 DT(x_\tau) d\tau (\bar{x}_2^c - \bar{x}_1^c + \bar{x}_2^s - \bar{x}_1^s + \bar{x}_2^u - \bar{x}_1^u). \end{aligned}$$

We obtain, from (4.14) and (4.7), for  $\alpha = c, s, u$ ,

$$\begin{aligned} |\hat{x}_2^\alpha - \hat{x}_1^\alpha| &= \left| \Pi_{\tilde{m}}^\alpha \int_0^1 DT(x_\tau) (\bar{x}_2^c - \bar{x}_1^c + \bar{x}_2^s - \bar{x}_1^s + \bar{x}_2^u - \bar{x}_1^u) d\tau \right| \\ &\leq \frac{1}{\mu} (4BB_1\varepsilon + C_0\chi\delta_0 + C\delta_0^2) \leq \frac{\delta_0}{2}. \end{aligned}$$

From inequality (3.2),

$$|\psi(m_2) - \psi(m_1)| < \delta_0.$$

Let

$$h(\tilde{m}_2, \tilde{x}_2^s) = \psi(\tilde{m}_2) + \tilde{x}_2^s + \tilde{x}_2^u = \psi(\tilde{m}_1) + \tilde{x}_*^c + \tilde{x}_*^s + \tilde{x}_*^u.$$

Since  $h \in \Gamma$ , we have

$$(4.15) \quad |\tilde{x}_*^u - \tilde{x}_1^u| \leq \mu(|\tilde{x}_*^c| + |\tilde{x}_*^s - \tilde{x}_1^s|).$$

On the other hand, for  $\alpha = c, s, u$ ,

$$\tilde{x}_*^\alpha - \tilde{x}_1^\alpha = \Pi_{\tilde{m}_1}^\alpha \int_0^1 DT(x_\tau) (\bar{x}_2^c - \bar{x}_1^c + \bar{x}_2^s - \bar{x}_1^s + \bar{x}_2^u - \bar{x}_1^u) d\tau.$$

We have

$$\begin{aligned} &|\tilde{x}_*^\alpha - \tilde{x}_1^\alpha - \Pi_{\tilde{m}}^\alpha DT(\psi(m_0))(\bar{x}_2^c - \bar{x}_1^c + \bar{x}_2^s - \bar{x}_1^s + \bar{x}_2^u - \bar{x}_1^u)| \\ &\leq (C\delta_0 + B\mathcal{A}(3\delta_0))(|\bar{x}_2^c - \bar{x}_1^c| + |\bar{x}_2^s - \bar{x}_1^s| + |\bar{x}_2^u - \bar{x}_1^u|) \end{aligned}$$

where  $\tilde{x}_1^c$  is understood as 0. This implies

$$\begin{aligned} |\tilde{x}_*^u - \tilde{x}_1^u| &\geq \left\| \left( \Pi_m^u DT(\psi(m_0)) \Big|_{x_{m_0}^u} \right)^{-1} \right\|^{-1} |\tilde{x}_2^u - \tilde{x}_1^u| \\ &\quad - (\sigma + C\delta_0 + B\mathcal{A}(3\delta_0)) (|\tilde{x}_2^c - \tilde{x}_1^c| + |\tilde{x}_2^s - \tilde{x}_1^s| + |\tilde{x}_2^u - \tilde{x}_1^u|) \end{aligned}$$

and

$$\begin{aligned} &|\tilde{x}_*^s - \tilde{x}_1^s| + |\tilde{x}_*^c| \\ &\leq \lambda |\tilde{x}_2^s - \tilde{x}_1^s| + \left\| \Pi_m^c DT(\psi(m_0)) \Big|_{x_{m_0}^c} \right\| |\tilde{x}_2^c - \tilde{x}_1^c| + 2BB_1 |\tilde{x}_2^u - \tilde{x}_1^u| \\ &\quad + (\sigma + C\delta_0 + B\mathcal{A}(3\delta_0)) (|\tilde{x}_2^c - \tilde{x}_1^c| + |\tilde{x}_2^s - \tilde{x}_1^s| + |\tilde{x}_2^u - \tilde{x}_1^u|). \end{aligned}$$

Therefore, from (4.15) and (H3), we obtain

$$\begin{aligned} &|\tilde{x}_2^u - \tilde{x}_1^u| \\ &\leq \frac{\lambda\mu + 2\sigma + C\delta_0 + 2B\mathcal{A}(3\delta_0)}{1 - 2BB_1\lambda\mu - 2\lambda\sigma - C\delta_0 - 2B\mathcal{A}(3\delta_0)} (|\tilde{x}_2^c - \tilde{x}_1^c| + |\tilde{x}_2^s - \tilde{x}_1^s|), \end{aligned}$$

which, along with condition (4.1) on  $\mu$ , implies (4.12).  $\square$

From Lemmas 4.5 and 4.6, we can define a transform  $\mathcal{F}$  on the space  $\Gamma$  as

$$\mathcal{F}(h) = \tilde{h}.$$

Since  $\tilde{h}$  is the unique map in  $\Gamma$  such that

$$T(\tilde{h}(X^s(\delta_0))) \subset h(X^s(\delta_0)),$$

a map  $h \in \Gamma$  is a fixed point of  $\mathcal{F}$  if and only if  $h(X^s(\delta_0))$  is positively invariant under  $T$ . We shall prove that  $\mathcal{F}$  is a contraction on  $\Gamma$ , the essential step being the following lemma.

Let  $h \in \Gamma$ ,  $(m_0, x_0^s) \in X^s(\delta_0)$ , and

$$\tilde{h}(m_0, x_0^s) = \psi(m_0) + x_0^s + x_0^u.$$

For any  $x_1^u \in X_{m_0}^u(2\varepsilon)$ , from Lemma 4.4 applied for a slightly different  $\varepsilon > 0$ , we can write

$$\begin{aligned} T(\psi(m_0) + x_0^s + x_1^u) &= \psi(\bar{m}_1) + \bar{x}_1^s + \bar{x}_1^u \\ \text{with } |\bar{x}_1^s| &< \frac{1+\lambda}{2}\delta_0 \text{ and } |\bar{x}_1^u| < \frac{\delta_0}{2}. \end{aligned}$$

Let  $h(\bar{m}_1, \bar{x}_1^s) = \psi(\bar{m}_1) + \bar{x}_1^s + \tilde{x}_0^u$ , under the conditions in Lemmas 4.5 and 4.6, we have

**Lemma 4.7.** *There exists  $\lambda_1 \in (\lambda, 1)$ , independent of  $h, m_0, x_0^s, x_0^u, x_1^u$ , such that*

$$|x_1^u - x_0^u| \leq \lambda_1 |\bar{x}_1^u - \tilde{x}_0^u|.$$

*Proof.* Let  $\tilde{m}_0 = u(m_0)$  and

$$T(\tilde{h}(m_0, x_0^s)) = h(\tilde{m}_0, \bar{x}_0^s) = \psi(\tilde{m}_0) + \bar{x}_0^s + \bar{x}_0^u = \psi(\tilde{m}_1) + \hat{x}_0^c + \hat{x}_0^s + \hat{x}_0^u,$$

where  $\hat{x}_0^\alpha \in X_{\tilde{m}_1}^\alpha$  and  $\bar{x}_0^\alpha \in X_{\tilde{m}_0}^\alpha(\delta_0)$ . From Lemma 4.4,

$$|\psi(\tilde{m}_0) - \psi(\tilde{m}_1)|, |\psi(\tilde{m}_0) - \psi(\tilde{m}_1)| \leq \frac{\delta_0}{2}.$$

For  $\alpha = c, s, u$ , write

$$\bar{x}_1^\alpha - \hat{x}_0^\alpha = \Pi_{\tilde{m}_1}^\alpha \int_0^1 DT(\psi(m_0) + x_0^s + (1-\tau)x_0^u + \tau x_1^u)(x_1^u - x_0^u) d\tau$$

and we have

$$|\bar{x}_1^\alpha - \hat{x}_0^\alpha - \Pi_{\tilde{m}_0}^\alpha DT(\psi(m_0))(x_1^u - x_0^u)| \leq (C\delta_0 + B\mathcal{A}(2\delta_0))|x_1^u - x_0^u|,$$

where it is understood that  $\bar{x}_1^c = 0$ . Therefore, from (H3), we obtain

$$\begin{aligned} |\hat{x}_0^c| + |\hat{x}_0^s - \bar{x}_1^s| &\leq (2BB_1 + C\delta_0 + B\mathcal{A}(2\delta_0))|x_1^u - x_0^u| \\ |\hat{x}_0^u - \bar{x}_1^u| &\geq (\lambda^{-1} - C\delta_0 - B\mathcal{A}(2\delta_0))|x_1^u - x_0^u|. \end{aligned}$$

On the other hand, since  $h$  has Lipschitz constant  $\mu$ ,

$$|\tilde{x}_0^u - \hat{x}_0^u| \leq \mu(|\hat{x}_0^c| + |\bar{x}_1^s - \hat{x}_0^s|) \leq \mu(2BB_1 + C\delta_0 + B\mathcal{A}(2\delta_0))|x_1^u - x_0^u|.$$

The lemma immediately follows from these inequalities and (4.1).  $\square$

**Corollary 4.8.**  $\mathcal{F}$  is a contraction on  $\Gamma$ , with Lipschitz constant  $\lambda_1$ .

To prove the corollary, taking any  $h, h_1 \in \Gamma$  and  $(m_0, x_0^s)$ , one obtains the estimate of

$$|\mathcal{F}(h_1)(m_0, x_0^s) - \mathcal{F}(h)(m_0, x_0^s)|$$

from Lemma 4.7 by letting  $x_1^u = \mathcal{F}(h_1)(m_0, x_0^s) - \psi(m_0) - x_0^s$ .

Thus, there exists a unique fixed point  $h_0 \in \Gamma$  of  $\mathcal{F}$  and

$$W^{cs} \equiv h_0(X^s(\delta_0))$$

is positively invariant under  $T$ . We call it the center-stable manifold around  $\psi(M)$ .

*Remark 4.9.* When (H2') holds,  $h_0$  is an embedding.

The higher order smoothness of  $W^{cs}$  follows from Theorem B and Theorem 7.3 in [BLZ2] and so the proof of Theorem 4.2 is complete.

To end this section, we present the following characterization of  $W^{cs}$ . Let  $U$  be the tubular neighborhood of  $\psi(M)$ ,

$$U = \{\psi(m) + x^s + x^u : x^s \in X_m^s(\delta_0), x^u \in X_m^u(2\varepsilon)\}.$$

**Proposition 4.10.** *For any  $x_0 \in U$ ,  $x_0 \in W^{cs}$  if and only if  $T^{(n)}(x_0) \in U$  for all  $n = 1, 2, \dots$ .*

*Proof.* Since  $W^{cs}$  is positively invariant, it is clear that if  $x_0 \in W^{cs}$ , then  $T^{(n)}(x_0) \in W^{cs} \subset U$  for all  $n > 0$ . On the other hand, if  $T^{(n)}(x_0) \in U$  for all  $n > 0$ , let

$$x_n = T^{(n)}(x_0) = m_n + x_n^s + x_n^u, \quad x_n^s \in X_{m_n}^s(\delta_0), \quad x_n^u \in \overline{X_{m_n}^u(\varepsilon)}, \quad n > 0.$$

From Lemma 4.7, we have

$$|x_n^u - h_0(m_0, x_0^s)| \leq \lambda_1^n |x_n^u - h_0(m_n, x_n^s)| \leq 3\lambda_1^n \varepsilon \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore,  $x_0 \in W^{cs}$ . □

## 5. Stable foliations

In this section, under the assumptions (H1)–(H4) and (H3'), we shall construct stable fibers inside the positively invariant center-stable manifold. In this construction, please note that no invariant center-manifold is assumed to exist in the center-stable manifold. Recall, from the notations used in Sect. 4, that  $W^{cs} = h_0(X^s(\delta_0))$  where  $h_0 \in \Gamma$ . Therefore,  $(m, x^s)$  can be used as a coordinate system on  $W^{cs}$ . Let

$$\tilde{W}^{cs} = \left\{ h_0(m, x^s) : \psi(\overline{B_c(m, \delta_0)}) \text{ is closed in } X, x^s \in X_m^s\left(\frac{\delta_0}{5}\right) \right\}.$$

We prove that, under the condition of Theorem 2.4 and for possibly smaller  $\delta_0$ , each point in  $\tilde{W}^{cs}$  belongs to a stable fiber. The stable fibers form a continuous family of  $C^J$  submanifolds which are roughly in the stable direction. They are either identical or disjoint. Furthermore,  $T$  maps each fiber entirely into another one and is contractive. The reason that we introduce the open subset  $\tilde{W}^{cs}$  in  $W^{cs}$  is to avoid technical complications near its boundary.

In [BLZ3], stable (unstable) foliations inside the center-stable (center-unstable) manifold of a compact normally hyperbolic invariant manifold (without boundary) are constructed by a graph transform method. The same method also works here but instead, we use a Lyapunov–Perron type method based on series. Note that our normal hyperbolicity assumption is only pointwise.

For any  $y_0 = h_0(m_0, x_0^s) \in \tilde{W}^{cs}$  and  $n \geq 0$ , from Definition 2.1, Lemma 4.4, and the invariance of  $W^{cs}$ , there exists  $y_n = h_0(m_n, x_n^s) \in \tilde{W}^{cs}$  such that

$$(5.1) \quad T(y_{n-1}) = y_n = \psi(m_n) + x_n^s + x_n^u, \quad m_n \in B_c(u(m_{n-1}), r).$$



Fix

$$\rho = \frac{9}{10}, \quad \delta = \frac{4}{5}\delta_0,$$

and let

$$f_n : \overline{X_{m_n}^c(\rho\delta)} \oplus \overline{X_{m_n}^s(\rho\delta)} \rightarrow X_{m_n}^u$$

be the map defined in Lemma 4.3 at  $y_n$ . Since  $W^{cs}$  is  $C^J$ ,  $f_n$  is a  $C^J$  map. From the invariance of  $W^{cs}$ , for any  $n \geq 0$ ,  $T$  induces a map  $S_n$  such that

$$(5.2) \quad \begin{aligned} & T(\psi(m_n) + x_n^s + x^c + x^s + f_n(x^c, x^s)) \\ &= \psi(m_{n+1}) + x_{n+1}^s + S_n(x^c, x^s) + f_{n+1}(S_n(x^c, x^s)). \end{aligned}$$

The domain of  $S_n$  is at least a neighborhood of 0 in  $\overline{X_{m_n}^c(\rho\delta)} \oplus \overline{X_{m_n}^s(\rho\delta)}$  and  $S_n$  is as smooth as  $T$ . It is clear that  $S_n(0) = 0$ . Let for  $\alpha = c, s$ ,

$$\begin{aligned} S_n^\alpha &= \Pi_{m_{n+1}}^\alpha S_n, \quad A_n^\alpha = DS_n^\alpha(0)|_{X_{m_n}^\alpha}, \\ R_n &= (R_n^c, R_n^s) \equiv (S_n^c - A_n^c, S_n^s - A_n^s), \end{aligned}$$

where we slightly abused the notation by also using  $A_n^\alpha$  to represent its 0 extension to  $X_{m_n}^c \oplus X_{m_n}^s$ . For  $DS_n$  and related linear operators, we use the operator norm

$$\|DS_n\| \equiv \sup \{ |DS_n^c(\bar{x}^c, \bar{x}^s)| + |DS_n^s(\bar{x}^c, \bar{x}^s)| : |\bar{x}^c| + |\bar{x}^s| = 1 \}.$$

Let  $\bar{\lambda} = \frac{1+\lambda}{2}$ . We have the following technical lemma.

**Lemma 5.1.** *With preamble (S), the domain of  $S_n$  contains  $\overline{X_{m_n}^c(\frac{\rho\delta}{2BB_1})} \oplus \overline{X_{m_n}^s(\rho\delta)}$  and for  $2 \leq k \leq J$ ,*

$$\begin{aligned} \|DS_n^\alpha\| &\leq 2BB_1, \quad \|A_n^s\| \leq \bar{\lambda} \min \{1, \|(A_n^c)^{-1}\|^{-1}\}, \quad \frac{a}{2} \leq \|(A_n^c)^{-1}\|^{-1}, \\ \|DR_n^\alpha\| &\leq \sigma + C\delta_0 + 3BB_1\mu + 3B\mathcal{A}(2\delta_0), \quad \|D^k R_n^\alpha\| \leq C. \end{aligned}$$

*Remark 5.2.* In this section, the conditions on parameter are always proved to be independent of the specific orbit  $\{y_n\}$  unless otherwise specified.

*Proof.* From (5.2), we have, for  $\alpha = c, s$ ,

$$(5.3) \quad \begin{aligned} & DS_n^\alpha(x^c, x^s) \\ &= \Pi_{m_{n+1}}^\alpha DT(\psi(m_n) + x_n^s + x^c + x^s + f_n(x^c, x^s)) \circ (I + Df_n(x^c, x^s)) \end{aligned}$$

where  $I$  represents the identity operator on  $X_{m_n}^c \oplus X_{m_n}^s$ . The conclusion concerning the domain of  $S_n$  and the estimate on  $DS_n^\alpha$  follow immediately

from (5.3). Since  $|\psi(m_{n+1}) - \psi(u(m_n))| \leq \frac{\delta_0}{2}$  by Lemma 4.4, substituting  $x^c = 0$  and  $x^s = 0$  into (5.3), we obtain

$$\|A_n^\alpha - \Pi_{u(m_n)}^\alpha DT(\psi(m_n))|_{X_{m_n}^\alpha}\| \leq BB_1\mu + C\delta_0 + B\mathcal{A}(\delta_0),$$

which, along with (H3'), implies the estimates on  $A_n^\alpha$ . The estimate on  $\|DR_n^\alpha\|$  follows similarly. From Theorem 7.3 in [BLZ2],  $D^k f_n$  is bounded with an upper bound independent of the orbit  $\{y_n\}$ . Differentiating (5.3) multiple times, we obtain the estimate on  $\|D^k R_n^\alpha\|$ .  $\square$

Suppose

$$(5.4) \quad \left\{ (\bar{x}_n^c, \bar{x}_n^s) \in \overline{X_{m_n}^c \left( \frac{\rho\delta}{2BB_1} \right)} \oplus \overline{X_{m_n}^s(\rho\delta)} : n \geq 0 \right\}$$

is an orbit of  $\{S_n : n \geq 0\}$ , i.e.,

$$(5.5) \quad (\bar{x}_{n+1}^c, \bar{x}_{n+1}^s) = S_n(\bar{x}_n^c, \bar{x}_n^s), \quad n \geq 0.$$

Then, for  $\alpha = c, s$ , and  $n_2 \geq n_1 \geq 0$ ,

$$\begin{aligned} \bar{x}_{n_2}^\alpha &= A_{n_2-1}^\alpha \bar{x}_{n_2-1}^\alpha + R_{n_2-1}^\alpha(\bar{x}_{n_2-1}^c, \bar{x}_{n_2-1}^s) \\ &= A_{n_2-1}^\alpha A_{n_2-2}^\alpha \bar{x}_{n_2-2}^\alpha + A_{n_2-1}^\alpha R_{n_2-2}^\alpha(\bar{x}_{n_2-2}^c, \bar{x}_{n_2-2}^s) + R_{n_2-1}^\alpha(\bar{x}_{n_2-1}^c, \bar{x}_{n_2-1}^s) \\ &= \left( \prod_{n_1 \leq k \leq n_2-1} A_k^\alpha \right) \bar{x}_{n_1}^\alpha + \sum_{n_1 \leq k \leq n_2-1} \left( \prod_{k+1 \leq l \leq n_2-1} A_l^\alpha \right) R_k^\alpha(\bar{x}_k^c, \bar{x}_k^s). \end{aligned}$$

Therefore, for  $n_0 > n \geq 0$ ,

$$(5.6) \quad \bar{x}_n^s = \left( \prod_{0 \leq k \leq n-1} A_k^s \right) \bar{x}_0^s + \sum_{0 \leq k \leq n-1} \left( \prod_{k+1 \leq l \leq n-1} A_l^s \right) R_k^s(\bar{x}_k^c, \bar{x}_k^s)$$

$$(5.7) \quad \bar{x}_n^c = \left( \prod_{n \leq k \leq n_0-1} A_k^c \right)^{-1} \bar{x}_{n_0}^c - \sum_{n \leq k \leq n_0-1} \left( \prod_{n \leq l \leq k} A_l^c \right)^{-1} R_k^c(\bar{x}_k^c, \bar{x}_k^s).$$

If we also assume this orbit satisfies

$$\lim_{n_0 \rightarrow \infty} \left( \prod_{0 \leq k \leq n_0-1} A_k^c \right)^{-1} \bar{x}_{n_0}^c = 0$$

then we have

$$(5.8) \quad \bar{x}_n^c = - \sum_{n \leq k < +\infty} \left( \prod_{n \leq l \leq k} A_l^c \right)^{-1} R_k^c(\bar{x}_k^c, \bar{x}_k^s).$$

Therefore, such orbits are solutions to systems of equations (5.6) and (5.8). We will use the contraction mapping principle to find such orbits.

Fix  $\gamma \in (\frac{1}{3} + \frac{2\bar{\lambda}}{3}, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ . For any

$$(5.9) \quad z = \left\{ (\bar{x}_n^c, \bar{x}_n^s) \in \overline{X_{m_n}^c \left( \frac{\rho\delta}{2BB_1} \right)} \oplus \overline{X_{m_n}^s(\rho\delta)} : n \geq 0 \right\},$$

let

$$(5.10) \quad \|z\|_\gamma = \sup_{n \geq 0, n \geq j \geq 0} \gamma^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) (|\bar{x}_n^c| + |\bar{x}_n^s|).$$

Define

$$\Omega_\gamma^s = \left\{ z = \left\{ (\bar{x}_n^c, \bar{x}_n^s) \in \overline{X_{m_n}^c \left( \frac{\rho\delta}{2BB_1} \right)} \oplus \overline{X_{m_n}^s(\rho\delta)} : n \geq 0 \right\} : \right. \\ \left. \|z\|_\gamma < \infty \right\},$$

$$\Gamma_\gamma^s = \{z \in \Omega_\gamma^s : \|z\|_\gamma \leq \rho\delta\}.$$

It is easy to see that  $\Omega_\gamma^s$  is a Banach space and  $\Gamma_\gamma^s$  is a closed subset in  $\Omega_\gamma^s$ . For any  $x^s \in X_{m_0}^s(\frac{\delta_0}{2})$  and any sequence  $z = \{(\bar{x}_n^c, \bar{x}_n^s) : n \geq 0\} \in \Gamma_\gamma^s$ , for  $n \geq 0$  let

$$(5.11) \quad \tilde{x}_n^s = \left( \prod_{0 \leq k \leq n-1} A_k^s \right) x^s + \sum_{0 \leq k \leq n-1} \left( \prod_{k+1 \leq l \leq n-1} A_l^s \right) R_k^s(\bar{x}_k^c, \bar{x}_k^s)$$

$$(5.12) \quad \tilde{x}_n^c = - \sum_{n \leq k < +\infty} \left( \prod_{n \leq l \leq k} A_l^c \right)^{-1} R_k^c(\bar{x}_k^c, \bar{x}_k^s)$$

$$(5.13) \quad \mathcal{F}^s(x^s, z) = \tilde{z} = \{(\tilde{x}_n^c, \tilde{x}_n^s) : n \geq 0\}.$$

We shall prove that, for any  $x^s \in X_{m_0}^s(\frac{\delta_0}{2})$ ,  $\mathcal{F}^s(x^s, \cdot)$  is a contraction on  $\Gamma_\gamma^s$ . Since  $\{R_n^\alpha\}$  has a uniform Lipschitz constant according to Lemma 5.1, it is easy to verify that  $z \in \Gamma_\gamma^s$  is an orbit of  $\{S_n\}$  if and only if it is a fixed point of  $\mathcal{F}^s(\bar{x}_0^s, \cdot)$  where  $\bar{x}_0^s$  is the 0-th stable component of  $z$ .

**Lemma 5.3.** *With preamble (S), for any  $x^s \in X_{m_0}^s(\frac{\delta_0}{2})$ ,  $\mathcal{F}^s(x^s, \cdot)$  maps  $\Gamma_\gamma^s$  to itself and has a Lipschitz constant bounded by*

$$\frac{12}{a(1-\lambda)}(\sigma + C\delta_0 + 3BB_1\mu + 3B\mathcal{A}(2\delta_0)).$$

Note that, from the proof, the conditions on the parameters are independent of  $\gamma$ .

*Proof.* Let  $z_i = \{(\bar{x}_{i,n}^c, \bar{x}_{i,n}^s)\} \in \Gamma_\gamma^s$  and  $\tilde{z}_i = \{(\tilde{x}_{i,n}^c, \tilde{x}_{i,n}^s)\}$  be defined as in (5.11)–(5.13) for  $i = 1, 2$ . From (5.11) and Lemma 5.1, we have, for any

$n > 0, 0 \leq j \leq n,$

$$\begin{aligned}
& \gamma^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\tilde{x}_{2,n}^s - \tilde{x}_{1,n}^s| \\
& \leq \gamma^{-n} \sum_{0 \leq k \leq n-1} \bar{\lambda}^{n-k-1} \left( \prod_{j \leq l \leq k} \|(A_l^c)^{-1}\| \right) |R_k^s(\bar{x}_{2,k}^c, \bar{x}_{2,k}^s) - R_k^s(\bar{x}_{2,k}^c, \bar{x}_{1,k}^s)| \\
& \leq \frac{2\gamma}{a} \sum_{0 \leq k \leq n-1} \left( \frac{\bar{\lambda}}{\gamma} \right)^{n-k-1} \|DR_k^\alpha\| \|z_2 - z_1\|_\gamma \\
& \leq \frac{12(\sigma + C\delta_0 + 3BB_1\mu + 3B\mathcal{A}(2\delta_0))}{a(1-\lambda)} \|z_2 - z_1\|_\gamma.
\end{aligned}$$

For  $n = 0$ , it is clear that  $\tilde{x}_1^s = \tilde{x}_2^s = x^s$ . On the other hand, for  $n \geq 0$ ,  $0 \leq j \leq n$ , we also have

$$\begin{aligned}
& \gamma^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\tilde{x}_{2,n}^c - \tilde{x}_{1,n}^c| \\
& \leq \gamma^{-n} \sum_{n \leq k < +\infty} \left( \prod_{j \leq l \leq k} \|(A_l^c)^{-1}\| \right) |R_k^c(\bar{x}_{2,k}^c, \bar{x}_{2,k}^s) - R_k^c(\bar{x}_{1,k}^c, \bar{x}_{1,k}^s)| \\
& \leq \frac{2}{a} \sum_{n \leq k < +\infty} \gamma^{k-n} \|DR_k^\alpha\| \|z_2 - z_1\|_\gamma \\
& \leq \frac{12(\sigma + C\delta_0 + 3BB_1\mu + 3B\mathcal{A}(2\delta_0))}{a(1-\lambda)} \|z_2 - z_1\|_\gamma.
\end{aligned}$$

Finally, using Lemma 5.1, it is easy to show that  $\|\mathcal{F}^s(x^s, 0)\|_\gamma \leq \frac{\delta_0}{2}$ . Therefore, one finds that  $\mathcal{F}^s(x^s, \cdot)$  maps  $\Gamma_\gamma^s$  to itself and is a contraction on  $\Gamma_\gamma^s$ .  $\square$

Lemma 5.3 implies that, for any  $x^s \in X_{m_0}^s(\frac{\delta_0}{2})$ , there exists a unique  $g(x^s) \in \Gamma_\gamma^s$  such that  $\mathcal{F}^s(x^s, g(x^s)) = g(x^s)$ . Let  $g(x^s) = \{(\bar{x}_n^c, \bar{x}_n^s)\}$ . In the following, we will investigate the dependence of  $g(x^s)$  on  $x^s$ . Firstly,  $\mathcal{F}^s$  is linear in  $x^s$  and

$$(5.14) \quad D_{x^s} \mathcal{F}^s = \left\{ \left( 0, \prod_{0 \leq k \leq n-1} A_k^s \right) \right\} \in L(X_{m_0}^s, \Omega_\gamma^s).$$

In addition, we have

**Lemma 5.4.**  $\mathcal{F}^s(x^s, \cdot) \in C^{J-1,1}(\Gamma_\gamma^s)$ . Furthermore, for any  $\gamma_1 \in (\gamma, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ ,  $\mathcal{F}^s(x^s, \cdot) \in C^J(\Gamma_{\gamma_1}^s, \Gamma_{\gamma_1}^s)$ .

Recall that  $T$  is assumed to be a  $C^J$  map on  $X$ . We also notice that  $\Gamma_\gamma^s \subset \Gamma_{\gamma_1}^s$  when  $\gamma_1 > \gamma$ .

*Proof.* Let  $z = \{(\bar{x}_n^c, \bar{x}_n^s)\} \in \Gamma_\gamma^s$  and  $\hat{z}_i = \{(\hat{x}_{i,n}^c, \hat{x}_{i,n}^s)\} \in \Omega_\gamma^s$  for  $i = 1, 2, \dots, K \leq J$ . Let

$$(5.15) \quad \bar{x}_n^s = \sum_{0 \leq k \leq n-1} \left( \prod_{k+1 \leq l \leq n-1} A_l^s \right) D^K R_k^s(\bar{x}_k^c, \bar{x}_k^s) [(\hat{x}_{1,k}^c, \hat{x}_{1,k}^s) \otimes \cdots \otimes (\hat{x}_{K,k}^c, \hat{x}_{K,k}^s)]$$

and

$$(5.16) \quad \bar{x}_n^c = - \sum_{n \leq k < +\infty} \left( \prod_{n \leq l \leq k} A_l^c \right)^{-1} D^K R_k^c(\bar{x}_k^c, \bar{x}_k^s) [(\hat{x}_{1,k}^c, \hat{x}_{1,k}^s) \otimes \cdots \otimes (\hat{x}_{K,k}^c, \hat{x}_{K,k}^s)].$$

Then, formally, we have

$$(5.17) \quad D_z^K \mathcal{F}^s(x^s, z)(\hat{z}_1, \dots, \hat{z}_K) = \tilde{z} = \{(\tilde{x}_n^c, \tilde{x}_n^s)\}.$$

We first show that  $D_z^K \mathcal{F}^s$  is a well-defined bounded multilinear operator for  $1 \leq K \leq J$ . Thus,  $\mathcal{F}^s$  is  $C^{J-1,1}$ . From (5.15) and Lemma 5.1, we have, for any  $n \geq 0$  and  $0 \leq j \leq n$ ,

$$\begin{aligned} & \gamma^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\bar{x}_n^s| \\ & \leq C \gamma^{-n} \sum_{0 \leq k \leq n-1} \bar{\lambda}^{n-k-1} \left( \prod_{j \leq l \leq k} \|(A_l^c)^{-1}\| \right) (|\hat{x}_{1,k}^c| + |\hat{x}_{1,k}^s|) \\ & \quad \times \gamma^{k(K-1)} \|\hat{z}_2\|_\gamma \cdots \|\hat{z}_K\|_\gamma \\ & \leq C \sum_{0 \leq k \leq n-1} \gamma^{-n} \bar{\lambda}^{n-k-1} \gamma^{Kk} \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma \leq \frac{C}{\gamma - \bar{\lambda}} \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \gamma^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\bar{x}_n^c| \\ & \leq C \gamma^{-n} \sum_{n \leq k < +\infty} \left( \prod_{j \leq l \leq k} \|(A_l^c)^{-1}\| \right) (|\hat{x}_{1,k}^c| + |\hat{x}_{1,k}^s|) \gamma^{(K-1)k} \|\hat{z}_2\|_\gamma \cdots \|\hat{z}_K\|_\gamma \\ & \leq \frac{C}{1 - \gamma} \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma. \end{aligned}$$

These estimates imply that  $D_z^K \mathcal{F}^s$  is a bounded  $K$ -linear operator for  $1 \leq K \leq J$  and thus  $\mathcal{F}^s$  is  $C^{J-1,1}$  in  $z$ .

We next prove the continuity of  $D_z^K \mathcal{F}^s$  in  $z$  when considered as a  $K$ -linear map from  $\Omega_\gamma^s$  to  $\Omega_{\gamma_1}^s$  for  $\gamma_1 \in (\gamma, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ . For  $z_i = \{(\bar{x}_{i,n}^c, \bar{x}_{i,n}^s)\} \in \Gamma_\gamma^s$ ,  $i = 1, 2$ , let

$$D_z^K \mathcal{F}^s(x^s, z_i)(\hat{z}_1, \dots, \hat{z}_K) = \tilde{z}_i = \{(\tilde{x}_{i,n}^c, \tilde{x}_{i,n}^s)\},$$

defined as in (5.15) and (5.16). Much as in the above estimates, for any  $n \geq 0$  and  $0 \leq j \leq n$ , one has

$$\begin{aligned}
 (5.18) \quad & \gamma_1^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\tilde{x}_{2,n}^s - \tilde{x}_{1,n}^s| \\
 & \leq \frac{2}{a\bar{\lambda}} \left( \frac{\gamma}{\gamma_1} \right)^n \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma \\
 & \quad \times \sum_{0 \leq k \leq n-1} \left( \frac{\bar{\lambda}}{\gamma} \right)^{n-k} \|D^K R_k^s(\tilde{x}_{2,k}^c, \tilde{x}_{2,k}^s) - D^K R_k^s(\tilde{x}_{1,k}^c, \tilde{x}_{1,k}^s)\|
 \end{aligned}$$

$$\begin{aligned}
 (5.19) \quad & \gamma_1^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) |\tilde{x}_{2,n}^c - \tilde{x}_{1,n}^c| \\
 & \leq \frac{2}{a} \left( \frac{\gamma}{\gamma_1} \right)^n \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma \\
 & \quad \times \sum_{n \leq k \leq \infty} \gamma^{k-n} \|D^K R_k^s(\tilde{x}_{2,k}^c, \tilde{x}_{2,k}^s) - D^K R_k^s(\tilde{x}_{1,k}^c, \tilde{x}_{1,k}^s)\|.
 \end{aligned}$$

Note that the right sides of the above two inequalities are both bounded by

$$C \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma \left( \frac{\gamma}{\gamma_1} \right)^n.$$

Therefore, for fixed  $z_1$  and any  $\omega > 0$ , there exists  $N > 0$  such that

$$\begin{aligned}
 & \gamma_1^{-n} \left( \prod_{j \leq k \leq n-1} \|(A_k^c)^{-1}\| \right) (|\tilde{x}_{2,n}^s - \tilde{x}_{1,n}^s| + |\tilde{x}_{2,n}^c - \tilde{x}_{1,n}^c|) \\
 & \leq \omega \|\hat{z}_1\|_\gamma \cdots \|\hat{z}_K\|_\gamma
 \end{aligned}$$

when  $n > N$ . For  $n \leq N$ , from the dominant convergence theorem, the above inequality still holds when  $\|z_2 - z_1\|_\gamma$  is sufficiently small. This proves that  $\mathcal{F}^s$  is  $C^J$  from  $\Gamma_\gamma^s$  to  $\Gamma_{\gamma_1}^s$ .  $\square$

It is clear from Lemma 5.4 that the fixed point  $g(x^s)$  of  $\mathcal{F}^s(x^s, \cdot)$  is  $C^{J-1,1}$  in  $x^s$ . A less obvious consequence of Lemma 5.4 is

**Lemma 5.5.** *For any  $\gamma \in (\frac{1}{3} + \frac{2\bar{\lambda}}{3}, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ ,  $g \in C^J(X_{m_0}^s(\frac{\delta_0}{2}), \Gamma_\gamma^s)$ .*

*Proof.* We shall only prove this for the case for  $J = 1$ ; the higher order smoothness follows similarly. Fix  $\gamma_0 \in (\frac{1}{3} + \frac{2\bar{\lambda}}{3}, \gamma)$ . Note, from Lemma 5.4,

$$\begin{aligned}
 & [I - D_z \mathcal{F}^s(x^s, g(x^s))]^{-1} = \sum_{k=0}^{\infty} D_z \mathcal{F}^s(x^s, g(x^s))^k \\
 & \in C^0\left(X_{m_0}^s\left(\frac{\delta_0}{2}\right), L(\Gamma_{\gamma_0}^s, \Gamma_{\gamma_1}^s)\right) \cap L^\infty\left(X_{m_0}^s\left(\frac{\delta_0}{2}\right), L(\Gamma_{\gamma_0}^s)\right),
 \end{aligned}$$

for any  $\gamma_1 \in (\gamma_0, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ . Therefore,

$$\begin{aligned} Z(x^s) &\equiv [I - D_z \mathcal{F}^s(x^s, g(x^s))]^{-1} D_{x^s} \mathcal{F}^s \\ &\in C^0\left(X_{m_0}^s\left(\frac{\delta_0}{2}\right), L(X_{m_0}^s, \Gamma_{\gamma_1}^s)\right) \cap L^\infty\left(X_{m_0}^s\left(\frac{\delta_0}{2}\right), L(X_{m_0}^s, \Gamma_{\gamma_0}^s)\right). \end{aligned}$$

For any  $x^s, \tilde{x}^s \in X_{m_0}^s(\frac{\delta_0}{2})$ ,

$$\begin{aligned} g(\tilde{x}^s) - g(x^s) &= \mathcal{F}^s(\tilde{x}^s, g(\tilde{x}^s)) - \mathcal{F}^s(x^s, g(x^s)) \\ &= D_{x^s} \mathcal{F}^s(\tilde{x}^s - x^s) + D_z \mathcal{F}^s(x^s, g(x^s))(g(\tilde{x}^s) - g(x^s)) \\ &\quad + \mathcal{F}^s(x^s, g(\tilde{x}^s)) - \mathcal{F}^s(x^s, g(x^s)) \\ &\quad - D_z \mathcal{F}^s(x^s, g(x^s))(g(\tilde{x}^s) - g(x^s)). \end{aligned}$$

Note that Lemma 5.4 implies that  $g$  is Lipschitz, but we also obtain, for any  $\gamma_1 > \gamma_0$ ,

$$\frac{\|g(\tilde{x}^s) - g(x^s) - Z(x^s)(\tilde{x}^s - x^s)\|_{\gamma_1}}{|\tilde{x}^s - x^s|} \rightarrow 0 \quad \text{as } \tilde{x}^s \rightarrow x^s,$$

which means that  $Z(x^s)$  is the derivative of  $g(x^s)$ .  $\square$

Recall the above construction is based on the orbit  $\{y_n\}$  starting at  $y_0 = h_0(m_0, x_0^s)$ . For any  $x^s \in X_{m_0}^s(\frac{\delta_0}{2})$ , let

$$\tilde{g}_{y_0}(x^s) = x_0^c,$$

where  $x_0^c$  is the 0-th center component of  $g(x^s)$ . From Lemma 5.5,  $\tilde{g}_{y_0} \in C^J(X_{m_0}^s(\frac{\delta_0}{2}), X_{m_0}^c)$ . In fact, it is clear that  $\tilde{g}_{y_0}(0) = 0$  and, from Lemma 5.3 and (5.14),

$$(5.20) \quad D_{x^s} \tilde{g}_{y_0} \leq \frac{24(\sigma + C\delta_0 + 3BB_1\mu + 3B\mathcal{A}(2\delta_0))}{a(1 - \lambda)} \equiv \bar{\mu}.$$

Let

$$W_{y_0}^{ss} \equiv \left\{ \psi(m_0) + x_0^s + x^s + \tilde{g}_{y_0}(x^s) + f_0(\tilde{g}_{y_0}(x^s), x^s) : x^s \in X_{m_0}^s\left(\frac{\delta_0}{2}\right) \right\}.$$

$W_{y_0}^{ss}$  is called the stable fiber passing through  $y_0$  and it is a  $C^J$  submanifold. The point  $y_0$  is said to be a representative or a base point of the fiber. Furthermore, we have

- Lemma 5.6.** (1) Let  $y_1 = h_0(m_1, x_1^s)$  be defined as in (5.1) for  $y_0$ , then we have  $T(W_{y_0}^{ss}) \subset W_{y_1}^{ss}$ .  
 (2) For  $y, \tilde{y} \in \tilde{W}^{cs}$ , “ $\tilde{y} \in W_y^{ss}$ ” is an equivalence relation.



The idea of the proof is that  $\tilde{y} \in W_y^{ss}$  implies the orbits of  $y$  and  $\tilde{y}$  exponentially converge to each other at a certain rate. We omit the details of the proof. It follows from the lemma that  $\tilde{W}^{cs}$  can be foliated into this invariant family of stable fibers.

Next, we prove that the fibers are locally Hölder with respect to the base points. For any base point  $y_0 \in \tilde{W}^{cs}$ , we still use the above notations,  $y_n = h_0(m_n, x_n^s)$ ,  $f_n$ ,  $S_n$ , etc. For any  $(\bar{x}^c, \bar{x}^s) \in X_{m_n}^c(\rho\delta) \oplus X_{m_n}^s(\rho\delta)$ , let

$$g_n(\bar{x}^c, \bar{x}^s, \cdot) : X_{m_n}^s \rightarrow X_{m_n}^c$$

be the local representation of the stable fiber through  $\bar{y}_n = \psi(m_n) + x_n^s + \bar{x}^c + \bar{x}^s + f_n(\bar{x}^c, \bar{x}^s)$ , i.e.,  $g_n(\bar{x}^c, \bar{x}^s, 0) = \bar{x}^c$  and

$$\psi(m_n) + x_n^s + g_n(\bar{x}^c, \bar{x}^s, x^s) + \bar{x}^s + x^s + f_n(g_n(\bar{x}^c, \bar{x}^s, x^s), \bar{x}^s + x^s) \in W_{\bar{y}}^{ss}.$$

From (5.20), the domain of  $g_n(\bar{x}^c, \bar{x}^s, \cdot)$  is a subset of  $X_{m_n}^s(\rho\delta)$  containing at least a neighborhood of 0 in  $X_{m_n}^s$ .

**Lemma 5.7.** (1) *There exist  $\beta > 0$  and  $C_0 > 0$  depending only on  $B$ ,  $B_1$ ,  $\lambda$ ,  $a$ , and  $y_0$ , and  $\omega_0 > 0$  independent of  $y_0$  such that*

$$|g_0(\bar{x}^c, \bar{x}^s, x^s) - g_0(0, 0, x^s)| \leq C_0(|\bar{x}^c| + |\bar{x}^s|)^\beta$$

*whenever  $|\bar{x}^c| + |\bar{x}^s| \leq \omega_0$  and  $|x^s| < \frac{\delta}{4}$ .*

(2)  *$D_{x^s}g_0(\bar{x}^c, \bar{x}^s, 0)$  is continuous in  $\bar{x}^c$  and  $\bar{x}^s$ .*

This lemma means that stable fibers are locally Hölder continuous in the base point and the tangent spaces of the fibers vary continuously.

*Proof.* From (5.20), when  $\omega_0$  is small enough, the domain of  $g_n(\bar{x}^c, \bar{x}^s, \cdot)$  contains  $X_{m_0}^s(\frac{\delta}{4})$ . Since  $W^{cs}$  and the family of stable fibers are invariant under the map  $T$ , we may define  $\bar{x}_n^c \in X_{m_n}^c$ ,  $\bar{x}_n^s, \tilde{x}_n^s \in X_{m_n}^s$  inductively so that

$$\begin{aligned} S_n(\bar{x}_n^c, \bar{x}_n^s) &= \bar{x}_{n+1}^c + \bar{x}_{n+1}^s, \\ S_n(g_n(\bar{x}_n^c, \bar{x}_n^s, \tilde{x}_n^s), \bar{x}_n^s + \tilde{x}_n^s) &= g_{n+1}(\bar{x}_{n+1}^c, \bar{x}_{n+1}^s, \tilde{x}_{n+1}^s) + \bar{x}_{n+1}^s + \tilde{x}_{n+1}^s \\ \hat{x}_n^c &= g_n(\bar{x}_n^c, \bar{x}_n^s, \tilde{x}_n^s) - g_n(0, 0, \tilde{x}_n^s), \end{aligned}$$

where  $\tilde{x}_0^s = x^s$ ,  $\bar{x}_0^s = \bar{x}^s$ . These quantities are well defined as long as  $\bar{x}_k^c, \bar{x}_k^s + \tilde{x}_k^s \in \overline{X_{m_k}^s(\rho\delta)}$  and  $\bar{x}_k^c, g_k(\bar{x}_k^c, \bar{x}_k^s, \tilde{x}_k^s) \in \overline{X_{m_k}^c(\rho\delta)}$  for  $k = 1, 2, \dots, n$ . It may happen that they leave the coordinate charts  $\overline{X_{m_n}^c(\rho\delta)} \oplus \overline{X_{m_n}^s(\rho\delta)}$  and stop being well defined for large  $n$ . From Lemma 5.1, we have

$$|\bar{x}_n^c| + |\bar{x}_n^s| \leq 2BB_1(|\bar{x}_{n-1}^c| + |\bar{x}_{n-1}^s|) \leq (2BB_1)^n(|\bar{x}^c| + |\bar{x}^s|).$$

On the other hand, from Lemma 5.1 and (5.20),

$$\begin{aligned}
 |\tilde{x}_{n+1}^s| &= |S_n^s(g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s), \tilde{x}_n^s + \tilde{x}_n^s) - S_n^s(\tilde{x}_n^c, \tilde{x}_n^s)| \\
 &= |A_n^s \tilde{x}_n^s + R_n^s(g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s), \tilde{x}_n^s + \tilde{x}_n^s) - R_n^s(\tilde{x}_n^c, \tilde{x}_n^s)| \\
 &\leq \|A_n^s\| |\tilde{x}_n^s| + \|DR_n^s\| (|\tilde{x}_n^s| + |g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s) - \tilde{x}_n^c|) \\
 &\leq \prod_{k=0}^n (\|A_k^s\| + 2\|DR_k^s\|) |x^s|.
 \end{aligned}$$

Therefore,

(5.21)

$$|g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s)| \leq (2BB_1)^n (|\tilde{x}^c| + |\tilde{x}^s|) + \bar{\mu} \prod_{k=0}^{n-1} (\|A_k^s\| + 2\|DR_k^s\|) |x^s|.$$

Let  $\hat{x}_{n+1}^s$  satisfy

$$g_{n+1}(0, 0, \hat{x}_{n+1}^s) + \hat{x}_{n+1}^s = S_n(g_n(0, 0, \tilde{x}_n^s), \tilde{x}_n^s).$$

Then we have

$$\begin{aligned}
 |\hat{x}_{n+1}^s - \tilde{x}_{n+1}^s| &\leq |\tilde{x}_{n+1}^s| + |S_n^s(g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s), \tilde{x}_n^s + \tilde{x}_n^s) \\
 &\quad - S_n^s(g_n(0, 0, \tilde{x}_n^s), \tilde{x}_n^s)| \\
 &\leq 2(2BB_1)^{n+1} (|\tilde{x}^c| + |\tilde{x}^s|) + \|DR_n^s\| |\hat{x}_n^c|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 |\hat{x}_{n+1}^c| &\geq |S_n^c(g_n(\tilde{x}_n^c, \tilde{x}_n^s, \tilde{x}_n^s), \tilde{x}_n^s + \tilde{x}_n^s) - S_n^c(g_n(0, 0, \tilde{x}_n^s), \tilde{x}_n^s)| \\
 &\quad - \bar{\mu} |\hat{x}_{n+1}^s - \tilde{x}_{n+1}^s| \\
 &\geq [\|(A_n^c)^{-1}\|^{-1} - 2\|DR_n^s\|] |\hat{x}_n^c| - 3(2BB_1)^{n+1} (|\tilde{x}^c| + |\tilde{x}^s|),
 \end{aligned}$$

which implies

$$|\hat{x}_n^c| \geq \prod_{k=0}^{n-1} [\|(A_k^c)^{-1}\|^{-1} - 2\|DR_k^s\|] |\hat{x}_0^c| - 3n\bar{\mu}(2BB_1)^n (|\tilde{x}^c| + |\tilde{x}^s|).$$

From the definition of  $\hat{x}_n^c$  and using (5.21) and the estimate on  $\tilde{x}_n^s$ , we obtain

$$(5.22) \quad |\hat{x}_0^c| \leq \left( \frac{10BB_1}{a} \right)^n (|\tilde{x}^c| + |\tilde{x}^s|) + 2 \left( \frac{1 + \bar{\lambda}}{2} \right)^n |x^s|.$$

Let

$$\beta = \frac{\log \frac{2}{1+\bar{\lambda}}}{\log \frac{2}{1+\bar{\lambda}} + \log \frac{10BB_1}{a}} \quad n_0 = \frac{(\beta - 1) \log(|\bar{x}^c| + |\bar{x}^s|)}{\log \frac{10BB_1}{a}}$$

$$\omega_0 = \left( \frac{a\delta}{200BB_1} \right)^{\frac{1}{\beta}}$$

and take  $n_1$  be the integer in  $[n_0, n_0 + 1)$ , then one can verify that  $\bar{x}_n^\alpha, \tilde{x}_n^s$ , and  $\hat{x}_n^c$  are well defined for all  $n \leq n_1$ . Substituting  $n_1$  into (5.22), we immediately obtain

$$|g_0(\bar{x}^c, \bar{x}^s, x^s) - g_0(0, 0, x^s)| = |\hat{x}_0^c| \leq \frac{20BB_1}{a} (|\bar{x}^c| + |\bar{x}^s|)^\beta,$$

which gives the Hölder continuity of the fibers in the base point.

In order to prove conclusion (2), we only need to consider the continuity of  $D_{x^s} g_0(\bar{x}^c, \bar{x}^s, 0)$  at  $(\bar{x}^c, \bar{x}^s) = (0, 0)$ . For  $k > n \geq 0$ , let

$$L_{n,k}^\alpha = D(S_{k-1} \circ \cdots \circ S_n)^\alpha(\bar{x}_n^c, \bar{x}_n^s), \quad L_{n,k}^{\alpha,0} = D(S_{k-1} \circ \cdots \circ S_n)^\alpha(0, 0)$$

and

$$G_n = D_{x^s} g_n(\bar{x}_n^c, \bar{x}_n^s, 0), \quad G_n^{(0)} = D_{x^s} g_n(0, 0, 0),$$

for  $\alpha = c, s$ , where  $\bar{x}_n^\alpha$  is defined as in the first part of the proof. From the invariance of the family of fibers, one can calculate that

$$(5.23) \quad (L_{n,k}^c - G_k L_{n,k}^s) G_n = (G_k L_{n,k}^s - L_{n,k}^c) \Big|_{X_{m_0}^s}.$$

The same equation also holds if  $\bar{x}_n^\alpha$  and  $\bar{x}_k^\alpha$  are replaced by 0. Though these operators may not be defined for all  $n$  and  $k$ , the smaller  $|\bar{x}_n^c| + |\bar{x}_n^s|$  is, the larger  $n$  and  $k$  can be taken so that  $G$  and  $L$  are well defined. Since  $\|G_n\| \leq \bar{\mu}$  by (5.20), identity (5.23) implies

$$(L_{0,k}^{c,0} - G_k^{(0)} L_{0,k}^{s,0})(G_0 - G_0^{(0)}) = (G_k - G_k^{(0)}) L_{0,k}^{s,0} (id_{X_{m_0}^s} + G_0) + o(1),$$

where  $o(1)$  term converges to 0 as  $|\bar{x}_0^c| + |\bar{x}_0^s| \rightarrow 0$  for any fixed  $k > 0$ . Therefore, from Lemma 5.1 and (5.20), we obtain

$$\|G_0 - G_0^{(0)}\| \leq 3\bar{\mu} \left( \frac{1 + \bar{\lambda}}{2} \right)^k + o(1).$$

For any  $\omega > 0$ , fix  $k$  so large that the first term is less than  $\frac{\omega}{2}$ . Then, when  $|\bar{x}_0^c| + |\bar{x}_0^s|$  is sufficiently small, the left side is bounded by  $\omega$ . This estimate proves the continuity of  $D_{x^s} g_0(\bar{x}^c, \bar{x}^s, 0)$  in  $\bar{x}^c$  and  $\bar{x}^s$ .  $\square$

## 6. Extensions

In the previous sections, we studied approximate inflowing manifolds for a  $C^J$  map  $T$  defined on a Banach space. We now extend these results in several directions.

**6.1. Semiflows.** Let  $T \in C^0([0, +\infty) \times X, X)$  be a  $C^J$  semiflow on  $X$  for some  $J \geq 1$ , i.e.,

$$T^t \in C^J(X, X), \quad T^0 = I, \quad T^{t+s} = T^t \circ T^s, \quad \text{for all } t, s \geq 0.$$

Suppose there exist  $t_0 > 0$ , a Banach manifold  $M$ , and an immersion  $\psi : M \rightarrow X$  such that conditions (H1)–(H4) are satisfied for  $\psi(M)$  and  $T^{t_0}$ . In that case we say that  $\psi(M)$  is an approximate inflowing manifold for the semiflow  $T$ . Under the conditions of Theorem 2.2, there exists a  $C^J$  immersed inflowing invariant submanifold  $W^{cs}$  for the map  $T^{t_0}$ , which is also the unique center stable manifold of the map  $T^{nt_0}$  for any integer  $n \geq 0$ . In order to prove that  $W^{cs}$  is positively invariant under the semiflow  $T^t$ , we further assume

(H5) There exists an integer  $k \geq 0$ , such that for any  $\xi > 0$ , there exists  $\zeta > 0$ , such that for any  $x \in B(\psi(M), r)$  and  $t \in [kt_0, kt_0 + \zeta]$ , we have

$$|T^t(x) - T^{kt_0}x| < \xi.$$

With this weak uniform continuity condition in time, from Lemma 4.4 and Proposition 4.10 we obtain the local positive invariance of  $W^{cs}$  under the semiflow.

**Theorem 6.1.** *Assume (H5) holds. For any  $x \in W^{cs}$  we have*

- $T^{nt_0}(x) \in W^{cs}$  for any integer  $n \geq 0$  and
- $T^t(x) \in W^{cs}$  for all  $t \in [0, t_1]$  if  $T^t(x) \in U$  for all  $t \in [0, t_1]$ , where  $U$  is the neighborhood defined in Proposition 4.10.

The next issue is, under hypothesis (H3'), the invariance of the family of stable fibers under the semiflow. For any two points  $x, \tilde{x} \in \tilde{W}^{cs}$ , from the construction of the stable fibers, it is clear that  $\tilde{x} \in W_x^{ss}$  if and only if  $|T^{nt_0}(x) - T^{nt_0}(\tilde{x})|$  satisfies the decay condition required in the definition of the set  $\Gamma_\gamma^s$ , defined based on  $\{T^{nt_0}(x)\}$ , for some  $\gamma \in (\frac{1}{3} + \frac{2\bar{\lambda}}{3}, \frac{2}{3} + \frac{\bar{\lambda}}{3})$ . On the other hand, if this decay condition holds for an orbit for some  $\gamma$  in that interval, then it holds for all  $\gamma$  in that range. Using these properties, one can prove that, for any  $\tilde{x}, x \in \tilde{W}^{cs}$ , if  $\tilde{x} \in W_x^{ss}$ , then  $T^t(\tilde{x}) \in W_{T^t(x)}^{ss}$  for all  $t = nt_0$  with integer  $n \geq 0$  and  $t \in [0, t_1]$  where  $t_1$  satisfying that  $T^t(x) \in U$  for all  $t \in [0, t_1]$ .

**6.2. Overflowing manifolds.** In this subsection, we shall consider normally hyperbolic approximately overflowing invariant manifolds. The results are basically parallel to the case of approximate inflowing invariant manifolds.

**Definition 6.2.** An immersed manifold  $\psi(M) \subset X$  is said to be approximately overflowing invariant if the following hold

1. There exist an open subset  $M_1 \subset M$ , a homeomorphism  $v : M \rightarrow M_1$ , and  $\eta > 0$  such that, for all  $m \in M$

$$|T(\psi(v(m))) - \psi(m)| < \eta;$$

2. There exists  $r_0 \in (0, 1)$  such that  $\psi(\overline{B_c(m_0, r_0)})$  is closed in  $X$  for any  $m_0 \in u(M)$ .

Condition (1) means that the image of  $\psi(M_1)$  under  $T$  almost covers  $\psi(M)$  and  $v$  is approximately  $T^{-1}$  on  $\psi(M)$ . In particular, (2) means  $\psi(M_1)$  is “strictly smaller” than  $\psi(M)$ . Note here that  $v$  is assumed to be invertible.

In addition to (H1) and (H2), instead of (H3), we assume the following normal hyperbolicity condition

- (C3) There exists  $a \in (0, 1)$ ,  $\lambda \in (0, 1)$  and positive integer  $J$  such that, for any  $m_1 \in M$ ,  $m_0 = v(m_1)$ ,  $\alpha \in \{c, u\}$ ,  $\beta \in \{c, s, u\}$ ,  $\alpha \neq \beta$ ,

$$\begin{aligned} \|\Pi_{m_1}^\beta DT(\psi(m_0))|_{X_{m_0}^\alpha}\| &\leq \sigma, \quad \|(\Pi_{m_1}^c DT(\psi(m_0))|_{X_{m_0}^c})^{-1}\|^{-1} > a, \\ \lambda \|(\Pi_{m_1}^u DT(\psi(m_0))|_{X_{m_0}^u})^{-1}\|^{-1} &> 1, \\ \|\Pi_{m_1}^s DT(\psi(m_0))|_{X_{m_0}^s}\| &< \lambda \min\{1, \|(\Pi_{m_1}^c DT(\psi(m_0))|_{X_{m_0}^c})^{-1}\|^{-J}\}. \end{aligned}$$

Note (C3) implicitly assumes that, for any  $m_1 \in M$ ,  $m_0 = v(m_1)$ ,

$$\Pi_{m_1}^\alpha DT(\psi(m_0)) : X_{m_0}^\alpha \rightarrow X_{m_1}^\alpha$$

is an isomorphism for  $\alpha = c, u$ .

**Theorem 6.3.** Assume that (H1), (H2), (C3), and (H4) hold. Depending on  $r_0, B, B_1, \lambda, L$ , when  $\eta, \chi, \sigma$ , and  $\inf \mathcal{A}(\delta)$  are sufficiently small, there exists a  $C^J$  negatively invariant manifold  $W^{cs}$ , which is given as the image of a map

$$h : \{(m, x^u) : m \in M, x^u \in X_m^u(\delta_0)\} \rightarrow X.$$

Moreover  $h$  satisfies

$$h(m, x^u) - (\psi(m) + x^u) \in X_m^s(\delta_0).$$

When we say that  $W^{cu}$  is negatively invariant we mean that it satisfies Definition 6.2 for  $\eta = 0$ .  $W^{cu}$  is called the center unstable manifold associated with  $\psi(M)$ . In this way,  $T : W^{cu} \rightarrow T(W^{cu})$  is a homeomorphism and  $T^{-1}$  can be uniquely defined on  $W^{cu}$ . A precise statement on the smallness of the parameters  $\eta$ ,  $\chi$ ,  $\sigma$ , and  $\inf \mathcal{A}(\delta)$  can be found in Theorem 4.2.

*Remark 6.4.* When  $(H2')$  holds for  $M$ , then  $W^{cu}$  is an embedded submanifold which also satisfies  $(H2')$ .

In order to construct unstable fibers in the center unstable manifold, which extend in the unstable direction, we will further need

(C3') For any  $m_1 \in M$ , writing  $m_0 = v(m_1)$ ,

$$\lambda \left\| \left( \Pi_{m_1}^u DT(\psi(m_0)) \Big|_{X_{m_0}^u} \right)^{-1} \right\|^{-1} > \left\| \Pi_{m_1}^c DT(\psi(m_0)) \Big|_{X_{m_0}^c} \right\|.$$

Let  $\tilde{W}^{cu}$  be an open subset of  $W^{cu}$  away from its boundary:

$$\tilde{W}^{cu} = \left\{ h(m, x^u) : \psi(\overline{B_c(m, \delta_0)}) \text{ is closed in } X, x^u \in X_m^u \left( \frac{\delta_0}{5} \right) \right\},$$

where  $h$  is the map in Theorem 6.3 representing  $W^{cu}$ .

**Theorem 6.5.** Assume that  $(H1)$ ,  $(H2)$ ,  $(C3)$ ,  $(H4)$ , and  $(C3')$  hold. Depending on  $r_0$ ,  $B$ ,  $B_1$ ,  $\lambda$ ,  $L$ , when  $\eta$ ,  $\chi$ ,  $\sigma$ , and  $\inf \mathcal{A}(\delta)$  are sufficiently small, for any  $y \in \tilde{W}^{cu}$ , there exists a unique  $C^J$  submanifold containing  $y$ ,  $W_y^{uu} \subset W^{cu}$ , such that

- (1)  $T : T^{-1}(W_{T(y)}^{uu}) \cap W_y^{uu} \rightarrow W_{T(y)}^{uu}$  is a diffeomorphism;
- (2) For  $y, \tilde{y} \in \tilde{W}^{cu}$ , " $\tilde{y} \in W_y^{uu}$ " is an equivalence relation;
- (3) for any  $\tilde{y} \in W_y^{uu}$ ,  $|T^{(-n)}(\tilde{y}) - T^{(-n)}(y)| \rightarrow 0$  exponentially, as  $n \rightarrow +\infty$ ;
- (4)  $W_y^{uu}$  is Hölder continuous in  $y$  and  $T_y W_y^{uu}$  is continuous in  $y$ .

*Remark 6.6.* Finally, if an approximately normally hyperbolic invariant manifold  $\mathcal{M}$  is both inflowing and overflowing, then the intersection of its center stable and center unstable manifolds gives a true  $C^J$  normally hyperbolic invariant manifold  $\tilde{\mathcal{M}}$  close to  $\mathcal{M}$ . Moreover, if the dynamics is defined by a semiflow, then Theorem 6.1 implies  $\tilde{\mathcal{M}}$  is invariant.

In fact, if an approximately normally hyperbolic invariant manifold is both inflowing and overflowing, in the spirit of Lemma 3.7, it has to be a closed manifold without boundary.

**6.3. Smoothness in external parameters.** In practical applications of invariant manifold theory, it is often the case that the map  $T$  depends on

an external parameter. It is important to determine the smoothness of the invariant objects with respect to this parameter.

Let  $X$  and  $Y$  be Banach spaces and  $T : X \times Y \rightarrow X$ . Suppose that  $T \in C^J(X \times Y, X)$  and that  $\psi(M)$  is an approximately inflowing invariant manifold for  $T(\cdot, 0)$ . Assume conditions (H1)–(H3) hold for  $T(\cdot, 0)$  and (H4) for  $T$  in both  $x$  and  $y$  in a neighborhood of  $\psi(M) \times \{0\}$  in  $X \times Y$ . Under these assumptions,  $\psi(M)$  is actually an approximately inflowing invariant manifold for  $T(\cdot, y)$  for all small  $y \in Y$ . Therefore, there exists a positively invariant center stable manifold given by a map  $h(\cdot, y)$  for each small  $y$ . One would like to know how smoothly does  $h$  depend on  $y$ .

If  $T$  and  $D_x T$  are only continuous in  $y \in Y$ , we refer to Theorem A and Theorem 6.2 in [BLZ2]. Basically, the result is, under certain conditions, if  $T$  is continuous from  $Y$  to  $C^J(X, X)$ , then  $h(y, \cdot)$  and its derivatives up to the  $J$ -th derivatives are uniformly continuous in  $y$ .

The idea to prove the smoothness of  $h$  in  $y$  is to extend the phase space of the dynamical system to  $X \times Y$ .

We need a technical assumption: The norm  $|\cdot|$  on the Banach space  $Y$  is a  $C^J$  function away from 0.

Let  $\phi : \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$  cut-off function,

$$\phi(s) = 1, \text{ for } s \in [-\xi, \xi], \quad \phi'(s) < 0, \text{ for } s > \xi,$$

for some  $\xi > 0$ . Extend  $T$  as

$$\tilde{T}(x, y) = (T(x, y), \phi(|y|)y).$$

It is easy to see that, when  $\xi$  is sufficiently small,  $\psi(M) \times B_Y(0, 2\xi)$  is an approximately normally hyperbolic inflowing invariant manifold for  $\tilde{T}$ . Therefore, there exists a  $C^J$  map  $\tilde{h}(m, x^s, y)$  such that its image is the positively invariant center stable manifold near  $\psi(M) \times B_Y(0, 2\xi)$ . On the other hand, for any fixed  $y \in B_Y(0, \xi)$ , the image of  $\tilde{h}(\cdot, y)$  obviously is the center stable manifold for  $T(\cdot, y)$ . Therefore, the center stable manifold is  $C^J$  in  $y$ . In addition, the stable fibers are Hölder continuous in  $y$ . Under additional spectral gap conditions, with some more effort which is not included in this paper, it is sometimes possible to prove that the stable fibers are  $C^{J-1}$  with respect to the base points (see for example, [CL]). The case of an approximately overflowing invariant manifold is similar.

This discussion is actually relevant to the fundamental issue of the smoothness of stable or unstable manifolds with respect to parameters even for a finite-dimensional ODE,

$$\dot{x} = f(x, y), \quad x \in \mathbb{R}^n, \quad f \in C^J, \quad f(0, y) = 0$$

near the fixed point  $x = 0$ . From the above argument, we obtain that the center stable and center unstable manifolds of  $x = 0$  are  $C^k$  in  $y$ . Thus, if  $x = 0$  is hyperbolic, its stable and unstable manifolds are also  $C^k$  in  $y$ . However, if  $x = 0$  has center directions for  $y = 0$ , we could not obtain the  $C^k$  smoothness of its stable and unstable manifolds, as they can only

be treated as stable and unstable fibers in the augmented system with  $y$  included. From the following example, one sees that this actually is not just a technical problem of our approach. Consider

$$(6.1) \quad \dot{x}_1 = x_1, \quad \dot{x}_2 = g(x_1, y)$$

where  $g \in C^1$  and  $g(0, y) = 0$ . The unstable manifold of this system is explicitly given by

$$x_2 = \int_0^1 \frac{1}{s} g(x_1 s, y) ds$$

whose right side may not be  $C^1$  in  $y$  if  $g$  is only  $C^1$ .

**6.4. Persistence of  $C^1$  normally hyperbolic invariant manifolds.** Finally, we consider persistence of an invariant manifold for a dynamical system under small perturbations and relax the  $C^2$  requirement for the result in [BLZ1].

Let  $X$  be a Banach space and let  $T \in C^J(X, X)$ . Suppose  $M \subset X$  is a  $C^J$  embedded invariant submanifold under  $T$ , i.e.,  $M = T(M)$ . One of the important questions is: When  $T$  is slightly perturbed to  $\tilde{T}$  in the  $C^J$  topology, does an invariant manifold  $\tilde{M}$  of  $\tilde{T}$  exist near  $M$ ? It has been shown (see [F1], [F2], [F3], [HPS], [KB], [Ku]) that in finite dimensions, a nearby invariant manifold  $\tilde{M}$  of  $\tilde{T}$  exists if  $M$  is a closed submanifold without boundary and it is normally hyperbolic. The problem in infinite dimensions has been studied in [He], etc. For the general problem of persistence of invariant manifolds, the dynamics near the boundary is subtle. Overflowing and inflowing are two robust situations for backward and forward semidynamics, respectively. In order to obtain a manifold invariant for both forward and backward directions, the unperturbed manifold should be both inflowing and overflowing, and thus without boundary. It is proved in [BLZ1] that if

- (A1)  $X$  is a Banach space and  $T \in C^1(X, X)$ ,  $M \subset X$  is a connected compact  $C^2$  submanifold and  $T(M) = M$ ;
- (A2) For any  $m \in M$ , there exist projections  $\Pi_m^c$ ,  $\Pi_m^s$ , and  $\Pi_m^u$ ,  $C^1$  in  $m$ , such that, for  $\alpha = c, u, s$ ,

$$\Pi_m^c + \Pi_m^u + \Pi_m^s = I, \quad X_m^c = T_m M, \quad DT(m)(X_m^\alpha) \subset X_{T(m)}^\alpha$$

where  $X_m^\alpha = \Pi_m^\alpha X$ . Furthermore, for  $\alpha = c, u$ ,  $DT(m) : X_m^\alpha \rightarrow X_{T(m)}^\alpha$  is an isomorphism;

- (A3) There exists  $\lambda \in (0, 1)$  such that, for any  $m \in M$ ,

$$\begin{aligned} \lambda \left\| (\Pi_{T(m)}^u DT(m)|_{X_m^u})^{-1} \right\|^{-1} &> \max \{1, \left\| \Pi_{T(m)}^c DT(m)|_{X_m^c} \right\|\}, \\ \left\| \Pi_{T(m)}^s DT(m)|_{X_m^s} \right\| &< \lambda \min \{1, \left\| (\Pi_{T(m)}^c DT(m)|_{X_m^c})^{-1} \right\|\}; \end{aligned}$$

then a compact  $C^1$  submanifold  $\tilde{M}$  exists for any small  $C^1$  perturbation  $\tilde{T}$  to  $T$ . Furthermore,



- $\tilde{M}$  is  $C^1$  close to  $M$ ;
- There exist projections  $\tilde{\Pi}_m^c$ ,  $\tilde{\Pi}_m^u$ , and  $\tilde{\Pi}_m^s$  for any  $m \in \tilde{M}$ , satisfying the above properties except they are  $C^0$  in  $m$ ;
- Center stable and center unstable manifolds of  $\tilde{M}$  are constructed. Invariant foliations in these two manifolds are constructed in [BLZ3].

One sees that the persistent manifold  $\tilde{M}$  is also normally hyperbolic, but does not satisfy all the conditions on  $M$ . Naturally, it is not ideal that only a  $C^1$  invariant manifold  $\tilde{M}$  persists near the  $C^2$  manifold  $M$ . The reason that  $M$  was assumed to be  $C^2$  was that we needed  $\Pi_m^\alpha$  to be  $C^1$  or Lipschitz, while the smoothness of  $X_m^c = \Pi_m^c X = T_n M$  is one order lower than  $M$ . We shall use the results proved in this paper to obtain

**Theorem 6.7.** *Suppose that  $M$  is a connected  $C^1$  compact submanifold in  $X$  which satisfies the above conditions (A1)–(A3), except the projections  $\Pi_m^\alpha$  are only assumed to be  $C^0$  in  $m$ . Then a compact  $C^1$  submanifold persists for any sufficiently small  $C^1$  perturbation  $\tilde{T}$  to  $T$  and  $\tilde{M}$  satisfies exactly the same properties as  $M$ .*

*Remark 6.8.* If  $T$  is  $C^J$ ,  $J \geq 1$ , then  $\tilde{M}$  is also  $C^J$  if conditions (H3) and (C3) hold. See Theorems 6.1 and 7.3 in [BLZ2].

It is clear that  $M$  is approximately invariant under  $\tilde{T}$ , both inflowing and overflowing since  $M$  does not have boundary. Furthermore, the compactness of  $M$  implies that (H4) is satisfied and

$$0 < a < \|(\Pi_{m_1}^c DT(\psi(m_0))|_{X_{m_0}^c})^{-1}\|^{-1}$$

holds for some  $a > 0$  as needed in (H3), (H3'), and (C3). In order to apply Theorems 2.2 and 6.3, the major difficulty is that  $\Pi_m^\alpha$  is not Lipschitz in  $m$  as required in (H2). Note that  $M$  is finite dimensional since it is compact. Our idea is to approximate  $\Pi_m^\alpha$  by projections Lipschitz in  $m$ . We need

**Theorem 6.9.** *Let  $M$  be an  $n$ -dimensional  $C^r$  manifold with countable topological basis and let  $N$  be a  $C^r$  Banach manifold with a metric  $d$  which induces an equivalent topology on  $N$ . Then for any  $f \in C^0(M, N)$ , there exists  $\tilde{f} \in C^r(M \times (0, 1], N)$  such that*

- $d(\tilde{f}_\varepsilon, \tilde{f}_{\varepsilon_0}) \rightarrow 0$  uniformly on  $M$  as  $\varepsilon \rightarrow \varepsilon_0 \in (0, 1]$ .
- $d(\tilde{f}_\varepsilon, f) \leq \varepsilon$  on  $M$ .
- $\tilde{f}_\varepsilon \rightarrow \tilde{f}_{\varepsilon_0}$  locally in  $C^r$  as  $\varepsilon \rightarrow \varepsilon_0 \in (0, 1]$ .

We will give the proof of the theorem later and first provide the proof of Theorem 6.7. Let

$$N = \{(P^c, P^u, P^s) \in L(X)^3 : P^c + P^u + P^s = I, (P^\alpha)^2 = P^\alpha, P^\alpha P^\beta = 0, \alpha, \beta = c, u, s, \alpha \neq \beta\},$$

i.e.,  $N$  is the collection of all projection triples corresponding to trichotomies of  $X$  into closed subspaces. One can verify that  $N$  is a Banach manifold. Therefore, we have a  $C^0$  map from  $M$  into  $N$  given by

$$\Pi(m) = (\Pi_m^c, \Pi_m^u, \Pi_m^s).$$

Note that, considering  $M$  as an approximately normally hyperbolic invariant manifold for any  $\tilde{T}$  so that  $\|\tilde{T} - T\|_{C^1(B(M,r))}$  is sufficiently small for some  $r > 0$ , the constant  $\sigma_0$  and  $\mathcal{A}_0$  in Theorems 2.2 and 6.3 are determined only by

$$B' = 2\|\Pi\|_{C^0}, \quad B'_1 = 2\|DT\|_{C^0(B(M,r))}, \quad \lambda' = \frac{1+\lambda}{2}, \quad a' = \frac{1}{2}a.$$

Applying Theorem 6.9, there exists a  $\tilde{\Pi} \in C^1(M, N)$  such that (H2), (H3), (H3'), (C3) and (C3') are satisfied for the following constants when  $\tilde{\Pi}$  and  $T$  are substituted in,

$$B'' = B', \quad \sigma'' = \frac{1}{3}\sigma_0, \quad \lambda'' = \frac{1}{3} + \frac{2\lambda}{3}, \quad a'' = \frac{4}{3}a', \quad L = \|\tilde{\Pi}\|_{C^1}.$$

Therefore, when  $\|\tilde{T} - T\|_{C^1(B(M,r))}$  is sufficiently small and  $\tilde{T}$  and  $\tilde{\Pi}$  are used, then Definition 2.1, Definition 6.2, and hypotheses (H1)–(H4), (H3'), (C3) and (C3') are satisfied with  $\lambda'$ ,  $B'$ ,  $B'_1$ ,  $L$ ,  $a'$  and

$$\eta = \|\tilde{T} - T\|_{C^0(M)}, \quad \sigma = \frac{\sigma_0}{2}.$$

Let  $\mathcal{A}_{\tilde{T}}(\delta)$  be defined for  $\tilde{T}$  as in (2.5) then  $\mathcal{A}_T(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  since  $M$  is compact. Moreover,

$$|\mathcal{A}_{\tilde{T}}(\delta) - \mathcal{A}_T(\delta)| \leq 2\|\tilde{T} - T\|_{C^1(B(M,r))}.$$

Therefore, when  $\|\tilde{T} - T\|_{C^1(B(M,r))}$  is sufficiently small, Theorems 2.2 and 6.3 imply that, for  $\tilde{T}$ , there exist  $C^1$  center stable manifold  $W^{cs}$  and center unstable manifold  $W^{cu}$ , whose common “width”  $\delta_0$  is uniform in  $\tilde{T}$ . It is clear that  $\tilde{M} = W^{cs} \cap W^{cu}$  is a  $C^1$  invariant manifold diffeomorphic to  $M$ . The corresponding invariant subspaces are given by the tangent spaces of the stable and unstable fibers, which are  $C^0$  in the base point  $\tilde{m} \in \tilde{M}$ .

Finally, we prove Theorem 6.9. The notation used below is completely independent of other parts of the paper.

*Proof.* The idea is fairly straightforward, even if the details are cumbersome: Use mollifiers and a partition of unity on subsets that form a locally finite cover. First, we construct a countable collection of coordinate charts covering  $M$ . Let  $\Lambda = \{W_i : i = 1, 2, \dots\}$  be a topological base of  $M$ . For any  $m \in M$ , since  $M$  is  $n$ -dimensional and thus locally compact, there

exists a neighborhood  $W$  of  $m$  such that  $\overline{W}$  is compact, and an element  $W_m \in \Lambda$  such that  $m \in W_m \subset W$ . Therefore,  $\overline{W_m}$  is compact. It is clear that  $\Lambda_1 = \{W_m : m \in M\} \subset \Lambda$  is a countable collection. We may label the elements of  $\Lambda_1$  as  $W_i^{(1)}$ ,  $i = 1, 2, \dots$ . Obviously  $M = \bigcup_i W_i^{(1)}$ .

Next, we inductively define a class of open sets  $\{K_i : i = 1, 2, \dots\}$  such that

$$\bigcup_{j=1}^i W_j^{(1)} \subset K_i, \quad \overline{K_i} \subset K_{i+1}, \quad \text{and} \quad \overline{K_i} \text{ is compact}$$

for  $i = 1, 2, \dots$ . Let  $K_1 = W_1^{(1)}$ . Suppose that we have defined  $K_1, \dots, K_i$  such that the above conditions hold. Let  $K = K_i \cup W_{i+1}^{(1)}$ , then  $\overline{K} \setminus K$  is compact and can be covered by the union of some finitely many elements  $W_{j_1}^{(1)}, \dots, W_{j_l}^{(1)}$  in  $\Lambda_1$ . Let

$$K_{i+1} = K_i \cup W_{i+1}^{(1)} \bigcup_{k=1}^l W_{j_k}^{(1)}.$$

It is clear that this construction gives sets satisfying our requirements. Let  $K_0 = K_{-1} = \emptyset$ .

For each  $i > 0$ , find finitely many coordinate charts  $(U_{i,j}, \phi_{i,j})$ ,  $j = 1, 2, \dots, l_i$ , where  $\phi_{i,j} : U_{i,j} \rightarrow B_n(0, 3) \subset \mathbb{R}^n$ , such that  $U_{i,j} \subset K_{i+1} \setminus \overline{K_{i-2}}$ ,  $f(U_{i,j})$  is within a coordinate neighborhood in  $N$  for each  $j$ , and  $\overline{K_i} \setminus K_{i-1} \subset \bigcup_{j=1}^{l_i} \phi_{i,j}^{-1}(B_n(0, 1))$ . Let  $\{(U_i, \phi_i) \mid \phi_i : U_i \rightarrow B_n(0, 3), i = 1, 2, \dots\}$  be the collection of all the above coordinate charts. Let  $Y$  be the model Banach space of  $N$  and, for each  $i$ , let  $\psi_i : O_i \rightarrow B_Y(0, 1)$  be a coordinate chart in  $N$  such that  $f(\overline{U_i}) \subset O_i$  and  $\psi_i(f(\phi_i^{-1}(0))) = 0$ . Let  $V_i = \phi_i^{-1}(B_n(0, 1))$ . From the above construction, it is clear that  $\bigcup_i V_i = M$ ,  $V_i$  is precompact, and  $\{j : \overline{U_j} \cap \overline{U_i} \neq \emptyset\}$  is finite for any  $i > 0$ .

With the above preliminaries, we start to construct a sequence of approximations

$$\{f_{\varepsilon,k} \in C^0(M, N) : \varepsilon \in [0, 1], k = 1, 2, \dots\}$$

such that

- For each  $k > 0$  and  $\varepsilon \in [0, 1]$ ,  $f_{\varepsilon,k}$  is  $C^r$  on  $\bigcup_{i=1}^k V_i$  and  $f_{\varepsilon,k+1} = f_{\varepsilon,k}$  except on  $U_{k+1}$ , and  $f_{\varepsilon,k}(\overline{U_i}) \subset O_i$  for all  $\varepsilon, k, i$ .
- $d(f_{\varepsilon,k}, f_{\varepsilon_0,k}) \rightarrow 0$  as  $\varepsilon \rightarrow \varepsilon_0$  for  $\varepsilon_0 \in [0, 1]$ ;
- $d(f_{\varepsilon,k}, f_{\varepsilon,k+1}) \leq 2^{-(k+1)}\varepsilon$  and  $f_{0,k} = f$ ,  $f_{\varepsilon,0} = f$ .

Let  $\sigma \in C_0^\infty(B_n(0, \frac{1}{2}), [0, +\infty))$  satisfy  $\int_{\mathbb{R}^n} \sigma = 1$  and define  $\sigma_\delta(x) = \delta^{-n} \sigma(\frac{x}{\delta})$ . Let  $\gamma \in C_0^\infty(B_n(0, \frac{3}{2}), [0, 1])$  satisfy  $\gamma|_{B_n(0,1)} = 1$ . Let  $f_{\varepsilon,0} = f$  and assume  $f_{\varepsilon,0}, \dots, f_{\varepsilon,k}$  have been defined. For any  $m \in \phi_{k+1}^{-1}(B_n(0, 2))$ ,

let  $x = \phi_{k+1}(m)$  and, for  $(\varepsilon, \delta) \in [0, 1] \times (0, 1]$ , define,

$$g_{\varepsilon, \delta}(m) = \psi_{k+1}^{-1} \left\{ \psi_{k+1}(f_{\varepsilon, k}(m)) + \gamma(x) \left[ \int_{B_n(0, 1)} \sigma_{\delta}(y) \psi_{k+1}(f_{\varepsilon, k}(\phi_{k+1}^{-1}(x + y))) dy - \psi_{k+1}(f_{\varepsilon, k}(m)) \right] \right\}.$$

Let  $g_{\varepsilon, \delta}(m) = f_{\varepsilon, k}(m)$  for  $m \notin \phi_{k+1}^{-1}(B_n(0, 2))$  and  $g_{\varepsilon, 0} = f_{\varepsilon, k}$ .

Clearly,  $g_{\varepsilon, \delta} \in C^0(M, N)$  and is  $C^r$  on  $V_{k+1}$ . We will first prove, for fixed  $(\varepsilon_0, \delta_0) \in [0, 1] \times [0, 1]$ ,  $d(g_{\varepsilon, \delta}, g_{\varepsilon_0, \delta_0}) \rightarrow 0$  as  $(\varepsilon, \delta) \rightarrow (\varepsilon_0, \delta_0)$ . On  $B_n(0, 2)$ , let

$$\bar{g}_{\varepsilon, \delta} = \psi_{k+1} \circ g_{\varepsilon, \delta} \circ \phi_{k+1}^{-1}, \quad \bar{f}_{\varepsilon} = \psi_{k+1} \circ f_{\varepsilon, k} \circ \phi_{k+1}^{-1}, \\ S_1 = \{z = \bar{f}_{\varepsilon}(x) : x \in \overline{B_n(0, \frac{3}{2})}, \varepsilon \in [0, 1]\}.$$

By the induction assumption (2),  $S_1$  is compact. Therefore,

$$S = \overline{\text{conv}(S_1)} \subset B_Y(0, 1) \subset Y$$

is also compact. This implies that, for any  $\eta > 0$ , there exists  $\xi > 0$  such that  $\|z_2 - z_1\|_Y < \eta$  whenever  $z_2, z_1 \in S$ ,  $d(\psi_{k+1}^{-1}(z_2), \psi_{k+1}^{-1}(z_1)) < \xi$ . By the definition,  $\bar{g}_{\varepsilon, \delta} \rightarrow \bar{g}_{\varepsilon_0, \delta_0}$  also uniformly on  $B_n(0, \frac{3}{2})$ , which implies that, restricted on  $\phi_{k+1}^{-1}(\overline{B_n(0, \frac{3}{2})})$ ,  $d(g_{\varepsilon, \delta}, g_{\varepsilon_0, \delta_0}) \rightarrow 0$  as  $(\varepsilon, \delta) \rightarrow (\varepsilon_0, \delta_0)$ . Since  $g_{\varepsilon, \delta} = f_{\varepsilon, k}$  except on  $\phi_{k+1}^{-1}(B_n(0, \frac{3}{2}))$ , therefore,  $d(g_{\varepsilon, \delta}, g_{\varepsilon_0, \delta_0}) \rightarrow 0$  as  $(\varepsilon, \delta) \rightarrow (\varepsilon_0, \delta_0)$ .

Let

$$D_i = \bigcup_{\varepsilon \in [0, 1]} f_{\varepsilon, k}(\phi_{k+1}^{-1}(\overline{B_n(0, \frac{3}{2})}) \cap \overline{U_i}),$$

for  $i = 1, 2, \dots$ . From the induction assumption,  $D_i$  is compact and  $d(D_i, O_i^c) > 0$ . The construction of  $U_i$  implies that there are only finitely many nonempty  $D_i$ . Therefore, there exists  $\eta > 0$  such that  $d(D_i, O_i^c) > \eta$  for all  $i$ . Let

$$h(\varepsilon, \delta) = \frac{d(g_{\varepsilon, \delta}, f_{\varepsilon, k})}{\varepsilon}$$

for  $(\varepsilon, \delta) \in (0, 1] \times [0, 1]$ . Obviously,  $h$  is continuous and  $h(\varepsilon, 0) = 0$ . By the continuity of  $h$ , there exists a smooth function  $\delta(\varepsilon) > 0$ , for  $\varepsilon \in (0, 1]$  such that

$$h(\varepsilon, \delta(\varepsilon)) < \min \left\{ 2^{-(k+1)}, \frac{\eta}{\varepsilon} \right\}$$

and  $\delta(\varepsilon) \rightarrow 0^+$  as  $\varepsilon \rightarrow 0^+$ . In fact,  $\delta(\varepsilon)$  can be set to be a constant on  $[\varepsilon_0, 1]$  for any  $\varepsilon_0 \in (0, 1]$ . Let  $f_{\varepsilon, k+1} = g_{\varepsilon, \delta(\varepsilon)}$ . One can verify the induction assumptions easily.

Let  $\tilde{f}_\varepsilon = \lim_{k \rightarrow \infty} f_{\varepsilon, k}$ . Because, for any  $i > 0$ , there are only finitely many  $j$ 's satisfying  $U_i \cap U_j \neq \emptyset$ ,  $\tilde{f}_\varepsilon$  is  $C^r$ . Clearly, all the conditions on  $\tilde{f}_\varepsilon$  hold.  $\square$

## 7. Dynamical spike solutions for a singular parabolic equation

We seek a manifold of evolving spike solutions of the semilinear parabolic equation

$$(7.1) \quad \begin{cases} u_t = \varepsilon^2 \Delta u - u + f(u), & x \in \Omega \subset \mathbb{R}^n \\ \frac{\partial u}{\partial N} = 0, & x \in \partial\Omega. \end{cases}$$

Here  $0 < \varepsilon \ll 1$ ,  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^n$ , and  $N$  is the outward unit normal vector field of  $\partial\Omega$ . The nonlinearity  $f$  is smooth and assumed to satisfy conditions (F1)–(F3) below, which guarantee certain properties of the ground states of the corresponding rescaled elliptic problem on  $\mathbb{R}^n$ .

In this section, the abstract theorems established in the previous sections will be applied to the nonlinear parabolic equation (7.1). We will prove that, under certain conditions on  $f$ , there exists a normally hyperbolic invariant manifold in the phase space  $W^{2,q}(\Omega)$  of (7.1), which is diffeomorphic to  $\partial\Omega$ . Moreover, this invariant manifold consists entirely of single-boundary-peak states. By a single-boundary-peak state, we mean a function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  which achieves its maximum at some  $p \in \partial\Omega$  and decays like  $O(e^{-\frac{C|x-p|}{\varepsilon}})$ . Moreover, the speed of the peak as it moves along  $\partial\Omega$  is of order  $O(\varepsilon^3)$ . The analysis could be extended to any fixed number of peaks moving on  $\partial\Omega$ , so long as the peaks remain separated by a distance of order  $O(\varepsilon^\gamma)$  with  $\gamma < 1$ , but that, being merely a technical exercise, will not be pursued here.

**Background on stationary peak solutions.** The stationary problem for (7.1) has been studied by many authors, especially for the case where  $f(u) = u^p$  with superlinear but subcritical growth. In [NT1], the Gierer–Meinhardt system was investigated in the asymptotic limit as the diffusivity of the inhibitor becomes unbounded. In that limit, one is lead to (7.1), referred to as the ‘shadow equation’. It was observed that no positive stationary solutions exist when  $\varepsilon$  is large. In [LNT] the system and the single equation were again studied. For (7.1) it was shown that positive solutions must have peaks with exponentially decaying tails as  $\varepsilon \downarrow 0$ . The paper [NT2] studies (7.1) with  $f(u) = u^p$ . Using a mountain pass argument, the authors obtain a positive solution that has a single peak, the so-called least energy solution. They show that this peak must actually lie on  $\partial\Omega$  and the profile of the solution is a modification of the ground state on  $\mathbb{R}^n$ , translated and

rescaled by  $\varepsilon$ . Z.-Q. Wang, in [WZ1], gave a topological lower bound on the number of such solutions. In further work W.-M. Ni and I. Takagi, in [NT3], proved that the peak location tended, as  $\varepsilon \rightarrow 0$ , to the point of  $\partial\Omega$  where the mean curvature achieved is maximum. The analysis involved an asymptotic expansion of the energy, in terms of  $\varepsilon$ , about that modified ground state discussed above. The modification to which we refer, involved smoothly mapping a neighborhood of the half plane, centered at the peak of the ground state, to a neighborhood of a point of  $\partial\Omega$  lying within  $\bar{\Omega}$ , and it is this mapping that introduces the mean curvature when one computes the expansion of the energy. Other papers followed, providing for solutions with spikes at any collection of nondegenerate (in some cases only topologically nontrivial) critical points of the mean curvature, and even multiple spikes accumulating at local minimal points of the mean curvature, or solutions to the Cahn–Hilliard and other singularly perturbed equations and systems (see, e.g., [FW], [Wax], [We1], [DFW], [O1], [O2], [Li], [BDS], [DY1], [BSh], [GWW], and [WW2]).

There are a number of related works which give multi-peaked stationary solutions to (7.1) and to related equations, with the peaks interior to  $\Omega$  or with some peaks interior and some on the boundary (see, e.g., [BFi], [WW1], [BFu], and [GW2]).

Likewise, the Dirichlet problem has also attracted some attention, with results providing detailed information about the existence and location of a stationary peak (see, e.g., [J], [NW], and [D]).

The case of critical growth is quite different, due to a scale invariance and related lack of compactness. Still there are some results and the interested reader is referred to [WZ2], [WZ3], [APY1], [AMY], and [APY2], for example.

All of the papers cited above concerned stationary solutions and most of the techniques are designed for that situation, i.e, static rather than dynamic. Our aim in this paper is to describe, in some precision, the dynamics of peaks as they move along  $\partial\Omega$ , including the location of stationary points. That viewpoint was espoused in [FH] and [CP1], where the motion of interfaces for the one-dimensional Allen–Cahn equation (a bistable version of (7.1)) was shown to be exponentially slow. Later, in [CP2], an invariant manifold of layered states was shown to exist, on which this slow motion evolved. Extensions of this approach to the one-dimensional Cahn–Hilliard equation were given in [ABF], [BX1], and [BX2]. Related equations and systems on higher dimensional domains were analyzed from this dynamic point of view in [BFu], [AF], and [Ko]. Furthermore, when one studies the Morse index of the stationary spike states, one is actually studying local dynamics in a neighborhood of the critical point, and so the dynamical point of view is actually contained in the work of [WW3] and [BSh], for instance. In all these cases, a (local) approximately invariant manifold is constructed which is shown to be approximately normally hyperbolic, although not necessarily using that terminology. The ‘normal hyperbolicity’ allows for a reduction to a finite-dimensional manifold upon which the dynamics is estimated and the

stationary state examined for stability. This reduction is sometimes referred to as a Lyapunov–Schmidt or saddle-point reduction, depending upon the point of view or techniques employed by the authors. We believe that this dynamical systems approach in a tubular neighborhood of the approximate invariant manifold unifies, clarifies, and extends the previous results.

**7.1. Existence of a normally hyperbolic invariant manifold consisting of spike profiles.** The profile of the peak solutions are roughly given by translations of the rescaled solution  $w$  of following elliptic equation

$$(7.2) \quad \begin{cases} \Delta w - w + f(w) = 0, & y \in \mathbb{R}^n, \\ w(0) = \max w(y), & w > 0, \\ w(y) \rightarrow 0, & y \rightarrow \infty. \end{cases}$$

It is easy to see that  $w$  is the ground state of the rescaled ( $x = \varepsilon y$ ) stationary equation of (7.1) considered on  $\mathbb{R}^n$ .

Let  $L_0 = \Delta - 1 + f'(w) : H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ . Assume

- (F1)  $f \in C^m(\mathbb{R}, \mathbb{R})$ ,  $m \geq 1$ , with  $f(0) = f'(0) = 0$ .  
 (F2) Equation (7.2) has a radially symmetric solution  $w \in C^{m+2}(\mathbb{R}^n)$  such that, for each integer  $0 \leq k \leq m + 2$ ,

$$|\partial_r^k w(y)| \leq C e^{-\mu|y|}, \quad y \in \mathbb{R}^n,$$

for some  $C, \mu > 0$ , where  $\partial_r$  is differentiation in the radial direction.

- (F3) For some  $b \in (0, 1)$ , assume  $\sigma(L_0) \cap [-b, \infty) = \{\lambda_1, 0\}$ , where  $\sigma(L_0)$  is the spectrum of  $L_0$ , and  $\lambda_1 > 0$  is the (simple) principle eigenvalue with a radially symmetric eigenfunction  $v_0 \in C^{m+1}(\mathbb{R}^n)$  which satisfies the same decaying property for  $0 \leq k \leq m + 1$  as in (F2). Moreover, assume the eigenspace of 0 is spanned by

$$\left\{ \frac{\partial w}{\partial y_j} : j = 1, 2, \dots, n \right\}.$$

*Remark 7.1.* Even though  $f$  is only assumed to be  $C^m$ , instead of  $C^{m,\beta}$  with  $\beta > 0$ , since  $w$  and  $v_0$  are radially symmetric, one may actually prove that  $w \in C^{m+2}(\mathbb{R}^n)$  and  $v_0 \in C^{m+1}(\mathbb{R}^n)$  from the ODEs they satisfy.

The necessary smoothness of  $\partial\Omega$  is not the main thrust of this paper and so we do not provide details of a careful analysis along these lines. Certainly, it is sufficient for  $\partial\Omega$  to be of class  $C^{m+3}$ .

Under assumptions (F1)–(F3), the proof of the main result (Theorem 1.1) will be given in the rest of this section.

**Preparations.** Define, for any  $q \in [1, \infty)$ , positive integer  $k$ , and  $u : \Omega \rightarrow \mathbb{R}$ ,

$$|u|_{W_{\varepsilon}^{k,q}(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \varepsilon^{|\alpha| - \frac{n}{q}} |\partial^\alpha u|_{L^q(\Omega)} = \sum_{0 \leq |\alpha| \leq k} \varepsilon^{|\alpha|} |\partial^\alpha u|_{L^q(\Omega, \varepsilon^{-n} d\mu)}.$$

In particular,  $|\cdot|_{W_\varepsilon^{0,q}(\Omega)} = |\cdot|_{L^q(\Omega, \varepsilon^{-n} d\mu)}$ . It is clear that the norm  $|\cdot|_{W_\varepsilon^{k,q}(\Omega)} = |\cdot|_{W^{k,q}(\Omega_\varepsilon, x_0)}$  where  $x_0 \in \mathbb{R}^n$  and  $\Omega_\varepsilon, x_0 = \{\frac{x-x_0}{\varepsilon} : x \in \Omega\}$ .

In order to obtain estimates uniform in sufficiently small  $\varepsilon > 0$ , we first introduce some local coordinate systems near  $\partial\Omega$  and prove a lemma on the extension to  $\mathbb{R}^n$  of functions defined on  $\bar{\Omega}$ .

Define  $\Phi : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}^n$  by

$$\Phi(p, d) = p + dN(p), \quad p \in \partial\Omega.$$

Since  $\Omega$  is compact with smooth boundary, there exists  $\delta_0 > 0$  such that  $\Phi$  is a diffeomorphism on  $\partial\Omega \times [-\delta_0, \delta_0]$ . By the compactness of  $\partial\Omega$  again, for some  $C > 0$  and small enough  $\delta_1 > 0$  and any  $p \in \partial\Omega$ , there exists  $\phi \in C_0^\infty(T_p\partial\Omega, \mathbb{R})$  such that

$$\begin{aligned} \{x \in \Omega : |x - p| \leq 3\delta_1\} &\subset \{p + y + dN(p) : y \in T_p\partial\Omega, d < \phi(y)\}, \\ \{x \in \partial\Omega : |x - p| \leq 3\delta_1\} &\subset \{p + y + dN(p) : y \in T_p\partial\Omega, d = \phi(y)\}, \end{aligned}$$

and

$$|\nabla\phi(y)| \leq C \min\{|y|, \delta_1\}.$$

This induces a local coordinate diffeomorphism near  $p$ ,

$$(7.3) \quad \bar{\Phi}(y, d) \equiv \Phi(p + y + \phi(y)N(p), d), \quad y \in T_p\partial\Omega,$$

where  $\partial\Omega$  is given by  $\bar{\Phi}(y, 0)$  locally and  $\partial_d\bar{\Phi}(y, 0) = N(\bar{\Phi}(y, 0))$ . In this local coordinate system, the Laplacian can be written in terms of  $y$  and  $d \equiv y_n$  as

$$(7.4) \quad \Delta = g^{ij}\partial_{ij} + \frac{1}{\sqrt{G}}\partial_i(g^{ij}\sqrt{G})\partial_j$$

where  $(g^{ij})_{n \times n}$  is the inverse of the matrix  $(g_{ij})_{n \times n}$  with  $g_{ij} = \partial_{y_i}\bar{\Phi} \cdot \partial_{y_j}\bar{\Phi}$  and  $G = \det(g_{ij})$ . From the construction, one finds that, for  $|d|, |y| \leq 2\delta_1$ ,

$$(7.5) \quad |\sqrt{G} - 1|_{C^0} + |g^{ij} - \delta_{ij}|_{C^0} \leq C(|y| + |d|) \quad \text{and} \quad |Dg^{ij}|_{C^k} + |DG|_{C^k} \leq C.$$

With the help of this local coordinate system, one can prove the following extension lemma.

**Lemma 7.2.** *For any  $\delta > 0$  there exists a linear extension operator  $E$  satisfying*

$$|E|_{L(W_\varepsilon^{k,q}(\Omega), W_\varepsilon^{k,q}(\mathbb{R}^n))} \leq C, \quad k = 0, 1$$

for any  $q \in (1, \infty)$  and some  $C > 0$  independent of  $\varepsilon$  and



(1) for any  $u \in W^{2,q}(\Omega)$  with  $\frac{\partial u}{\partial N}|_{\partial\Omega} \equiv 0$ , one has  $Eu \in W^{2,q}(\mathbb{R}^n)$  and

$$|E(\varepsilon^2 \Delta u) - \varepsilon^2 \Delta Eu|_{W_\varepsilon^{0,q}(\mathbb{R}^n)} \leq \delta |u|_{W_\varepsilon^{2,q}(\Omega)};$$

(2) for any  $p \in \partial\Omega$  and any function  $\gamma_e \in W^{2,\infty}(\mathbb{R}^n, \mathbb{R})$ , even about the hyperplane  $T_p \partial\Omega$  and satisfying  $|D^l \gamma_e(\tilde{x})| \leq C_0 e^{-\frac{|\tilde{x}|}{C_0}}$  for  $l = 0, 1, 2$ , one has

$$\left| \gamma_e \left( \frac{\cdot - p}{\varepsilon} \right) Eu - E \left( \gamma_e \left( \frac{\cdot - p}{\varepsilon} \right) u \right) \right|_{W_\varepsilon^{k,q}(\mathbb{R}^n)} \leq C \varepsilon |u|_{W_\varepsilon^{k,q}(\Omega)}, \quad k = 0, 1,$$

and

$$\left| \int_{\mathbb{R}^n} \gamma_e \left( \frac{x - p}{\varepsilon} \right) (Eu)(x) \varepsilon^{-n} dx - 2 \int_{\Omega} \gamma_e \left( \frac{x - p}{\varepsilon} \right) u(x) \varepsilon^{-n} dx \right| \leq C \varepsilon |u|_{W_\varepsilon^{0,q}(\Omega)},$$

for some  $C > 0$  depending only on  $q$  and  $C_0$ . For a function  $\gamma_o : \mathbb{R}^n \rightarrow \mathbb{R}$ , odd about the hyperplane  $T_p \partial\Omega$  and satisfying the same decay property, one has

$$\left| \int_{\mathbb{R}^n} \gamma_o \left( \frac{x - p}{\varepsilon} \right) (Eu)(x) \varepsilon^{-n} dx \right| \leq C \varepsilon |u|_{W_\varepsilon^{0,q}(\Omega)}.$$

*Proof.* We give a sketch of the proof, which follows from the standard construction of extension operators, while the  $\varepsilon$  scaling yields most of the estimates. Let  $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$  be a cut-off function satisfying  $\eta|_{[-1,1]} \equiv 1$ ,  $\eta|_{\mathbb{R} \setminus [-2,2]} \equiv 0$ , and  $|\eta'|_{C^0} \leq 2$ . For  $\delta_1 \in (0, \frac{1}{2}\delta_0]$ , define the extension in terms of the local coordinate map  $\Phi$

$$(Eu)(p, d) = \eta \left( \frac{d}{\delta_1} \right) u(p, -d), \quad d > 0.$$

When considered as an operator from  $W_\varepsilon^{k,q}(\Omega)$  to itself,  $k = 0, 1$ , it is easy to verify that  $E$  has a bound  $C > 0$  independent of  $\varepsilon \in (0, 1]$ .

In the coordinate system given by  $\bar{\Phi}$ , near any  $p \in \partial\Omega$ ,  $(Eu)(\bar{\Phi}(y, d)) = (Eu)(\bar{\Phi}(y, -d))$ ,  $d \in [0, \delta_1]$ , and then it is clear that  $Eu \in W^{2,q}(\mathbb{R}^n)$  if  $u \in W^{2,q}(\Omega)$  and  $\frac{\partial u}{\partial N}|_{\partial\Omega} \equiv 0$ . The estimate for the commutator of  $E$  and  $\varepsilon^2 \Delta$  follows from (7.4) and (7.5) by taking sufficiently small  $\delta_1$  and a partition of unity.

For the given function  $\gamma_e$ , even about the hyperplane  $T_p \partial\Omega$ , let  $\bar{\gamma}_e(y, d) = \gamma_e(\frac{1}{\varepsilon}(\bar{\Phi}(y, d) - p))$ . We have

$$\begin{aligned} & \bar{\gamma}_e(y, d) - \bar{\gamma}_e(y, -d) \\ &= \gamma_e \left( \frac{1}{\varepsilon}(\bar{\Phi}(y, d) - p) \right) - \gamma_e \left( \frac{1}{\varepsilon}(\bar{\Phi}(y, d) - p - 2\sigma) \right), \end{aligned}$$

where

$$\sigma = \phi(y)N(p) - \frac{d\nabla\phi(y)}{\sqrt{1 + |\nabla\phi(y)|^2}} = O(|y|(|y| + |d|)).$$

The desired inequalities for  $E$ , which show that  $Eu$  is almost even about  $\partial\Omega$ , follow from a rescaling and this symmetry-breaking estimate.  $\square$

From the extension lemma and rescaling, we find that there exists a uniform  $C > 0$ , independent of sufficiently small  $\varepsilon > 0$ , such that

$$(7.6) \quad |u|_{W_\varepsilon^{0, \frac{nq}{n-q}}(\Omega)} \leq C|u|_{W_\varepsilon^{1,q}(\Omega)}$$

if  $q < n$  and

$$(7.7) \quad |u|_{C^0(\Omega)} \leq C|u|_{W_\varepsilon^{1,q}(\Omega)}, \quad |u_1 u_2|_{W_\varepsilon^{1,q}(\Omega)} \leq C|u_1|_{W_\varepsilon^{1,q}(\Omega)}|u_2|_{W_\varepsilon^{1,q}(\Omega)}$$

if  $q > n$ .

In order to produce the manifold of spike-like solutions with the peak moving on  $\partial\Omega$ , we will also need to discuss the calculus for maps defined on  $\partial\Omega$ . Thus, the covariant derivative  $\mathcal{D}$  along  $\partial\Omega$  and the second fundamental form  $\Pi$  get involved naturally. For any  $v$  defined on  $\partial\Omega$  and tangential vector fields  $X, Y, Z, \dots$  along  $\partial\Omega$ , we use the notation

$$\begin{aligned} \mathcal{D}v(X) &= \mathcal{D}_X v = \nabla_X v, & \mathcal{D}^2 v(X, Y) &= \mathcal{D}_X \mathcal{D}_Y v - \mathcal{D}_{\mathcal{D}_X Y} v, \\ \mathcal{D}^3 v(X, Y, Z) &= \dots \end{aligned}$$

and the higher order derivatives, as multilinear forms (maps) at each  $p \in \partial\Omega$ , are calculated similarly. Occasionally,  $\mathcal{D}$  will appear with a subscript  $\tau$  from the tangent space, indicating the directional derivative in that direction. At other times  $\mathcal{D}_p$  is used to emphasize that the derivative is with respect to  $p \in \partial\Omega$ . Define

$$|u|_{W_\varepsilon^{k,q}(\partial\Omega)} = \sum_{0 \leq i \leq k} \varepsilon^{i - \frac{n-1}{q}} |\mathcal{D}^i u|_{L^q(\partial\Omega)} = \sum_{0 \leq i \leq k} \varepsilon^i |\mathcal{D}^i u|_{L^q(\partial\Omega, \varepsilon^{1-n} d\mu)}.$$

The Sobolev space  $W_\varepsilon^{s,q}(\partial\Omega)$  can be defined by interpolation for non-integer  $s$ . From elliptic estimates, the norm  $|\cdot|_{W_\varepsilon^{k,q}(\partial\Omega)}$  can also be defined by the Beltrami-Laplacian  $\Delta_{\partial\Omega}$  on  $\partial\Omega$ . By a localization argument as in the proof of Lemma 7.2, the trace estimates with constants uniform in  $\varepsilon$  hold as well.

Let  $L_\varepsilon = \varepsilon^2 \Delta - 1$  with the domain

$$D(L_\varepsilon) = \left\{ u \in W^{2,q}(\Omega) : \frac{\partial u}{\partial N}(x) = 0, x \in \partial\Omega \right\} \subset L^q(\Omega).$$

Elliptic estimates on  $L_\varepsilon^{-1}$  uniform in  $\varepsilon$  follow directly from the estimate given in Lemma 7.2 on the commutator of  $\varepsilon^2 \Delta$  and  $E$ . Clearly,  $L_\varepsilon$  is

dissipative and for  $1 < q < \infty$

(7.8)

$$|e^{tL_\varepsilon}|_{L(W_\varepsilon^{0,q}(\Omega))} \leq e^{-t}, \quad |e^{tL_\varepsilon}|_{L(W_\varepsilon^{0,q}(\Omega), W_\varepsilon^{1,q}(\Omega))} \leq C(1+t^{-\frac{1}{2}})e^{-t} \quad \text{for } t > 0$$

for some  $C > 0$  independent of  $\varepsilon$  and  $t$ . In order to guarantee that the composition with  $f$  defines a smooth mapping in the phase space, we fix  $q_0 > n$  and take the phase space of (7.1) as

$$(7.9) \quad X = W_\varepsilon^{1,q_0}(\Omega) = W_\varepsilon^{1,q_0}(\Omega) \cap W_\varepsilon^{1,2}(\Omega) \\ \text{with norm } |\cdot|_X = |\cdot|_{W_\varepsilon^{1,q_0}(\Omega)} + |\cdot|_{W_\varepsilon^{1,2}(\Omega)}.$$

Note that the embedding constant from  $W_\varepsilon^{1,q_0}(\Omega)$  to  $W_\varepsilon^{1,2}(\Omega)$  depends on  $\varepsilon$  and thus  $|\cdot|_{W_\varepsilon^{1,2}(\Omega)}$  can not be omitted in  $|\cdot|_X$ .

In order to apply our general theorem, we modify  $f$  so that the evolution defines a semiflow globally in time. Thus, we consider

$$(7.10) \quad u_t = \varepsilon^2 \Delta u - u + \tilde{f}(u), \quad \text{where } \tilde{f}(u) = \eta(u)f(u).$$

Here,  $\eta(s) \geq 0$  is a  $C^\infty$  cut-off function satisfying

$$\eta(s) = 1, \quad |s| \leq 1 + 2s_0; \quad \eta(s) = 0, \quad |s| \geq 2 + 4s_0, \\ \text{where } s_0 = \max_{\mathbb{R}^n} |w(x)|.$$

Note that  $|\tilde{f}|_{C^m(\mathbb{R})} < \infty$ . By composition,  $\tilde{f}$  induces a mapping (with a slight abuse of notation)  $\tilde{f} : X \rightarrow W_\varepsilon^{0,q}(\Omega)$ ,  $q \in [2, q_0]$ , with

$$(7.11) \quad |\tilde{f}|_{C^m(X, W_\varepsilon^{0,q}(\Omega))} \leq C, \quad q \in [2, q_0]$$

for some  $C > 0$  independent of  $\varepsilon$ . Moreover, since the scalar function  $\tilde{f}$  is compactly supported and  $X \hookrightarrow L^\infty$ , we have

- $D^m \tilde{f} : X \rightarrow L(\otimes^m X, W_\varepsilon^{0,q}(\Omega))$  is uniformly continuous.

From (7.8), the initial value problem of (7.10) with Neumann boundary condition is well-posed globally in time in  $X$  and it generates a semiflow  $T_\varepsilon^t$  on  $X$ . Moreover, for any time  $T_0 > 0$ , there exists  $B_1 > 0$  independent of sufficiently small  $\varepsilon > 0$  such that

$$(7.12) \quad |T_\varepsilon^t|_{C^m(X, X)} \leq B_1 \quad \text{for all } t \in [0, T_0]$$

and  $D^m T_\varepsilon^t$  is uniformly continuous:

- for any  $\xi > 0$ , there exists  $\zeta > 0$  such that

$$|D^m T_\varepsilon^t(u_1) - D^m T_\varepsilon^t(u_2)|_{L(\otimes^m X, X)} \leq \xi$$

for any  $t \in [0, T_0]$  and  $u_1, u_2 \in X$  with  $|u_1 - u_2|_X \leq \zeta$ .

**Construction of the approximate invariant manifold.** For any  $p \in \partial\Omega$ , let

$$\tilde{w}_{\varepsilon,p}(x) = w\left(\frac{x-p}{\varepsilon}\right),$$

where  $w$  is the ground state given by (7.2). We first modify  $\tilde{w}_{\varepsilon,p}$  so that it satisfies the boundary condition. Given any  $v : \partial\Omega \rightarrow \mathbb{R}$ , let  $h = \mathcal{H}(v) : \Omega \rightarrow \mathbb{R}$  be the solution of

$$(7.13) \quad L_\varepsilon h = 0 \text{ in } \Omega, \quad \frac{\partial h}{\partial N} = v \text{ on } \partial\Omega.$$

For any  $p \in \partial\Omega$ , let

$$(7.14) \quad W_{\varepsilon,p} = \tilde{w}_{\varepsilon,p} - \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right).$$

Clearly,  $W_{\varepsilon,p} \in D(L_\varepsilon)$ . Define the map  $\psi_\varepsilon$  and the approximate invariant manifold  $M_\varepsilon$  as

$$(7.15) \quad \psi_\varepsilon(p) = W_{\varepsilon,p} \quad \text{and} \quad M_\varepsilon = \psi_\varepsilon(\partial\Omega).$$

It is important to obtain estimates for the correction term  $\mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right)$  and we will start with  $\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}$ .

**Lemma 7.3.** *Let  $u$  be a  $C^J$  radially symmetric function defined on  $\mathbb{R}^n$  which satisfies the decaying property in (F2) for  $0 \leq k \leq J$ . Let  $l, k \geq 0$  be integers with  $l + k \leq J - 1$ . Then there exists  $C > 0$  independent of  $\varepsilon > 0$ ,  $p \in \partial\Omega$ , and  $\tau_1, \dots, \tau_l \in T_p \partial\Omega$ , such that*

$$\left| (\mathcal{D}_p)^l \left( \frac{\partial u_{\varepsilon,p}}{\partial N} \right) (\tau_1, \dots, \tau_l) \right|_{W_\varepsilon^{k,q}(\partial\Omega)} \leq C \left| \frac{\tau_1}{\varepsilon} \right| \cdots \left| \frac{\tau_l}{\varepsilon} \right|, \quad q \in [1, \infty)$$

where  $u_{\varepsilon,p}(x) = u\left(\frac{x-p}{\varepsilon}\right)$ .

*Proof.* Since  $u$  is radially symmetric, write  $u(x) = \theta(|x|)$ . Clearly,  $\theta(\rho)$  is even, smooth, and its derivatives up to the  $J$ -th order decay as  $O(e^{-\mu\rho})$ . We have

$$(7.16) \quad \frac{\partial u_{\varepsilon,p}}{\partial N}(x) = \frac{1}{\varepsilon} \theta' \left( \frac{|x-p|}{\varepsilon} \right) \frac{x-p}{|x-p|} \cdot N(x), \quad \text{for all } x \in \partial\Omega.$$

From

$$\left| \frac{x-p}{|x-p|} \cdot N(x) \right| \leq C|x-p| \quad \text{for } x \in \partial\Omega,$$

we obtain

$$\left| \frac{\partial u_{\varepsilon,p}}{\partial N} \right|_{W_\varepsilon^{0,q}(\partial\Omega)} \leq C.$$

Therefore, the lemma holds for  $l = 0$  and  $k = 0$ . To estimate  $\mathcal{D} \frac{\partial u_{\varepsilon,p}}{\partial N}$ , we take an arbitrary tangential vector field  $\tau(x)$  on  $\partial\Omega$  and compute

$$\begin{aligned} \nabla_{\varepsilon\tau} \left( \frac{\partial u_{\varepsilon,p}}{\partial N} \right) &= \theta'' \left( \frac{|x-p|}{\varepsilon} \right) \frac{(x-p) \cdot N(x)}{\varepsilon|x-p|} \frac{(x-p) \cdot \tau(x)}{|x-p|} \\ &\quad + \theta' \left( \frac{|x-p|}{\varepsilon} \right) \left( \frac{(x-p) \cdot \nabla_{\tau(x)} N(x)}{|x-p|} \right. \\ &\quad \left. - \frac{(x-p) \cdot N(x)}{|x-p|^2} \frac{(x-p) \cdot \tau(x)}{|x-p|} \right). \end{aligned}$$

Applying the same estimate to  $(x-p) \cdot N$  and noting  $|\nabla_{\tau} N| \leq C|\tau|$  with  $C$  being an upper bound of the second fundamental form of  $\partial\Omega$ , we derive the estimate for  $l = 0$  and  $k = 1$ . A similar calculation further shows that for any integer  $i \geq 0$  with  $2i + 1 \leq J$ ,

$$|\varepsilon^{2i} D^{2i+1} u_{\varepsilon,p}(N, \dots, N)|_{W_{\varepsilon}^{0,q}(\partial\Omega)} \leq C.$$

Using this estimate and the identity

$$\Delta_{\partial\Omega} g(x) = \Delta g(x) - H(x) \cdot \nabla g(x) - D^2 g(x)(N(x), N(x)) \quad \text{for } x \in \partial\Omega,$$

where  $\Delta_{\partial\Omega}$  is the Beltrami-Laplacian on  $\partial\Omega$  and  $H(x)$  is the mean curvature vector of  $\partial\Omega$ , we obtain the estimate

$$\left| \varepsilon^2 \Delta_{\partial\Omega} \frac{\partial u_{\varepsilon,p}}{\partial N} \right|_{W_{\varepsilon}^{0,q}(\partial\Omega)} \leq C \left( 1 + \left| \varepsilon^2 \frac{\partial \Delta u_{\varepsilon,p}}{\partial N} \right|_{W_{\varepsilon}^{0,q}(\partial\Omega)} \right).$$

Since  $\varepsilon^2 \frac{\partial \Delta u_{\varepsilon,p}}{\partial N} = \frac{\partial(\Delta u)_{\varepsilon,p}}{\partial N}$  and  $\Delta u$  is radially symmetric and satisfies the same decay property for derivatives up to the  $(J-2)$ -th order, the  $W_{\varepsilon}^{2,q}(\partial\Omega)$  estimate of  $\frac{\partial u_{\varepsilon,p}}{\partial N}$  follows immediately. The estimate of  $|\frac{\partial u_{\varepsilon,p}}{\partial N}|_{W_{\varepsilon}^{k,q}}$  with general  $k > 0$  follows in a similar manner inductively.

The estimate of  $(\mathcal{D}_p)^l(\frac{\partial u_{\varepsilon,p}}{\partial N})$  can be derived in the same fashion. As an illustration for  $l = 1$ , we obtain from differentiating (7.16)

$$\begin{aligned} & -\mathcal{D}_p \left( \frac{\partial u_{\varepsilon,p}}{\partial N} \right) (\varepsilon \tau_1)(x) \\ &= \theta'' \left( \frac{|x-p|}{\varepsilon} \right) \frac{(x-p) \cdot N(x)}{\varepsilon|x-p|} \frac{(x-p) \cdot \tau_1}{|x-p|} \\ &\quad + \theta' \left( \frac{|x-p|}{\varepsilon} \right) \left( \frac{\tau_1 \cdot N(x)}{|x-p|} - \frac{(x-p) \cdot N(x)}{|x-p|^2} \frac{(x-p) \cdot \tau_1}{|x-p|} \right) \end{aligned}$$

for any  $x \in \partial\Omega$ . Since

$$|\tau_1 \cdot N(x)| \leq C|\tau_1| |x-p|$$

the estimate of  $|\mathcal{D}_p(\frac{\partial u_{\varepsilon,p}}{\partial N})(\tau_1)|_{W_{\varepsilon}^{0,q}(\partial\Omega)}$  follows immediately. The estimates for the higher order Sobolev norms are obtained in a similar manner.  $\square$

Combining this lemma with the elliptic estimates, we obtain that, for  $q \in (1, \infty)$  and any integers  $k, l \geq 0$  with  $k + l \leq m + 2$ ,

$$(7.17) \quad \left| (\varepsilon \mathcal{D}_p)^l \psi_\varepsilon \right|_{L(\otimes^l(T_p \partial \Omega), W_\varepsilon^{k,q}(\Omega))} + \frac{1}{\varepsilon} \left| (\varepsilon \mathcal{D}_p)^l \left[ \mathcal{H} \left( \frac{\tilde{w}_{\varepsilon,p}}{\partial N} \right) \right] \right|_{L(\otimes^l(T_p \partial \Omega), W_\varepsilon^{k,q}(\Omega))} \leq C.$$

In particular, for  $l \leq m + 1$

$$(7.18) \quad \left| (\varepsilon \mathcal{D}_p)^l \psi_\varepsilon \right|_{L(\otimes^l(T_p \partial \Omega), X)} + \frac{1}{\varepsilon} \left| (\varepsilon \mathcal{D}_p)^l \left[ \mathcal{H} \left( \frac{\tilde{w}_{\varepsilon,p}}{\partial N} \right) \right] \right|_{L(\otimes^l(T_p \partial \Omega), X)} \leq C$$

and thus  $\psi_\varepsilon : \partial \Omega \rightarrow W_\varepsilon^{k,q}(\Omega)$ , for  $k \leq m + 1$  and  $q > 1$ , is a smooth imbedding. Moreover,  $\psi_\varepsilon : \partial \Omega \rightarrow M_\varepsilon \subset W_\varepsilon^{k,q}$  almost preserves the metric up to a scaling of  $\varepsilon$ . In fact, for any  $p \in \partial \Omega$  and  $\tau \in T_p \partial \Omega$ ,

$$\begin{aligned} |\varepsilon \mathcal{D}_\tau \tilde{w}_{\varepsilon,p}|_{W_\varepsilon^{k,q}(\Omega)} &= \left| \nabla w \left( \frac{x-p}{\varepsilon} \right) \cdot \tau \right|_{W_\varepsilon^{k,q}(\Omega)} \\ &= |\nabla w \cdot \tau|_{W^{k,q}(\Omega_{\varepsilon,p})} = \left( 2^{-\frac{1}{q}} |\partial_{x_1} w|_{W^{k,q}(\mathbb{R}^n)} + O(\varepsilon) \right) |\tau|, \end{aligned}$$

where  $\Omega_{\varepsilon,p} = \{ \frac{x-p}{\varepsilon} : x \in \Omega \}$ . The last step, obtaining  $|\partial_{x_1} w|_{W^{k,q}(\mathbb{R}^n)}$ , is based on the radial symmetry of  $w$  and a localization argument as in the proof of Lemma 7.2. Along with Lemma 7.3, this implies

$$(7.19) \quad \left| |\varepsilon \mathcal{D}_\tau \psi_\varepsilon(p)|_{W_\varepsilon^{k,q}(\Omega)} - 2^{-\frac{1}{q}} |\partial_{x_1} w|_{W^{k,q}(\mathbb{R}^n)} |\tau| \right| \leq C\varepsilon |\tau|, \\ 0 \leq k \leq m + 1, \quad q \in (1, \infty).$$

**$M_\varepsilon$  is approximately stationary.** From the construction of  $W_{\varepsilon,p} = \psi_\varepsilon(p)$ , the flow there is expected to be almost stationary. In fact, we have

**Lemma 7.4.** *There exists  $C > 0$  such that, for any sufficiently small  $\varepsilon > 0$  and  $p \in \partial \Omega$ , the solution  $u(t, x)$  of (7.10) with initial data  $u(0, x) = W_{\varepsilon,p}(x)$  satisfies*

$$|u(t, \cdot) - W_{\varepsilon,p}|_X \leq C\varepsilon e^{Ct}.$$

*Proof.* Let  $v(t, x) = u(t, x) - W_{\varepsilon,p}(x)$ . Clearly,  $v(0, \cdot) = 0$  and

$$(7.20) \quad \begin{cases} v_t = L_\varepsilon v + \tilde{f} \left( \tilde{w}_{\varepsilon,p} - \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) + v \right) - \tilde{f}(\tilde{w}_{\varepsilon,p}) \equiv L_\varepsilon v + g(v), \\ \frac{\partial v}{\partial N} \Big|_{\partial \Omega} = 0, \quad v(t, 0) = 0, \end{cases}$$

where we recall  $L_\varepsilon = \varepsilon^2 \Delta - 1$ . For  $q > 1$ , Lemma 7.3 implies

$$|g(v)|_{W_\varepsilon^{0,q}(\Omega)} \leq C(\varepsilon + |v|_{W_\varepsilon^{0,q}(\Omega)}).$$

The desired estimate on  $v = u - W_{\varepsilon,p}$  follows from (7.8).  $\square$

*Remark 7.5.* Assumption (2) in Definition 2.1 of approximate invariant manifolds is satisfied automatically due to the fact that  $M_\varepsilon$  is compact.

**Splitting along the manifold  $M_\varepsilon$ .** Let  $v_0(x)$ , with  $|v_0|_{L^2(\mathbb{R}^n)} = 1$ , be the first eigenfunction (corresponding to the eigenvalue  $\lambda_1$ ) of the linearized operator  $L_0$ , defined prior to the Assumptions (F1)–(F3). It is well known that  $v_0$  is also radially symmetric and decays exponentially in  $x$  much as  $w$  in (F2). By our assumption that the unstable subspace of  $L_0$  is  $\text{span}\{v_0\}$  and the center subspace of  $L_0$  is  $\text{span}\{\partial_{x_1} w, \dots, \partial_{x_n} w\}$ , we will use  $v_0$  and  $T_{\psi_\varepsilon(p)} M_\varepsilon \sim \text{span}\{\partial_\tau w\}$ , where  $\tau \in T_p \partial\Omega$ , to construct the unstable and center subspaces along  $M_\varepsilon$ , respectively.

For any  $p \in \partial\Omega$ , let

$$\tilde{v}_{\varepsilon,p}(x) = v_0\left(\frac{x-p}{\varepsilon}\right) \quad \text{and} \quad \bar{v}_{\varepsilon,p} = \tilde{v}_{\varepsilon,p} - \mathcal{H}\left(\frac{\partial \tilde{v}_{\varepsilon,p}}{\partial N}\right).$$

For any  $\tau \in T_p \partial\Omega$ , let

$$a_{\varepsilon,p}(\tau) = \int_{\Omega} v_0\left(\frac{x-p}{\varepsilon}\right) \nabla w\left(\frac{x-p}{\varepsilon}\right) \cdot \tau \varepsilon^{-n} dx.$$

Since  $\int_{x_n > 0} v_0 \partial_{x_j} w dx = 0$ ,  $1 \leq j \leq n-1$ , we obtain from a rescaling argument that  $|a_{\varepsilon,p}(\tau)| \leq C\varepsilon$ . Here  $a_{\varepsilon,p}$  can be viewed as a linear functional on  $T_p \partial\Omega$ . Moreover, it is not hard to see that

$$|(\varepsilon \mathcal{D}_p)^l a_{\varepsilon,p}| \leq C\varepsilon, \quad 0 \leq l \leq m+1.$$

In fact, for any  $\tau_1 \in T_p \partial\Omega$ ,

$$\begin{aligned} (\mathcal{D}_{\varepsilon\tau_1} a_{\varepsilon,p})(\tau) &= - \int_{\Omega} \tau_1 \cdot \nabla \left\{ v_0\left(\frac{x-p}{\varepsilon}\right) \nabla w\left(\frac{x-p}{\varepsilon}\right) \cdot \tau \right\} \varepsilon^{1-n} dx \\ &= - \int_{\partial\Omega} \left\{ v_0\left(\frac{x-p}{\varepsilon}\right) \nabla w\left(\frac{x-p}{\varepsilon}\right) \cdot \tau \right\} N \cdot \tau_1 \varepsilon^{1-n} dS \end{aligned}$$

and then the desired inequality for  $m = 1$  follows from  $N \cdot \tau_1 = O(|x-p||\tau_1|)$ . Simply repeating this process yields the inequality for general  $m$ .

From Lemma 7.3, the correction term  $\mathcal{H}\left(\frac{\partial \tilde{v}_{\varepsilon,p}}{\partial N}\right)$  is smooth in  $p$  and satisfies

$$\left| (\varepsilon \mathcal{D}_p)^l \mathcal{H}\left(\frac{\partial \tilde{v}_{\varepsilon,p}}{\partial N}\right) \right|_{L(\otimes^m(T_p \partial\Omega), W_\varepsilon^{k,q}(\Omega))} \leq C\varepsilon, \quad 0 \leq k+l \leq m+1, \quad q > 1,$$

which along with the estimate for  $a_{\varepsilon,p}$  implies that  $\bar{v}_{\varepsilon,p}$  is smooth in  $p$  and almost perpendicular to  $T_{W_{\varepsilon,p}}M_{\varepsilon}$ . Let  $V_{\varepsilon,p}$  be the normalization of the part of  $\bar{v}_{\varepsilon,p}$  orthogonal (in the sense of  $W_{\varepsilon}^{0,2}(\Omega)$ ) to  $T_{W_{\varepsilon,p}}M_{\varepsilon}$ , i.e.,  $|V_{\varepsilon,p}|_{W_{\varepsilon}^{0,2}(\Omega)} = 1$  and for some scalar  $b_{\varepsilon,p}$ ,

$$V_{\varepsilon,p} - b_{\varepsilon,p}\bar{v}_{\varepsilon,p} \in T_{W_{\varepsilon,p}}M_{\varepsilon}, \quad \int_{\Omega} V_{\varepsilon,p} \mathcal{D}_{\varepsilon\tau} \psi_{\varepsilon}(p) \varepsilon^{-n} dx = 0, \\ \text{for all } \tau \in T_p \partial\Omega.$$

From the above property of  $a_{\varepsilon,p}$ , it is clear that

$$b_{\varepsilon,p} = 2 + O(\varepsilon), \quad |(\varepsilon \mathcal{D}_p)^l b_{\varepsilon,p}| \leq C\varepsilon, \quad 0 \leq l \leq m+1.$$

Therefore, for  $q \geq 1$  and integers  $k+l \leq m+1$ ,

$$(7.21) \quad |(\varepsilon \mathcal{D}_p)^l V_{\varepsilon,p}|_{L(\otimes^l(T_p \partial\Omega), W_{\varepsilon}^{k,q}(\Omega))} \leq C \quad \text{and} \\ |(\varepsilon \mathcal{D}_p)^l (V_{\varepsilon,p} - \bar{v}_{\varepsilon,p})|_{L(\otimes^l(T_p \partial\Omega), W_{\varepsilon}^{k,q}(\Omega))} \leq C\varepsilon.$$

Define

$$X_{\varepsilon,p}^u = \text{span}\{V_{\varepsilon,p}\}, \quad X_{\varepsilon,p}^c = T_{W_{\varepsilon,p}}M_{\varepsilon} = \mathcal{D}\psi_{\varepsilon}(p)T_p\partial\Omega, \quad \text{and} \\ X_{\varepsilon,p}^s = \{v \in X : \int_{\Omega} v \tilde{v} dx = 0, \text{ for all } \tilde{v} \in X_{\varepsilon,p}^c \oplus X_{\varepsilon,p}^u\}.$$

Let  $\Pi_{\varepsilon,p}^{\alpha}$ ,  $\alpha = u, s, c$ , be the projections associated with this splitting. Since  $X_{\varepsilon,p}^{u,c}$  are finite dimensional with bases consisting of functions in  $W_{\varepsilon}^{0,q}(\Omega)$  for any  $q \geq 1$ , these projections can be extended to  $W_{\varepsilon}^{0,q}(\Omega)$  for any  $q > 1$ . These are orthogonal projections on  $W_{\varepsilon}^{0,2}(\Omega) = L^2(\Omega, \varepsilon^{-n} d\mu)$ . From the smoothness of  $\psi_{\varepsilon}$  and  $V_{\varepsilon,p}$  in  $p$ , it is clear that there exists  $B > 0$  independent of sufficiently small  $\varepsilon > 0$  such that

$$(7.22) \quad |\Pi_{\varepsilon,p}^{\alpha}|_{C^m((\partial\Omega, \frac{1}{\varepsilon^2} \langle \cdot, \cdot \rangle), L(X))}, \quad |\Pi_{\varepsilon,p}^{\alpha}|_{C^{m+1}((\partial\Omega, \frac{1}{\varepsilon^2} \langle \cdot, \cdot \rangle), L(W_{\varepsilon}^{0,q}(\Omega)))} \leq B, \\ \alpha = u, s, c,$$

where  $(\partial\Omega, \frac{1}{\varepsilon^2} \langle \cdot, \cdot \rangle)$  denotes the Riemannian manifold  $\partial\Omega$  with the inner product scaled by  $\frac{1}{\varepsilon^2}$ .

**$M_{\varepsilon}$  is approximately normally hyperbolic.** In order to prove that the above splitting along  $M_{\varepsilon}$  is approximately normally hyperbolic, we start with the estimate of the linearization of (7.10). Fix  $p \in \partial\Omega$  and let  $u(t, x)$  be the solution of (7.10) with initial value  $u(0, \cdot) = \psi_{\varepsilon}(p) = W_{\varepsilon,p}$ . Let  $\mathbf{w}(t, x)$  be the solution of the linearized equation

$$(7.23) \quad \mathbf{w}_t = L_{\varepsilon} \mathbf{w} + \tilde{f}'(u(t, \cdot)) \mathbf{w} \quad \text{with } \mathbf{w}(0, \cdot) \in X.$$



It follows from (7.8) and (7.11) that

$$(7.24) \quad |\mathbf{w}(t, \cdot)|_X \leq C e^{Ct} |\mathbf{w}(0, \cdot)|_X, \\ |\mathbf{w}(t, \cdot)|_{W_\varepsilon^{0,q}(\Omega)} \leq C e^{Ct} |\mathbf{w}(0, \cdot)|_{W_\varepsilon^{0,q}(\Omega)}, \quad q \in (1, \infty).$$

Let

$$\bar{L}_{\varepsilon,p} = L_\varepsilon + f'(\tilde{w}_{\varepsilon,p}) = L_\varepsilon + \tilde{f}'(\tilde{w}_{\varepsilon,p}) \quad \text{and} \quad \bar{w}(t, \cdot) = e^{t\bar{L}_{\varepsilon,p}} w(0, \cdot).$$

By writing  $\mathbf{w}$  using the variation of constants formula

$$\mathbf{w}(t, \cdot) = \bar{w}(t, \cdot) + \int_0^t e^{(t-s)\bar{L}_{\varepsilon,p}} (\tilde{f}'(u(s, \cdot)) - \tilde{f}'(\tilde{w}_{\varepsilon,p})) \mathbf{w}(s, \cdot) ds$$

and applying Lemmas 7.3 and 7.4, we obtain a refined estimate for  $q \in (1, \infty)$

$$(7.25) \quad |\mathbf{w}(t) - \bar{w}(t)|_X \leq C \varepsilon e^{Ct} |\mathbf{w}(0)|_{W_\varepsilon^{0,q}(\Omega)}, \quad \text{if } m > 1.$$

If  $m = 1$ , then for any  $\xi > 0$ , there exists  $\varepsilon_0 > 0$  such that

$$(7.26) \quad |\mathbf{w}(t) - \bar{w}(t)|_X \leq C \xi e^{Ct} |\mathbf{w}(0)|_{W_\varepsilon^{0,q}(\Omega)} \quad \forall \varepsilon < \varepsilon_0.$$

To study the behavior of  $\bar{w}(t, \cdot)$  in each of the three directions, we first consider the action of  $\bar{L}_{\varepsilon,p}$  in these directions. Firstly, for any  $\tilde{\tau} \in T_p \partial \Omega$ , since  $\partial w$  is in the kernel of  $L_0$ , we have

$$\bar{L}_{\varepsilon,p} \mathcal{D}\psi_\varepsilon(p)(\varepsilon \tilde{\tau}) = -f'(\tilde{w}_{\varepsilon,p}) \mathcal{H} \left( \mathcal{D}_{\varepsilon \tilde{\tau}} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right).$$

From Lemma 7.3, we obtain

$$(7.27) \quad |\bar{L}_{\varepsilon,p} \mathcal{D}\psi_\varepsilon(p)(\varepsilon \tilde{\tau})|_{W_\varepsilon^{0,q}(\Omega)} \leq C \varepsilon |\tilde{\tau}| \quad \text{for all } q > 1.$$

Secondly, from inequality (7.21) and the fact that  $v_0$  is an eigenvector of  $L_0$  with the eigenvalue  $\lambda_1 > 0$ , we have

$$(7.28) \quad |(\bar{L}_{\varepsilon,p} - \lambda_1) V_{\varepsilon,p}|_{W_\varepsilon^{0,q}(\Omega)} \leq C \varepsilon \quad \text{for all } q > 1.$$

Finally, for any  $w^s \in X_{\varepsilon,p}^s \cap D(\bar{L}_\varepsilon)$ , taking the  $L^2(\Omega, \varepsilon^{-n} d\mu)$  inner product of  $\bar{L}_{\varepsilon,p} w^s$  with  $V_{\varepsilon,p}$  and  $\mathcal{D}\psi_\varepsilon(p)(\varepsilon \tau)$  and using the orthogonality of the decomposition, inequalities (7.27) and (7.28), and the fact that  $\bar{L}_{\varepsilon,p}$  is self-adjoint in  $L^2(\Omega, \varepsilon^{-n} d\mu)$ , we obtain for any  $q > 1$

$$(7.29) \quad |\Pi_{\varepsilon,p}^u \bar{L}_{\varepsilon,p} w^s|_X + |\Pi_{\varepsilon,p}^c \bar{L}_{\varepsilon,p} w^s|_X \leq C \varepsilon |w^s|_{W_\varepsilon^{0,q}(\Omega)}.$$

Note that here we use the finite-dimensionality of the center and unstable subspaces, where all norms are equivalent.

Coming back to  $\bar{w}(t, \cdot)$ , we look at its decomposition

$$(7.30) \quad \bar{w}(t, \cdot) = a(t)V_{\varepsilon,p} + \mathcal{D}\psi_\varepsilon(p)(\varepsilon\tau(t)) + w^s(t).$$

From the uniform bound on  $\Pi_{\varepsilon,p}^\alpha$ ,  $\alpha = u, c, s$ , and inequality (7.8), we have

$$(7.31) \quad |a(t)|, |\tau(t)|, |w^s(t)|_{W_\varepsilon^{0,q}(\Omega)} \leq Ce^{Ct}|\mathbf{w}(0)|_{W_\varepsilon^{0,q}(\Omega)} \quad \text{for all } q > 1$$

and

$$(7.32) \quad |w^s(t)|_X \leq Ce^{Ct}|\mathbf{w}(0)|_X.$$

Taking the  $L^2(\Omega, \varepsilon^{-n}d\mu)$  inner product of (7.30) with  $\bar{L}_{\varepsilon,p}\mathcal{D}\psi_\varepsilon(p)(\varepsilon\tau_t(t))$  and  $\bar{L}_{\varepsilon,p}V_{\varepsilon,p}$  and using inequalities (7.27), (7.28), and (7.31), and the facts that  $\bar{L}_{\varepsilon,p}$  is self-adjoint and  $\bar{L}_{\varepsilon,p}\bar{w} = \bar{w}_t$ , we obtain for any  $q > 1$

$$(7.33) \quad |a_t - \lambda_1 a| + |\tau_t| \leq Ce^{Ct}\varepsilon|\mathbf{w}(0)|_{W_\varepsilon^{0,q}(\Omega)}.$$

Finally, we estimate the evolution of  $w^s(t)$ , the stable component of  $\bar{w}(t)$ . Since we only made assumptions on the spectrum of the operator

$$L_0 = \Delta - 1 + f'(w) \quad \text{on } \mathbb{R}^n,$$

our strategy is to extend the domain of  $w^s(t)$  to  $\mathbb{R}^n$  in an appropriate way and to estimate its evolution. For  $q > 1$ , let  $\tilde{\Pi}^s$  be the spectral projection operator to the eigenspace  $X^s \subset L^q(\mathbb{R}^n)$  of  $L_0$  corresponding to the subset  $\sigma(L_0) \setminus \{\lambda_1, 0\} \subset (-\infty, -b)$ . Let  $S$  be the operator defined by  $(S\phi)(\tilde{x}) = \phi(p + \varepsilon\tilde{x})$  and define the modified ‘extension’ operator

$$\Lambda_E = \tilde{\Pi}^s S E : \tilde{X}_{\varepsilon,p}^s \rightarrow X^s$$

where  $E$  is defined in Lemma 7.2 for a  $\delta > 0$  to be determined and

$$\tilde{X}_{\varepsilon,p}^s = \left\{ \gamma \in W_\varepsilon^{0,q}(\Omega) : \int_\Omega \gamma v dx = 0, \text{ for all } v \in X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^c \right\}.$$

Obviously  $\Lambda_E$  can not be an isomorphism. However, we will define a left inverse of  $\Lambda_E$ , i.e., a modified ‘restriction’ operator  $\Lambda_R$ . Since, for any  $\gamma : \Omega \rightarrow \mathbb{R}$ ,

$$|\gamma|_{W_\varepsilon^{0,q}(\Omega)} \leq |SE\gamma|_{L^q(\mathbb{R}^n)} \leq C|\gamma|_{W_\varepsilon^{0,q}(\Omega)},$$

$SE(\tilde{X}_{\varepsilon,p}^s)$  is a closed subspace of  $L^q(\mathbb{R}^n)$ . For  $\gamma \in \tilde{X}_{\varepsilon,p}^s$ , write

$$SE\gamma = \gamma^s + \gamma^{cu} \quad \text{where } \gamma^s = \tilde{\Pi}^s SE\gamma.$$

By taking the inner product of  $SE\gamma$  with  $v_0$  and  $\partial_\tau w$  and using Lemmas 7.2 and 7.3, we obtain

$$|\gamma^{cu}|_{W_\varepsilon^{k,q}(\mathbb{R}^n)} \leq C\varepsilon|\gamma|_{W_\varepsilon^{0,q}(\Omega)}, \quad 0 \leq k \leq m+1.$$

Therefore,

$$\frac{1}{2}|SE\gamma|_{L^q(\mathbb{R}^n)} \leq |\tilde{\Pi}^s SE\gamma|_{L^q(\mathbb{R}^n)} \leq 2|SE\gamma|_{L^q(\mathbb{R}^n)}$$

and

$$A : \tilde{\Pi}^s SE\tilde{X}_{\varepsilon,p}^s \rightarrow X^{cu} = (I - \tilde{\Pi}^s)L^q(\mathbb{R}^n), \quad \text{as } A(\gamma^s) = \gamma^{cu}$$

is a bounded operator with  $|A|_{L(\tilde{\Pi}^s SE\tilde{X}_{\varepsilon,p}^s, X^{cu})} \leq C\varepsilon$ . By the Hahn–Banach theorem,  $A$  can be extended to an operator on  $X^s$  with the same bound. From the construction,  $A$  satisfies that, if  $\gamma^s = \tilde{\Pi}^s SE\gamma$  with  $\gamma \in \tilde{X}_{\varepsilon,p}^s$ , then  $(I + A)\gamma^s = SE\gamma$ . Let  $\Lambda_R : X^s \rightarrow \tilde{X}_{\varepsilon,p}^s$  be given by

$$\Lambda_R \gamma^s = (S^{-1}(I + A)\gamma^s)|_{\Omega}.$$

Clearly,  $\Lambda_R \Lambda_E = I$  and  $|\Lambda_R|_{L(X^s, W_{\varepsilon}^{0,q}(\Omega))}, |\Lambda_E|_{L(\tilde{X}_{\varepsilon,p}^s, L^q(\mathbb{R}^n))}$ , and

$$\inf \left\{ |\Lambda_E \gamma|_{L^q(\mathbb{R}^n)} : \gamma \in \tilde{X}_{\varepsilon,p}^s, |\gamma|_{W_{\varepsilon}^{0,q}(\Omega)} = 1 \right\}^{-1}$$

are bounded uniformly in sufficiently small  $\varepsilon > 0$ .

Let

$$\tilde{w}^s(t) = \Lambda_E w^s(t) = \Lambda_E \Pi_{\varepsilon,p}^s \tilde{w}(t).$$

Since  $\frac{\partial w^s}{\partial N} = 0$  and  $w^s = \Lambda_R \tilde{w}^s$ , we have

$$w^s = w^s - \mathcal{H} \left( \frac{\partial w^s}{\partial N} \right) = \Lambda_R \tilde{w}^s - \mathcal{H} \left( \frac{\partial}{\partial N} (\Lambda_R \tilde{w}^s) \right).$$

From the definition of  $w^s(t)$  and  $\tilde{w}^s(t)$ , we have

$$\begin{aligned} \tilde{w}_t^s &= L_0 \tilde{w}^s + (\Lambda_E \Pi_{\varepsilon,p}^s \bar{L}_{\varepsilon,p} - L_0 \Lambda_E) \left( \Lambda_R \tilde{w}^s - \mathcal{H} \left( \frac{\partial}{\partial N} (\Lambda_R \tilde{w}^s) \right) \right) \\ &\quad + \Lambda_E \Pi_{\varepsilon,p}^s \bar{L}_{\varepsilon,p} (\tilde{w} - w^s), \end{aligned}$$

where  $\mathcal{H}$  is defined in (7.13). On the one hand, due to inequalities (7.27), (7.28), and (7.31),

$$|\Lambda_E \Pi_{\varepsilon,p}^s \bar{L}_{\varepsilon,p} (\tilde{w} - w^s)|_{W_{\varepsilon}^{0,q}(\mathbb{R}^n)} \leq C e^{Ct} \varepsilon |\mathbf{w}(0)|_{W_{\varepsilon}^{0,q}(\Omega)}.$$

On the other hand, for any  $\gamma \in W_{\varepsilon}^{2,q}(\Omega)$  with  $\frac{\partial \gamma}{\partial N}|_{\partial\Omega} = 0$ ,

$$\begin{aligned} L_0 \Lambda_E \gamma - \Lambda_E \bar{L}_{\varepsilon,p} \gamma &= \varepsilon^2 \tilde{\Pi}^s S(\Delta E \gamma - E \Delta \gamma) \\ &\quad + \tilde{\Pi}^s S(f(\tilde{w}_{\varepsilon,p}) E \gamma - E(f(\tilde{w}_{\varepsilon,p}) \gamma)). \end{aligned}$$

Therefore, Lemma 7.2 and the above estimates on  $\Lambda_E$  and  $\Lambda_R$  imply

$$|L_0 \Lambda_E \gamma - \Lambda_E \bar{L}_{\varepsilon,p} \gamma|_{L^q(\mathbb{R}^n)} \leq C \delta |\gamma|_{W_{\varepsilon}^{2,q}(\Omega)},$$

for some  $C > 0$  independent of  $\varepsilon$  and  $\delta$ . Since  $\Lambda_R \tilde{w}^s - \mathcal{H}(\frac{\partial}{\partial N}(\Lambda_R \tilde{w}^s))$  satisfies the boundary condition of  $L_\varepsilon$  if  $\tilde{w}^s \in W^{2,q}(\mathbb{R}^n)$ , we have

$$\begin{aligned} & \left| \tilde{\Pi}^s(L_0 \Lambda_E - \Lambda_E \Pi_{\varepsilon,p}^s \bar{L}_{\varepsilon,p}) \left( \Lambda_R \tilde{w}^s - \mathcal{H} \left( \frac{\partial}{\partial N} (\Lambda_R \tilde{w}^s) \right) \right) \right|_{L^q(\mathbb{R}^n)} \\ & \leq C\delta |\Lambda_R \tilde{w}^s|_{W_{\varepsilon}^{2,q}(\Omega)} \leq C\delta |\tilde{w}^s|_{W^{2,q}(\mathbb{R}^n)}. \end{aligned}$$

By our Assumption (F3) on the spectrum of  $L_0$ , standard perturbation theory of operators, and the above estimates, we obtain

$$\begin{aligned} (7.34) \quad |w^s(t)|_X & \leq C(|\tilde{w}^s(t)|_{W^{1,q_0}(\mathbb{R}^n)} + |\tilde{w}^s(t)|_{W^{1,2}(\mathbb{R}^n)}) \\ & \leq C e^{-bt} |w^s(0)|_X + C\varepsilon e^{Ct} (|\mathbf{w}(0)|_{W_{\varepsilon}^{0,2}(\Omega)} + |\mathbf{w}(0)|_{W_{\varepsilon}^{0,q_0}(\Omega)}). \end{aligned}$$

The combination of inequalities (7.25)–(7.29), (7.33), and (7.34) implies

$$(7.35) \quad |a(t) - e^{\lambda_1 t} a(0)| + |\tau(t)| \leq C\varepsilon e^{Ct} (|\mathbf{w}|_{W_{\varepsilon}^{0,2}(\Omega)} + |\mathbf{w}|_{W_{\varepsilon}^{0,q_0}(\Omega)}) \quad \text{if } m > 1.$$

As a consequence, for  $\alpha = u, s, c$  and  $w \in X$  with  $\Pi_{\varepsilon,p}^\alpha w = 0$ , we have

$$(7.36) \quad |\Pi_{\varepsilon,p}^\alpha D T_\varepsilon'(\psi_\varepsilon(p)) \mathbf{w}|_X \leq C\varepsilon e^{Ct} (|\mathbf{w}|_{W_{\varepsilon}^{0,2}(\Omega)} + |\mathbf{w}|_{W_{\varepsilon}^{0,q_0}(\Omega)}) \quad \text{if } m > 1.$$

If  $m = 1$ , then for any  $\xi > 0$ , there exists  $\varepsilon_0 > 0$  such that

$$(7.37) \quad \begin{aligned} & |a(t) - e^{\lambda_1 t} a(0)| + |\tau(t)|, |\Pi_{\varepsilon,p}^\alpha D T_\varepsilon'(\psi_\varepsilon(p)) \mathbf{w}|_X \\ & \leq C\xi e^{Ct} (|\mathbf{w}|_{W_{\varepsilon}^{0,2}(\Omega)} + |\mathbf{w}|_{W_{\varepsilon}^{0,q_0}(\Omega)}), \quad \forall \varepsilon < \varepsilon_0. \end{aligned}$$

Therefore, the splitting  $X = X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^c \oplus X_{\varepsilon,p}^s$  is approximately invariant and approximately hyperbolic for the time- $t_0$  map  $T_\varepsilon^{t_0}$  of the semiflow defined by the parabolic equation (7.10) for some  $t_0 > 1$  chosen large enough but independent of sufficiently small  $\varepsilon$ .

**Dynamical peak solutions of (7.1).** We use the above construction of the approximately normally hyperbolic invariant manifold  $M_\varepsilon$ , the splitting

$$X = W_\varepsilon^{1,q_0}(\Omega) \cap W_\varepsilon^{1,2}(\Omega) = X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^c \oplus X_{\varepsilon,p}^s,$$

and their related estimates. Apply Theorems 2.2, 2.4, 4.2, 6.3, and 6.5, for sufficiently small  $\varepsilon > 0$ , to the time- $t_0$  map  $T_\varepsilon^{t_0}$  of the semiflow defined by the parabolic equation (7.10) for some  $t_0 > 1$  chosen large enough but independent of sufficiently small  $\varepsilon$ . They imply the following

- For the map  $T_\varepsilon^{t_0}$ , there exists a unique  $C^m$  normally hyperbolic invariant manifold  $M_\varepsilon^* = \Psi_\varepsilon(\partial\Omega) \subset X$ , where  $\Psi_\varepsilon \in C^m(\partial\Omega, X)$  satisfies  $\Psi_\varepsilon(p) - \psi_\varepsilon(p) \in X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^s$ .

- There exist unique  $C^m$  invariant center unstable manifold  $W_\varepsilon^{cu*}$  and center stable manifold  $W_\varepsilon^{cs*}$  of  $M_\varepsilon^*$ .
- There exists invariant foliations on  $W_\varepsilon^{cu*}$  and  $W_\varepsilon^{cs*}$  by  $C^m$  unstable and stable fibers, respectively.
- $M_\varepsilon^*$  is independent of  $q_0 > n$  due to the uniqueness of the invariant manifolds characterized by statements such as Proposition 4.10. Moreover, since the backward flow is well-defined on  $W_\varepsilon^{cu*}$ , for any  $u \in W_\varepsilon^{cu*}$ , we have  $u \in W_\varepsilon^{2,q}(\Omega)$  for all  $q \geq 2$ , and  $\frac{\partial u}{\partial N}|_{\partial\Omega} = 0$ .
- From Theorem 4.2 and its parallel version for the center unstable manifold, there exist  $\delta > 0$  and  $C > 0$  independent of  $\varepsilon > 0$  such that at any  $p \in \partial\Omega$ , in the frame given by  $X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^c \oplus X_{\varepsilon,p}^s$ , the manifolds  $M_\varepsilon^*$ ,  $W_\varepsilon^{cu*}$ , and  $W_\varepsilon^{cs*}$  can be written as graphs of appropriate  $C^m$  mappings whose  $C^m$  norms are bounded by  $C$  and whose domains contain balls of radius  $\delta$ . In particular  $\Psi_\varepsilon$  satisfies  $|\Psi_\varepsilon - \psi_\varepsilon|_{C^0(\partial\Omega, X)} \leq C\varepsilon$ . Moreover, by choosing  $\mu$  in Theorem 4.2 sufficiently small, we obtain  $|\Psi_\varepsilon - \psi_\varepsilon|_{C^1((\partial\Omega, \frac{1}{\varepsilon}(\cdot, \cdot)), X)} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , where  $(\partial\Omega, \frac{1}{\varepsilon}(\cdot, \cdot))$  indicates that the metric on  $\partial\Omega$  should be scaled by  $\frac{1}{\varepsilon}$ .

In order to demonstrate that  $M_\varepsilon^*$  is invariant under the semiflow  $T_\varepsilon^t$  generated by (7.10), we have to verify the type of weak uniform continuity condition required in (H5) given in Subsect. 6.1. This will be done for  $k = 1$  in (H5). In fact, we will prove that

- for any sufficiently small  $\varepsilon > 0$  and a bounded set  $\Gamma \subset X$ , given any  $\xi > 0$ , there exists  $\zeta > 0$  so that  $|T_\varepsilon^t(u) - T_\varepsilon^{t_0}(u)|_X \leq \xi$  for any  $t \in [t_0, t_0 + \zeta]$  and  $u \in \Gamma$ .

It is important to notice that  $\zeta > 0$  does not have to be independent of  $\varepsilon$ . In order to prove this statement, one first notices from the cut-off on  $f$  given in (7.10), the variation of constants formula, and the smoothing effect of  $e^{tL_\varepsilon}$ , that  $(-L_\varepsilon)^{\frac{1}{2}}T_\varepsilon^{t_0}(\Gamma) \subset X$  is bounded. For  $t \in [t_0, t_0 + \zeta]$ , from

$$T_\varepsilon^t(u) - T_\varepsilon^{t_0}(u) = \int_{t_0}^t [e^{(t-s)L_\varepsilon} L_\varepsilon T_\varepsilon^{t_0}(u) + e^{(t-s)L_\varepsilon} \tilde{f}(T_\varepsilon^s(u))] ds$$

and the same smoothing effect of  $e^{tL_\varepsilon}$  again, the above statement follows immediately. Therefore, from Theorem 6.1 and Remark 6.6, the manifold  $M_\varepsilon^*$  and its center unstable manifold, center stable manifold, unstable foliations, and stable foliations are all locally invariant under (7.10). Moreover, since  $M_\varepsilon^*$  is in an  $O(\varepsilon)$  neighborhood of  $M_\varepsilon$  in  $W_\varepsilon^{1,q_0}(\Omega)$  with  $q_0 > n$ , (7.10) is equivalent to the original equation (7.1) in this region. Therefore, these structures are locally invariant under (7.1) in the sense of Theorem 6.1 as well.

Qualitatively,  $M_\varepsilon^*$  consists of functions each of which achieves its maximum close to  $\partial\Omega$  (in the last subsection we show that the peak actually lies on  $\partial\Omega$ ) and decays exponentially in  $x$ . Dynamically,  $M_\varepsilon^*$  is almost stationary for the evolution governed by (7.1). Therefore, the manifold  $M_\varepsilon^*$  represents a collection of special solutions each with a single spike on  $\partial\Omega$  which moves slowly along  $\partial\Omega$  for all  $t \in \mathbb{R}$ .

**7.2. Dynamics on  $M_\varepsilon^*$ .** Due to the invariance of  $M_\varepsilon^*$  under (7.1), this parabolic equation defines a tangent vector field on the  $(n-1)$ -dimensional manifold  $M_\varepsilon^*$ . For any  $p \in \partial\Omega$ , there exists  $\tau_\varepsilon(p) \in T_p\partial\Omega$  such that

$$(7.38) \quad \mathcal{D}\Psi_\varepsilon(p)(\varepsilon\tau_\varepsilon(p)) = \varepsilon^2\Delta\Psi_\varepsilon(p) - \Psi_\varepsilon(p) + f(\Psi_\varepsilon(p)).$$

Let

$$\tilde{\psi}_\varepsilon(p) = \Psi_\varepsilon(p) - \psi_\varepsilon(p) \in X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^s,$$

which satisfies

$$(7.39) \quad |\tilde{\psi}_\varepsilon|_{C^0(\partial\Omega, X)} \leq C\varepsilon, \quad \lim_{\varepsilon \rightarrow 0} |\tilde{\psi}_\varepsilon|_{C^1((\partial\Omega, \frac{1}{\varepsilon}(\cdot, \cdot)), X)} = 0.$$

Since

$$\varepsilon^2\Delta\tilde{w}_{\varepsilon,p} - \tilde{w}_{\varepsilon,p} + f(\tilde{w}_{\varepsilon,p}) = 0,$$

(7.38) can be rewritten as

$$(7.40) \quad \mathcal{D}\Psi_\varepsilon(p)(\varepsilon\tau_\varepsilon(p)) = \bar{L}_{\varepsilon,p}\tilde{\psi}_\varepsilon(p) - f'(\tilde{w}_{\varepsilon,p})\mathcal{H}\left(\frac{\partial\tilde{w}_{\varepsilon,p}}{\partial N}\right) + g$$

where

$$\begin{aligned} g = & f\left(\tilde{w}_{\varepsilon,p} - \mathcal{H}\left(\frac{\partial\tilde{w}_{\varepsilon,p}}{\partial N}\right) + \tilde{\psi}_\varepsilon(p)\right) \\ & - f(\tilde{w}_{\varepsilon,p}) - f'(\tilde{w}_{\varepsilon,p})\left(\tilde{\psi}_\varepsilon(p) - \mathcal{H}\left(\frac{\partial\tilde{w}_{\varepsilon,p}}{\partial N}\right)\right). \end{aligned}$$

The remainder term  $g$  can be estimated by using Lemma 7.3 and (7.39) for  $q \geq 2$ , giving

$$(7.41) \quad |g|_{W_\varepsilon^{0,q}(\Omega)} \leq C\varepsilon^2 \quad \text{if } m > 1$$

and if  $m = 1$ , then for any  $\xi > 0$ , there exists  $\varepsilon_0 > 0$  such that

$$(7.42) \quad |g|_{W_\varepsilon^{0,q}(\Omega)} \leq C\xi\varepsilon \quad \forall \varepsilon < \varepsilon_0.$$

Moreover, if  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ , then

$$(7.43) \quad |g|_{W_\varepsilon^{0,q}(\Omega)} \leq C\varepsilon^{1+\beta}.$$

The estimate on  $\tau_\varepsilon$  will be based on (7.40).

To derive an equation for  $\tilde{\psi}$  not involving  $\tau_\varepsilon$ , let

$$(X_{\varepsilon,p}^{q,c})^\perp = \left\{ w \in W_\varepsilon^{0,q}(\Omega) : \int_\Omega w v dx = 0 \quad \forall v \in X_{\varepsilon,p}^c \right\}.$$

Consider the decomposition  $W_\varepsilon^{0,q}(\Omega) = (X_{\varepsilon,p}^{q,c})^\perp \oplus D\Psi_\varepsilon(p)(T_p\partial\Omega)$ . One may verify that the projection from  $W_\varepsilon^{0,q}(\Omega)$  to  $(X_{\varepsilon,p}^{q,c})^\perp$  associated to this

decomposition is given by

$$(7.44) \quad \begin{aligned} \Pi_{\varepsilon,p}^M &= I - \mathcal{D}\Psi_\varepsilon(p)(\mathcal{D}\psi_\varepsilon(p) + \Pi_{\varepsilon,p}^c \mathcal{D}\tilde{\psi}_\varepsilon(p))^{-1} \Pi_{\varepsilon,p}^c \\ &= (I - \Pi_{\varepsilon,p}^c)(I - \mathcal{D}\tilde{\psi}_\varepsilon(p)(\mathcal{D}\psi_\varepsilon(p) + \Pi_{\varepsilon,p}^c \mathcal{D}\tilde{\psi}_\varepsilon(p))^{-1} \Pi_{\varepsilon,p}^c) \end{aligned}$$

which is well-defined and bounded uniformly in  $p \in \partial\Omega$  and  $\varepsilon \ll 1$  due to (7.39). Moreover, since  $\mathcal{D}\Psi_\varepsilon(p)(T_p\partial\Omega) = T_{\Psi_\varepsilon(p)}M_\varepsilon^*$  is close to  $X_{\varepsilon,p}^c = T_{\psi_\varepsilon(p)}M_\varepsilon$ , it is clear that there exists  $C > 0$  independent of  $p$  and  $\varepsilon$  such that

$$(7.45) \quad |(\Pi_{\varepsilon,p}^M \bar{L}_{\varepsilon,p})^{-1}|_{L((X_{\varepsilon,p}^{q,c})^\perp, W_\varepsilon^{2,q}(\Omega) \cap D(L_\varepsilon))} \leq C.$$

Applying  $\Pi_{\varepsilon,p}^M$  to (7.40) and using (7.28), (7.29), and the fact that  $\tilde{\psi}_\varepsilon(p) \in X_{\varepsilon,p}^u \oplus X_{\varepsilon,p}^s$ , we have

$$(7.46) \quad \tilde{\psi}_\varepsilon(p) = (\Pi_{\varepsilon,p}^M \bar{L}_{\varepsilon,p}|_{(X_{\varepsilon,p}^{q,c})^\perp})^{-1} \Pi_{\varepsilon,p}^M \left( f'(\tilde{w}_{\varepsilon,p}) \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) - g \right).$$

**Approximation of  $\tilde{\psi}_\varepsilon$  and the scale of the motion on  $M_\varepsilon^*$**

**Lemma 7.6.** *If  $m = 1$ , then for any  $\xi > 0$  there exists  $\varepsilon_0$  uniform in  $p$  such that for any  $\varepsilon < \varepsilon_0$  and  $q \geq 2$*

$$\begin{aligned} & \left| \tilde{\psi}_\varepsilon(p) - ((I - \Pi_{\varepsilon,p}^c) \bar{L}_{\varepsilon,p}|_{(X_{\varepsilon,p}^{q,c})^\perp})^{-1} (I - \Pi_{\varepsilon,p}^c) \right. \\ & \quad \left. \times \left( f'(\tilde{w}_{\varepsilon,p}) \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) \right) \right|_{W_\varepsilon^{2,q}(\Omega)} \leq C\xi\varepsilon. \end{aligned}$$

*If  $m > 1$ , then there exists  $C > 0$  uniform in  $\varepsilon \ll 1$  and  $p$  such that for  $q \geq 2$  we have*

$$\begin{aligned} & |\mathcal{D}\tilde{\psi}_\varepsilon(p)(\tau)|_{W_\varepsilon^{2,q}(\Omega)} \leq C\varepsilon|\tau|, \quad \tau \in T_p\partial\Omega, \\ & \left| \tilde{\psi}_\varepsilon(p) - ((I - \Pi_{\varepsilon,p}^c) \bar{L}_{\varepsilon,p}|_{(X_{\varepsilon,p}^{q,c})^\perp})^{-1} (I - \Pi_{\varepsilon,p}^c) \right. \\ & \quad \left. \times \left( f'(\tilde{w}_{\varepsilon,p}) \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) \right) \right|_{W_\varepsilon^{2,q}(\Omega)} \leq C\varepsilon^2. \end{aligned}$$

*Proof.* The proof for the case  $m = 1$  is straightforward based on (7.39), (7.42), (7.44), and (7.46). If  $m \geq 2$ , then we have  $|\tilde{\psi}_\varepsilon|_{C^2((\partial\Omega, \frac{1}{\varepsilon}\langle \cdot, \cdot \rangle), X)} \leq C$ . The definition of  $g$  and  $\Pi_{\varepsilon,p}^M$  and direct computation show that, for some  $C$  uniform in  $\varepsilon$  and  $p$ ,

$$\frac{1}{\varepsilon} |\mathcal{D}_{\varepsilon\tau} g|_{W_\varepsilon^{0,q}(\Omega)}, |\mathcal{D}_{\varepsilon\tau} \Pi_{\varepsilon,p}^M|_{L(W_\varepsilon^{0,q}(\Omega))} \leq C\tau, \quad \forall \tau \in T_p\partial\Omega.$$

Therefore the estimate on  $\mathcal{D}\tilde{\psi}_\varepsilon$  follows immediately from (7.46) and Lemma 7.3 which in turn implies the remaining inequality.  $\square$

As  $\varepsilon \rightarrow 0$ , the scale of  $|\varepsilon \tau_\varepsilon|$  gives the scale of the velocity field on  $M_\varepsilon^*$ .

**Lemma 7.7.** *There exists  $C > 0$  independent of sufficiently small  $\varepsilon > 0$  such that for any  $q \geq 2$ ,*

$$|\tau_\varepsilon|, \left| \tilde{\psi}_\varepsilon(p) - \left( (I - \Pi_{\varepsilon,p}^c) \bar{L}_{\varepsilon,p} \Big|_{(X_{\varepsilon,p}^{q,c})^\perp} \right)^{-1} (I - \Pi_{\varepsilon,p}^c) \right. \\ \left. \times \left( f'(\tilde{w}_{\varepsilon,p}) \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) \right) \right|_{W_\varepsilon^{2,q}(\Omega)} \leq C(\varepsilon^2 + |g|_{W_\varepsilon^{0,q}(\Omega)}).$$

*Proof.* In order to estimate  $\tau_\varepsilon$ , we consider the  $L^2(\Omega, \varepsilon^{-n} d\mu)$  inner products of the terms on the right side of (7.40) with

$$-\mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} = \nabla w \left( \frac{x-p}{\varepsilon} \right) \cdot \tau = \nabla_{\varepsilon\tau} \tilde{w}_{\varepsilon,p}$$

for  $\tau \in T_p \partial\Omega$ . Recall that  $\mathcal{D}$  is with respect to  $p$  and  $\nabla$  is with respect to  $x$ . The following equations will be used frequently

$$(7.47) \quad (\varepsilon^2 \Delta - 1) \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) = 0 \\ \text{and} \quad (\varepsilon^2 \Delta - 1 + f'(\tilde{w}_{\varepsilon,p})) \left( \nabla w \left( \frac{x-p}{\varepsilon} \right) \cdot \tau \right) = 0.$$

Firstly, using the above equations and repeated integration by parts, we compute

$$(7.48) \quad \int_\Omega f'(\tilde{w}_{\varepsilon,p}) \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) \left( \nabla w \left( \frac{x-p}{\varepsilon} \right) \cdot \tau \right) \varepsilon^{-n} dx \\ = \int_\Omega \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) (\varepsilon^2 \Delta - 1) \mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} \varepsilon^{-n} dx \\ = \int_{\partial\Omega} \left\{ \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) \mathcal{D}_{\varepsilon\tau} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} - \left( \frac{\partial}{\partial N} \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) \right) \mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} \right\} \varepsilon^{2-n} dS.$$

We will first focus on the second term on the right side of the above equation. In fact, from (7.47) and repeated integration by parts again,

$$\int_{\partial\Omega} \left( \frac{\partial}{\partial N} \mathcal{H} \left( \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right) \right) \mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} \varepsilon^{2-n} dS = \int_{\partial\Omega} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} \varepsilon^{2-n} dS \\ = \mathcal{D}_{\varepsilon\tau} \int_\Omega \left( \frac{\varepsilon^2}{2} |\nabla \tilde{w}_{\varepsilon,p}|^2 + \frac{1}{2} \tilde{w}_{\varepsilon,p}^2 - F(\tilde{w}_{\varepsilon,p}) \right) \varepsilon^{-n} dx \\ = - \int_{\partial\Omega} e \left( \frac{x-p}{\varepsilon} \right) N \cdot \tau \varepsilon^{1-n} dS$$



where  $F(0) = 0$ ,  $F' = f$ , and  $e$  is radially symmetric and is actually the scaled energy density of this parabolic system at  $w$ :

$$e = \frac{1}{2}|\nabla w|^2 + \frac{1}{2}w^2 - F(w).$$

Therefore,

$$(7.49) \quad \begin{aligned} & \int_{\Omega} f'(\tilde{w}_{\varepsilon,p}) \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) \left(\nabla w\left(\frac{x-p}{\varepsilon}\right) \cdot \tau\right) \varepsilon^{-n} dx \\ &= \int_{\partial\Omega} e\left(\frac{x-p}{\varepsilon}\right) N \cdot \tau \varepsilon^{1-n} dS + \int_{\partial\Omega} \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) \mathcal{D}_{\varepsilon\tau} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \varepsilon^{2-n} dS. \end{aligned}$$

We also obtain from integrating by parts twice,

$$(7.50) \quad \int_{\Omega} \bar{L}_{\varepsilon,p} \tilde{\psi}_{\varepsilon}(p) \left(\nabla w\left(\frac{x-p}{\varepsilon}\right) \cdot \tau\right) \varepsilon^{-n} dx = \int_{\partial\Omega} \tilde{\psi}_{\varepsilon}(p) \mathcal{D}_{\varepsilon\tau} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \varepsilon^{2-n} dS.$$

The term  $\int_{\partial\Omega} e\left(\frac{x-p}{\varepsilon}\right) N \cdot \tau \varepsilon^{1-n} dS$  will be computed in the next lemma and thus it follows from Lemma 7.3 and estimates (7.19), (7.39), (7.41), (7.49), and (7.50) that

$$\begin{aligned} |\tau_{\varepsilon} \cdot \tau| &\leq C \left| \int_{\Omega} \mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p} (\mathcal{D}\psi_{\varepsilon}(p)(\varepsilon\tau_{\varepsilon}) + \mathcal{D}\tilde{\psi}_{\varepsilon}(p)(\varepsilon\tau_{\varepsilon})) \varepsilon^{-n} dx \right| \\ &\leq C(\varepsilon^2 + |g|_{W_{\varepsilon}^{0,q}(\Omega)}) |\tau| \end{aligned}$$

and the estimate on  $\tau_{\varepsilon}$  follows.

To complete the proof of the lemma, we apply  $I - \Pi_{\varepsilon,p}^c$  to (7.40). From (7.45), (7.28), and (7.29), we obtain for any  $q \geq 2$ ,

$$\begin{aligned} & \left| \tilde{\psi}_{\varepsilon}(p) - ((I - \Pi_{\varepsilon,p}^c) \bar{L}_{\varepsilon,p}|_{(X_{\varepsilon,p}^{q,c})^{\perp}})^{-1} (I - \Pi_{\varepsilon,p}^c) \right. \\ & \quad \left. \times \left( f'(\tilde{w}_{\varepsilon,p}) \mathcal{H}\left(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}\right) \right) \right|_{W_{\varepsilon}^{2,q}(\Omega)} \leq C |g|_{W_{\varepsilon}^{0,q}(\Omega)} + o(1) |\tau_{\varepsilon}|, \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  uniformly in  $p$ . This, along with the estimate on  $\tau_{\varepsilon}$ , yields the desired estimate.  $\square$

**Lemma 7.8.** *There exists  $C > 0$  uniform in  $\varepsilon \ll 1$  and  $p \in \partial\Omega$  such that*

$$\left| \int_{\partial\Omega} e\left(\frac{x-p}{\varepsilon}\right) N \cdot \tau \varepsilon^{1-n} dS - c_1 \varepsilon^2 \nabla \kappa(p) \cdot \tau \right| \leq C \varepsilon^3 |\tau|$$

where

$$c_1 = \frac{1}{2(n-1)} \int_{\mathbb{R}^{n-1}} e(\tilde{x}) |\tilde{x}|^2 d\tilde{x} = \frac{s_{n-2}}{n^2-1} \int_0^{\infty} r^n |\nabla w(r)|^2 dr > 0,$$

with  $s_{n-2}$  being the surface area of the unit sphere of dimension  $n-2$ .

Here the definition of the mean curvature  $\kappa$  is taken as the trace of the second fundamental form determined by  $N$ , not divided by  $n - 1$ , i.e.,  $\kappa(p) = H(p) \cdot N$ , where  $H(p)$  is the mean curvature vector.

*Proof.* Without loss of generality, we may assume  $p = 0$ ,  $T_p \partial \Omega = \{x_n = 0\}$ ,  $\tau = e_1$ , where  $(e_1, \dots, e_n)$  is the standard basis. Locally  $\Omega$  is given by  $x_n > \phi(\bar{x})$ ,  $\bar{x} = (x_1, \dots, x_{n-1})$  with  $|\bar{x}| \leq \delta$ . Clearly,  $\phi(0) = 0$ ,  $\nabla \phi(0) = 0$  and it is standard that  $\nabla \kappa(p) = \nabla \Delta \phi(0)$ . Due to the exponential decay of  $w$  and its derivatives, we only need to compute this integral in a  $\delta$ -neighborhood

$$\int_{\partial \Omega \cap \{|\bar{x}| < \delta\}} e\left(\frac{x-p}{\varepsilon}\right) N \cdot \tau \varepsilon^{1-n} dS = \int_{|\bar{y}| < \frac{\delta}{\varepsilon}} e\left(\bar{y} + \frac{1}{\varepsilon} \phi(\varepsilon \bar{y}) e_n\right) \phi_{x_1}(\varepsilon \bar{y}) d\bar{y}.$$

Since  $e$  is radially symmetric and decays exponentially and  $\phi(\bar{x}) = O(|\bar{x}|^2)$  for  $|\bar{x}| \leq \delta$ ,

$$\begin{aligned} & \left| e\left(\bar{y} + \frac{1}{\varepsilon} \phi(\varepsilon \bar{y}) e_n\right) - e(\bar{y}) \right| \\ & \leq \int_0^1 \left| \partial_r e\left(\bar{y} + \frac{s}{\varepsilon} \phi(\varepsilon \bar{y}) e_n\right) \right| \frac{\frac{s^2}{\varepsilon^2} \phi(\varepsilon \bar{y})^2}{\sqrt{|\bar{y}|^2 + \frac{1}{\varepsilon^2} \phi(\varepsilon \bar{y})}} ds \\ & \leq C \varepsilon^2 |\bar{y}|^3 e^{-\frac{\mu}{2} |\bar{y}|}, \quad \text{for } |\bar{y}| \leq \frac{\delta}{\varepsilon}. \end{aligned}$$

Therefore,

$$\left| \int_{\partial \Omega} e\left(\frac{x-p}{\varepsilon}\right) N \cdot \tau \varepsilon^{1-n} dS - \int_{|\bar{y}| < \frac{\delta}{\varepsilon}} e(\bar{y}) \phi_{x_1}(\varepsilon \bar{y}) d\bar{y} \right| \leq C \varepsilon^3.$$

The second integral in the above can be computed by using the Taylor expansion of  $\phi$ , the fact that  $\nabla \kappa(p) = \nabla \Delta \phi(0)$ , the radial symmetry, and the exponential decay of  $e$  and the desired inequality follows. Finally, since  $w$  is radially symmetric, it satisfies

$$\partial_{rr} w + \frac{n-1}{r} \partial_r w - w + f(w) = 0.$$

Multiplying this equation by  $r^{n+1} \partial_r w$  and integrating on  $[0, \infty)$  immediately lead to the second integral form of  $c_1$ .  $\square$

**Leading order approximation of  $\tau_\varepsilon$ .** In order to derive a refined estimate on  $\tau_\varepsilon$ , we need to consider  $h \triangleq \mathcal{H}(\frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N})$  and  $\tilde{\psi}$  more carefully in the local coordinate system  $\bar{\Phi}(y, d)$ ,  $(y, d) \in T_p \partial \Omega \times \mathbb{R}$ , given by (7.3). The key is that their principle parts are symmetric in  $y$ .

We start with a weighted norm estimate of  $h$  by considering  $L_\varepsilon((\frac{x-p}{\varepsilon})^\alpha h)$  and  $\frac{\partial}{\partial N}((\frac{x-p}{\varepsilon})^\alpha h)$  where  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index. For  $|\alpha| = 1$ , we

have from Lemma 7.3

$$\left| L_\varepsilon \left( \left( \frac{x-p}{\varepsilon} \right) h \right) \right|_{W_\varepsilon^{m,q}(\Omega)} = |2\varepsilon \nabla h|_{W_\varepsilon^{m,q}(\Omega)} \leq C\varepsilon$$

and from the exponential decay of  $w$

$$\left| \frac{\partial}{\partial N} \left( \left( \frac{x-p}{\varepsilon} \right) h \right) \right|_{W_\varepsilon^{m+1,q}(\partial\Omega)} = \left| \frac{1}{\varepsilon} hN + \frac{x-p}{\varepsilon} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N} \right|_{W_\varepsilon^{m+1,q}(\partial\Omega)} \leq C.$$

These inequalities imply that  $|(\frac{x-p}{\varepsilon})h|_{W_\varepsilon^{m+1,q}(\Omega)} \leq C\varepsilon$ . Inductively in  $|\alpha|$  and in a similar fashion, we obtain for any index  $\alpha$ ,

$$(7.51) \quad \left| \left( \frac{x-p}{\varepsilon} \right)^\alpha D^l h \right|_{W_\varepsilon^{0,q}(\Omega)} \leq C\varepsilon, \quad l = 0, 1, \dots, m+2,$$

for some  $C > 0$  independent of  $p$  and  $\varepsilon \ll 1$ .

Let  $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$  be a smooth cut-off function satisfying  $\eta|_{[-1,1]} \equiv 1$ ,  $\eta|_{\mathbb{R} \setminus [-\frac{3}{2}, \frac{3}{2}]} \equiv 0$ , and  $|\eta'|_{C^0} \leq 3$ . For  $\delta_1 > 0$  satisfying the requirement (7.3) in the definition of  $\bar{\Phi}$ , let

$$h_0 = \eta_{\delta_1} h \quad \text{and} \quad h_1 = (1 - \eta_{\delta_1})h \quad \text{where} \quad \eta_{\delta_1}(y, d) = \eta\left(\frac{\sqrt{|y|^2 + d^2}}{\delta_1}\right).$$

For any orthogonal reflection  $R$  on  $T_p \partial\Omega$ , let

$$\bar{h}_0(y, d) = h_0(\bar{\Phi}(y, d)) - h_0(\bar{\Phi}(Ry, d)).$$

**Lemma 7.9.** *There exists  $C > 0$  independent of  $p \in \partial\Omega$  and sufficiently small  $\varepsilon > 0$  such that*

$$|\bar{h}_0|_{W_\varepsilon^{m+1,2}((\mathbb{R}^n)^+)}, \quad |h_1|_{W_\varepsilon^{m+2,2}(\Omega)} \leq C\varepsilon^2,$$

where  $(\mathbb{R}^n)^+ = \{(y, d) : d > 0\}$ .

*Proof.* From its definition,  $h$  satisfies  $L_\varepsilon h = (\varepsilon^2 \Delta - 1)h = 0$  and  $\frac{\partial h}{\partial N}|_{\partial\Omega} = \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}$ . By considering  $L_\varepsilon h_1$  and  $\frac{\partial h_1}{\partial N}|_{\partial\Omega}$ , the estimate on  $h_1$  follows directly from (7.17), the  $\varepsilon$  scaling in the problem, the exponential decay of  $w$ , and standard elliptic estimates.

From the form (7.4) of the Laplacian in  $(y, d)$  coordinates,  $h_0$  is supported in  $\{|y| < 2\delta_1\}$  and satisfies

$$\begin{cases} L_\varepsilon h_0 = \frac{\varepsilon^2}{\sqrt{G}} \partial_i (g^{ij} \sqrt{G} \partial_j h_0) - h_0 = \frac{\varepsilon^2}{\sqrt{G}} \partial_i (g^{ij} \sqrt{G} h \partial_j \eta_{\delta_1}) + \varepsilon^2 \partial_i \eta_{\delta_1} g^{ij} \partial_j h \\ \partial_d h_0|_{d=0} = \eta_{\delta_1} \frac{\partial \tilde{w}_{\varepsilon,p}}{\partial N}, \end{cases}$$

where  $y_n$  is understood as  $d$ . We also used that locally  $\{d = 0\} = \partial\Omega$ ,  $\partial_n = \frac{\partial}{\partial N}$ , and  $\frac{\partial \eta_{\delta_1}}{\partial N}(y, 0) = 0$ . We compute the boundary term using the

notation in the construction of  $\bar{\Phi}$ ,

$$\begin{aligned} \eta_{\delta_1}(y, 0) \frac{\partial \tilde{w}_{\varepsilon, p}}{\partial N}(\bar{\Phi}(y, 0)) &= \frac{1}{\varepsilon} \eta \left( \frac{|y|}{\delta_1} \right) w_r \left( \frac{y}{\varepsilon} + \frac{\phi(y)}{\varepsilon} N(p) \right) \\ &\quad \times \frac{y + \phi(y) N(p)}{\sqrt{|y|^2 + |\phi(y)|^2}} \cdot \frac{N(p) - \nabla \phi(y)}{\sqrt{1 + |\nabla \phi(y)|^2}}. \end{aligned}$$

From the definition of  $\phi$  and the above identity, it is easy to verify that on  $\{d = 0\}$ ,

$$\begin{aligned} &|\partial_d \bar{h}_0|_{W_\varepsilon^{m,2}(\mathbb{R}^{n-1})} \\ &= \left| \eta_{\delta_1}(\cdot, 0) \frac{\partial \tilde{w}_{\varepsilon, p}}{\partial N}(\bar{\Phi}(\cdot, 0)) - \eta_{\delta_1}(R \cdot, 0) \frac{\partial \tilde{w}_{\varepsilon, p}}{\partial N}(\bar{\Phi}(R \cdot, 0)) \right|_{W_\varepsilon^{m,2}(\mathbb{R}^{n-1})} \\ &\leq C\varepsilon. \end{aligned}$$

From Lemma 7.3, one can obtain that the right side of  $L_\varepsilon h_0$  is bounded by  $C\varepsilon^2$  in the norm  $|\cdot|_{W_\varepsilon^{m+1,2}((\mathbb{R}^n)^+)}$ . Thus it is easy to verify

$$L_\varepsilon \bar{h}_0 = \varepsilon^2 (g^{ij}(Ry, d) - g^{ij}(y, d)) \partial_{ij} h_0(\bar{\Phi}(y, d)) + O(\varepsilon^2).$$

The desired estimate on  $\bar{h}_0$  follows immediately from the weighted norm estimate (7.51).  $\square$

From Lemma 7.7, the almost symmetry of  $\mathcal{H}(\frac{\partial \tilde{w}_{\varepsilon, p}}{\partial N})$  in turn implies the almost symmetry of  $\tilde{\psi}_\varepsilon(p)$ . In fact, let

$$\tilde{\psi}_{\varepsilon, 0}(p) = \eta_{\delta_1} \tilde{\psi}_\varepsilon(p) \quad \text{and} \quad \tilde{\psi}_{\varepsilon, 1}(p) = (1 - \eta_{\delta_1}) \tilde{\psi}_\varepsilon(p).$$

For any orthogonal reflection  $R$  on  $T_p \partial \Omega$ , let

$$\bar{\psi}_{\varepsilon, 0}(p, y, d) = \tilde{\psi}_{\varepsilon, 0}(p, \bar{\Phi}(y, d)) - \tilde{\psi}_{\varepsilon, 0}(p, \bar{\Phi}(Ry, d)).$$

**Lemma 7.10.** *For any  $\xi > 0$ , there exists  $\varepsilon_0 > 0$  such that*

$$|\bar{\psi}_{\varepsilon, 0}(p)|_{W_\varepsilon^{2,2}((\mathbb{R}^n)^+)}, |\tilde{\psi}_{\varepsilon, 1}(p)|_{W_\varepsilon^{2,2}(\Omega)} \leq \xi \varepsilon, \quad \forall \varepsilon < \varepsilon_0$$

where  $(\mathbb{R}^n)^+ = \{(y, d) : d > 0\}$ . Moreover, if  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ , then there exists  $C > 0$  independent of  $p \in \partial \Omega$  and sufficiently small  $\varepsilon > 0$  so that

$$|\bar{\psi}_{\varepsilon, 0}(p)|_{W_\varepsilon^{2,2}((\mathbb{R}^n)^+)}, |\tilde{\psi}_{\varepsilon, 1}(p)|_{W_\varepsilon^{2,2}(\Omega)} \leq C\varepsilon^{1+\beta}.$$

*Proof.* First, we notice that  $\partial_d \tilde{\psi}_{\varepsilon, 1}(p)(y, 0) = 0$ . For any  $\tau \in T_p \partial \Omega$ , since the function  $\mathcal{D}_{\varepsilon \tau} \tilde{w}_{\varepsilon, p}$  decays exponentially, it is easy to verify that

$$\left| \int_\Omega \tilde{\psi}_{\varepsilon, 1}(p) \mathcal{D}_{\varepsilon \tau} \tilde{w}_{\varepsilon, p} \varepsilon^{-n} dx \right| \leq C\varepsilon^l |\tau|, \quad l = 1, 2, \dots$$

which along with the fact that  $\Pi_{\varepsilon,p}^c \tilde{\psi}_\varepsilon(p) = 0$  implies that

$$(7.52) \quad \left| \Pi_{\varepsilon,p}^c \tilde{\psi}_{\varepsilon,1}(p) \right|_{W_\varepsilon^{k,q}(\Omega)}, \left| \Pi_{\varepsilon,p}^c \tilde{\psi}_{\varepsilon,0}(p) \right|_{W_\varepsilon^{k,q}(\Omega)} \leq C\varepsilon^l, \\ k \leq m+1, \quad l = 1, 2, \dots, \quad q \geq 2.$$

Therefore, from (7.27), (7.39), (7.40), Lemma 7.7, and the decay of  $w$ ,

$$\begin{aligned} |\tilde{\psi}_{\varepsilon,1}(p)|_{W_\varepsilon^{2,2}(\Omega)} &\leq C\varepsilon^l + |(I - \Pi_{\varepsilon,p}^c) \tilde{\psi}_{\varepsilon,1}(p)|_{W_\varepsilon^{2,2}(\Omega)} \\ &\leq C\varepsilon^l + C|\bar{L}_{\varepsilon,p}(I - \Pi_{\varepsilon,p}^c) \tilde{\psi}_{\varepsilon,1}(p)|_{W_\varepsilon^{0,2}(\Omega)} \\ &\leq C\varepsilon^l + C|\bar{L}_{\varepsilon,p}((1 - \eta_{\delta_1}) \tilde{\psi}_\varepsilon(p))|_{W_\varepsilon^{0,2}(\Omega)} \\ &\leq C(\varepsilon^2 + |g|_{W_\varepsilon^{0,q}(\Omega)}). \end{aligned}$$

The desired estimates on  $\tilde{\psi}_{\varepsilon,1}$  follows from (7.42) or (7.43).

The estimate for  $\tilde{\psi}_{\varepsilon,0}(p)$  is obtained in a similar fashion to that for  $\bar{h}_0$  which also begins with a weighted norm estimate. From (7.40), (7.39), Lemma 7.3, the exponential decay of  $w$ , and the fact that  $\frac{x-p}{\varepsilon} \leq C\varepsilon^{-1}$  on  $\Omega$ , we have

$$\begin{aligned} &\left| L_\varepsilon \left( \frac{x-p}{\varepsilon} \tilde{\psi}_\varepsilon(p)(x) \right) \right|_{W_\varepsilon^{0,2}(\Omega)} \\ &= \left| 2\varepsilon \nabla \tilde{\psi}_\varepsilon(p) + \frac{x-p}{\varepsilon} (D\Psi_\varepsilon(p)(\varepsilon\tau_\varepsilon) - g + f'(\tilde{w}_{\varepsilon,p})(h - \tilde{\psi}_\varepsilon(p))) \right|_{W_\varepsilon^{0,2}(\Omega)} \\ &\leq C\varepsilon + \varepsilon^{-1} |g|_{W_\varepsilon^{0,2}(\Omega)} \end{aligned}$$

and

$$\left| \frac{\partial}{\partial N} \left( \frac{x-p}{\varepsilon} \tilde{\psi}_\varepsilon(p)(x) \right) \right|_{W_\varepsilon^{1,2}(\partial\Omega)} = \left| \frac{1}{\varepsilon} \tilde{\psi}_\varepsilon(p)(x) N \right|_{W_\varepsilon^{1,2}(\partial\Omega)} \leq C.$$

Therefore, the elliptic estimate implies

$$(7.53) \quad \left| \frac{x-p}{\varepsilon} \tilde{\psi}_\varepsilon(p)(x) \right|_{W_\varepsilon^{2,2}(\Omega)} \leq C\varepsilon + \varepsilon^{-1} |g|_{W_\varepsilon^{0,2}(\Omega)}.$$

Since  $w$  is radially symmetric and decays exponentially, in the  $(y, d)$  coordinates it is easy to verify that, for any  $\tau \in T_p \partial\Omega$ ,

$$(7.54) \quad |\tilde{w}_{\varepsilon,p}(\cdot, \cdot) - \tilde{w}_{\varepsilon,p}(R \cdot, \cdot)|_{W_\varepsilon^{m,q}(\Omega)}, \\ |\mathcal{D}_{\varepsilon\tau} \tilde{w}_{\varepsilon,p}(R \cdot, \cdot) - \mathcal{D}_{\varepsilon R\tau} \tilde{w}_{\varepsilon,p}(\cdot, \cdot)|_{W_\varepsilon^{m,q}(\Omega)} \leq C\varepsilon|\tau|, \quad q \geq 2,$$

which along with (7.52) implies

$$(7.55) \quad \left| \Pi_{\varepsilon,p}^c \tilde{\psi}_{\varepsilon,0}(p) \right|_{W_\varepsilon^{k,q}(\Omega)} \leq C\varepsilon^2, \quad k \leq m+1.$$

Equation (7.40) implies that  $\tilde{\psi}_{\varepsilon,0}(p)$  satisfies  $\partial_d(\tilde{\psi}_{\varepsilon,0}(p))(y, 0) = 0$  and

$$\begin{aligned}\bar{L}_{\varepsilon,p}\tilde{\psi}_{\varepsilon,0}(p) &= \frac{\varepsilon^2}{\sqrt{G}}\partial_i(g^{ij}\sqrt{G}\partial_j\tilde{\psi}_{\varepsilon,0}(p)) + (-1 + f'(\tilde{w}_{\varepsilon,p}))\tilde{\psi}_{\varepsilon,0}(p) \\ &= \frac{\varepsilon^2}{\sqrt{G}}\partial_i(g^{ij}\sqrt{G}\tilde{\psi}_{\varepsilon}(p)\partial_j\eta_{\delta_1}) + \varepsilon^2\partial_i\eta_{\delta_1}g^{ij}\partial_j\tilde{\psi}_{\varepsilon}(p) \\ &\quad + \eta_{\delta_1}(D\Psi_{\varepsilon}(p)(\varepsilon\tau_{\varepsilon}) - g + f'(\tilde{w}_{\varepsilon,p})h).\end{aligned}$$

From Lemmas 7.7 and 7.9

$$(7.56) \quad |(\bar{L}_{\varepsilon,p}\tilde{\psi}_{\varepsilon,0}(p))(\cdot, \cdot) - (\bar{L}_{\varepsilon,p}\tilde{\psi}_{\varepsilon,0}(p))(R\cdot, \cdot)|_{W_{\varepsilon}^{2,2}(\Omega)} \\ \leq C(\varepsilon^2 + |g|_{W_{\varepsilon}^{0,2}(\Omega)} + \varepsilon|f'(\tilde{w}_{\varepsilon,p}(\cdot, \cdot)) - f'(\tilde{w}_{\varepsilon,p}(R\cdot, \cdot))|_{W_{\varepsilon}^{0,2}(\Omega)}).$$

Therefore, from (7.55), (7.56), and (7.53),

$$\begin{aligned}&|\tilde{\psi}_{\varepsilon,0}(p)|_{W_{\varepsilon}^{2,2}(\Omega)} \\ &\leq C(\varepsilon^2 + |\bar{L}_{\varepsilon,p}\tilde{\psi}_{\varepsilon,0}(p)|_{W_{\varepsilon}^{0,2}(\Omega)}) \\ &\leq C(\varepsilon^2 + |g|_{W_{\varepsilon}^{0,2}(\Omega)} + \varepsilon|f'(\tilde{w}_{\varepsilon,p}(\cdot, \cdot)) - f'(\tilde{w}_{\varepsilon,p}(R\cdot, \cdot))|_{W_{\varepsilon}^{0,2}(\Omega)} \\ &\quad + |(g^{ij}(\cdot, \cdot) - g^{ij}(R\cdot, \cdot))\varepsilon^2\partial_{ij}\tilde{\psi}_{\varepsilon,0}(p)|_{W_{\varepsilon}^{0,2}(\Omega)})\end{aligned}$$

and (7.42) and (7.43) yield the desired inequality.  $\square$

These two lemmas imply that  $\mathcal{H}(\frac{\partial\tilde{w}_{\varepsilon,p}}{\partial N})$  and  $\tilde{\psi}_{\varepsilon}(p)$  are almost radially symmetric in the directions of  $T_p\partial\Omega$ . Moreover (7.54) also implies that  $\mathcal{D}_{\varepsilon\tau}\frac{\tilde{w}_{\varepsilon,p}}{\partial N}$  is almost odd in the  $\tau$  direction for any  $\tau \in T_p\partial\Omega$ . This symmetry property allows us to refine the estimates on  $\tau_{\varepsilon}$ .

**Proposition 7.11.** *For any  $\xi > 0$ , there exists  $\varepsilon_0 > 0$  such that*

$$|\tau_{\varepsilon} - c_2\varepsilon^2\nabla\kappa(p)| \leq C\xi\varepsilon^2, \quad \forall \varepsilon < \varepsilon_0.$$

Moreover, if  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ , then there exists  $C > 0$  independent of  $p \in \partial\Omega$  and sufficiently small  $\varepsilon > 0$  so that

$$|\tau_{\varepsilon} - c_2\varepsilon^2\nabla\kappa(p)| \leq C\varepsilon^{2+\beta},$$

where  $c_2 > 0$  is a constant determined only by  $w$ .

*Proof.* For any  $\tau \in T_p\partial\Omega$ , take the  $L^2(\varepsilon^{-n}dx)$  inner product of (7.40) and  $\mathcal{D}_{\varepsilon\tau}\tilde{w}_{\varepsilon,p}$ . From (7.49), (7.50), (7.42), (7.43), (7.54), and Lemmas 7.9, 7.10, and 7.8, we obtain

$$\left| \int_{\Omega} \mathcal{D}_{\varepsilon\tau_{\varepsilon}}\Psi_{\varepsilon}(p)\mathcal{D}_{\varepsilon\tau}\tilde{w}_{\varepsilon,p}\varepsilon^{-n}dx - c_1\varepsilon^2\nabla\kappa(p) \cdot \tau \right| \leq \begin{cases} \xi\varepsilon^2, & \text{if } f \in C^1 \\ C\varepsilon^{2+\beta}, & \text{if } f \in C^{1,\beta}. \end{cases}$$

The conclusion of the proposition follows from estimate (7.19) and Lemmas 7.3 and 7.7, taking  $c_2 \equiv \frac{2c_1}{|\partial_{x_1} w|_{L^2(\mathbb{R}^n)}} > 0$ .  $\square$

**Locations of the peaks.** Fix  $p \in \partial\Omega$ , consider the locations of the peaks of  $\Psi_\varepsilon(p)$ , i.e., the set  $\{x : \Psi_\varepsilon(p)(x) = \max_{\bar{\Omega}} \Psi_\varepsilon(p)\}$ .

**Proposition 7.12.** *Assume  $f \in C^{1,\beta}$  with  $\beta \in (0, 1]$ . For each  $p \in \partial\Omega$  and  $\varepsilon \ll 1$ , there exists a unique  $\tilde{p} = \tilde{p}(p, \varepsilon) \in \partial\Omega$  such that it is a unique nondegenerate global max of  $\psi_\varepsilon(p)(x)$  on  $\bar{\Omega}$  and  $|p - \tilde{p}| \leq C\varepsilon^2$ .*

*Proof.* For this purpose, it is easier to work with the variable

$$z = \frac{x - p}{\varepsilon} \in \Omega_{\varepsilon,p}.$$

Using the definition of  $g$ , (7.39), (7.40), Lemmas 7.3 and 7.7, and standard elliptic estimates one may show that there exists  $C > 0$  independent of  $p$  and  $\varepsilon$  such that  $|\tilde{\psi}_\varepsilon(p)|_{C^{2,\beta}(\Omega_{\varepsilon,p})} \leq C\varepsilon$ . This along with Lemma 7.3 implies

$$|\Psi_\varepsilon(p) - w|_{C^{2,\beta}(\Omega_{\varepsilon,p})} \leq C\varepsilon.$$

Since  $w$  achieves its unique maximum at the non-degenerate critical point  $0 \in \partial\Omega_{\varepsilon,p}$ , the max of  $\Psi_\varepsilon(p)(z)$  has to be achieved in a neighborhood of 0

$$B(C\varepsilon^{\frac{1}{2}}) = \{z : |z| \leq C\varepsilon^{\frac{1}{2}}\}.$$

Moreover, the  $C^{2,\beta}$  estimate also implies that  $\Psi_\varepsilon(p)(z)$  is concave in  $z$  and for some  $a, C > 0$ , the Hessian as a quadratic form satisfies

$$D^2\Psi_\varepsilon(p)(z) \leq -a, \quad \forall z \in B(C\varepsilon^{\frac{1}{2}}).$$

We claim  $\Psi_\varepsilon(p)(z)$  achieves its max at a unique point. To see this, we notice that  $B(C\varepsilon^{\frac{1}{2}}) \cap \partial\Omega_{\varepsilon,p}$  is almost flat, and thus, for any  $z_1, z_2 \in B(C\varepsilon^{\frac{1}{2}}) \cap \Omega_{\varepsilon,p}$ , there exists a curve in  $B(C\varepsilon^{\frac{1}{2}}) \cap \Omega_{\varepsilon,p}$  with curvature bounded by  $C\varepsilon^2$  joining  $z_1$  and  $z_2$ . The above uniform concavity implies that, along such a curve  $\Psi_\varepsilon(p)(z)$  achieves its max only once. Therefore, there exists  $z_\varepsilon \in B(C\varepsilon^{\frac{1}{2}}) \cap \Omega_{\varepsilon,p}$  such that  $\Psi_\varepsilon(p)$  achieves its max only at  $z_\varepsilon$ .

From the Neumann boundary condition, it is clear that  $z_\varepsilon$  is a critical point of  $\Psi_\varepsilon(p)(z)$  no matter whether it is on  $\partial\Omega_{\varepsilon,p}$ . Moreover, since  $\Psi_\varepsilon(p)(z)$  is  $C^{2,\beta}$  close to  $w$  and 0 is a nondegenerate critical point of  $w$ ,  $z_\varepsilon$  is also a nondegenerate critical point of  $\Psi_\varepsilon(p)(z)$  and

$$|z_\varepsilon| \leq C\varepsilon.$$

Finally, we show  $z_\varepsilon \in \partial\Omega_{\varepsilon,p}$ . Otherwise, there exists a unique  $\hat{z} \in \partial\Omega_{\varepsilon,p}$  such that

$$|z_\varepsilon - \hat{z}| = \tilde{d} \equiv d(z_\varepsilon, \partial\Omega_{\varepsilon,p}) \leq C\varepsilon \quad \text{and also} \quad z_\varepsilon - \hat{z} \perp T_{\hat{z}}\partial\Omega_{\varepsilon,p}.$$

The boundary condition gives

$$\begin{aligned} 0 &= \frac{\partial \psi_\varepsilon(p)}{\partial N}(\hat{z}) = \int_0^1 D^2 \Psi_\varepsilon(p)((1-s)z_\varepsilon + s\hat{z})(\hat{z} - z_\varepsilon, N_{\hat{z}}) ds \\ &\leq \int_0^1 D^2 w((1-s)z_\varepsilon + s\hat{z})(\hat{z} - z_\varepsilon, N_{\hat{z}}) ds + C\varepsilon |\hat{z} - z_\varepsilon|. \end{aligned}$$

Since  $\hat{z} - z_\varepsilon = -bN_{\hat{z}}$  for some  $0 < b < C\varepsilon$  and, for all  $z$  satisfying  $|z| \leq C\varepsilon$ , we have

$$D^2 w(x) \leq -\frac{1}{C}$$

the above inequality immediately implies that  $z_\varepsilon = \hat{z} \in \partial\Omega_{\varepsilon, P}$ . □

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