The Brinkman model for fast, viscous, and turbulent flows in porous media

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Brinkman flow model

In many applications, fluid velocities are too large to model accurately with Darcy's equation and pore geometries are too complex to approximate by the Navier-Stokes equations (NSE) in pores with no-slip boundary conditions on the solid obstacles. The Brinkman equation is a physically viable and numerically efficient fluid flow model for these situations:

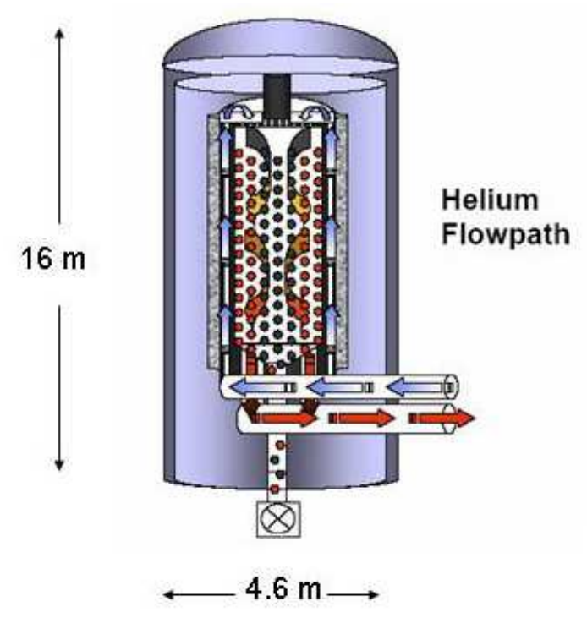
Problem 1. *In domain* $\Omega \subset \mathbb{R}^3$*, find velocity* u^δ *and pressure* p^δ *satisfying*

$$\begin{cases} u_t + u^\delta \cdot \nabla u^\delta = f + 2\nabla \cdot (\tilde{\nu}\mathbb{D}(u^\delta)) - \nabla p^\delta - \nu K^{-1} u^\delta \\ \nabla \cdot u^\delta = g \\ u|_{\partial\Omega} = \phi \end{cases}$$

Advantages of Brinkman flow model:

- Avoid meshing complicated internal geometries (alternative to NSE)
- Account for **viscous** and **inertial** flow characteristics (improvement to Darcy's equation)

Pebble bed nuclear reactors

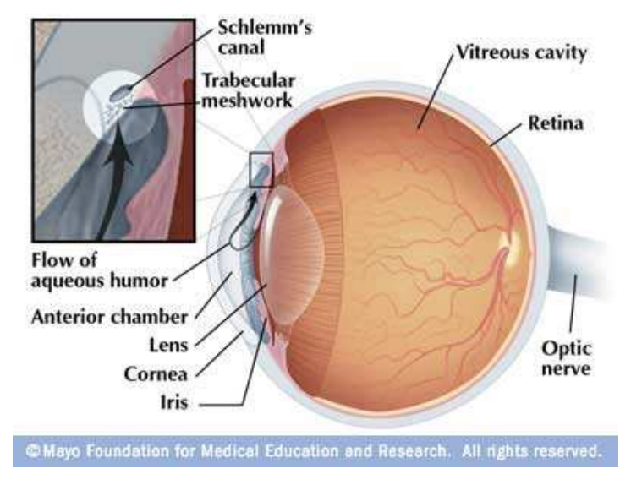


- $> 350,000$ uranium fuel pellets per core
- Small fuel pellets, $70mm$ diameter
- Helium gas coolant

Wind farms

- $\$114$ million DOE grant to develop in Texas
- Complicated flow obstacles (rotating turbine blades)

Open-angle glaucoma



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- $\sim 85\%$ of Glaucoma cases
- Related to flow obstruction from eye through the *trabecular meshwork*

Two perspectives

Brinkman as a porous media solver

- Internal domain geometry accounted for by

K (peremability tensor), $\tilde{\nu}$ (Brinkman viscosity)

- u^δ – *macroscopic* generalization of the true, pore velocity (e.g. Darcy velocity vs. NSE velocity)

Brinkman as a penalized approximation of NSE

- For small $\delta > 0$, let

$$\tilde{\nu}_{solid} = \delta^{-1}, \quad K_{fluid} = \delta^{-1}, \quad K_{solid} = \delta$$

- Recover NSE velocity as $\delta \to 0$

Finite element approximation

Definition 2. *(Equilibrium case)Fix* $f \in (H_0^1(\Omega))')^3$, $g \in L^2(\Omega)$, $\phi \in H^{1/2}(\partial\Omega)^3$ *such that* $\int_\Omega g = \int_{\partial\Omega} \phi \cdot \hat{n}$*. Find* $u^{\delta,h} \in X^h$, $p^{\delta,h} \in Q^h$ *satisfying*

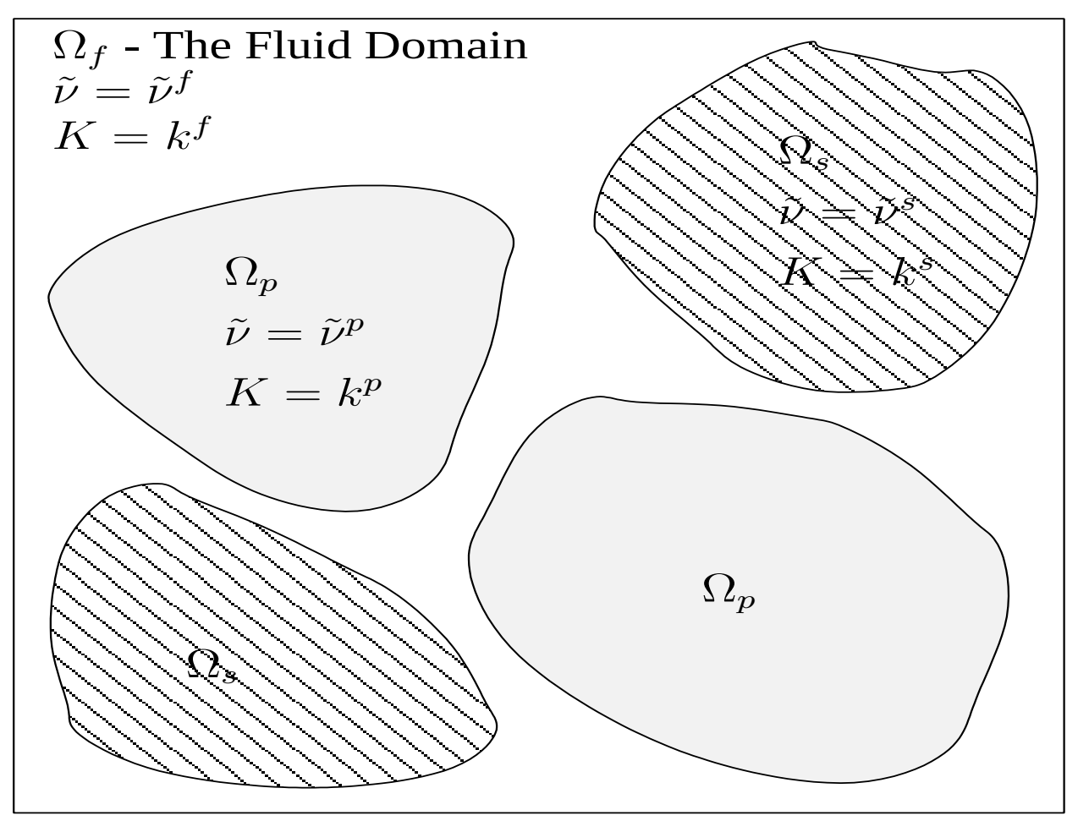
$$\forall v \in X_0^h, \quad \int_\Omega \tilde{\nu} \nabla u^{\delta,h} : \nabla v + \int_\Omega \nu k^{-1} u^{\delta,h} \cdot v + c^h(u^{\delta,h} u^{\delta,h} v) - \int_\Omega p^{\delta,h} (\nabla \cdot v) = \int_\Omega f \cdot v - \int_\Omega g(\nabla \cdot v)$$

$$\forall q \in Q^h, \quad \int_\Omega q(\nabla \cdot u^{\delta,h}) = \int_\Omega gq$$

$X^h \subset H^1(\Omega)$, $X_0^h \subset H_0^1(\Omega)$, $Q^h \subset L_0^2(\Omega)$ *are piecewise polynomial spaces.*

Sample flow domain Ω

$SampleDomain - \Omega = \Omega_f \cup \bar{\Omega}_p \cup \bar{\Omega}_s$



Problem data

- Ω_f - purely fluid domain (no flow obstruction)
- Ω_p - porous domain (some flow obstruction)
- Ω_s - purely solid domain (complete flow obstruction)
- $\tilde{u}$ - extension of boundary data ϕ (preserving $\nabla \cdot \tilde{u} = g$)

Skew-symmeterization of nonlinear term

Definition 3. *Let* $u, v, w \in H^1(\Omega)$ *satisfy* $\int_\Omega (\nabla \cdot u) q^h = \int_\Omega g q^h$ *for any* $q^h \in Q^h$*, then*

$$c^h(u, v, w) := \frac{1}{2}(c(u, v, w) - c(u, w, v)) - \frac{1}{2}\int_\Omega g\,(v \cdot w)$$

Proposition 4. *For* u, v, w *as above,*

$$\begin{aligned} c^h(u, v, w) &\leq C||u||_1||\nabla v||||w||_1 \quad \textit{(continuous)} \\ c^h(u, v, v) &= -\frac{1}{2}\int_\Omega g|v|^2, \quad \textit{(pseudo-skew-symmetric)} \\ \nabla \cdot u = g \Rightarrow c^h(u, v, w) &= \int_\Omega u \cdot \nabla v \cdot w \quad \textit{(consistent)} \end{aligned}$$

Existence and uniqueness

Theorem 5. *For data satisfying*

- $g \equiv 0$*, or*
- g *compact support in* Ω, $\int_\Omega g = 0$, g *small enough, or*
- g *compact support in* Ω, g *and* ϕ *is small enough*

there exists $u^{\delta,h} \in H^1(\Omega)$, $p^{\delta,h} \in L^2(\Omega)$ *and*

$$||\nabla u^{\delta,h}|| \leq C_{ns}\nu^{-1}, \qquad ||u^{\delta,h}||_{H^1(\Omega_s)} \leq C_{ns}\nu^{-1}\delta$$

where $C_{ns} > 0$ *(independent of* ν, δ, h*). There is at most one such solution when*

$$C_L||g||_p + 2\sqrt{C_L}||\tilde{u}||_{L^4(\Omega_p)} + 2\nu^{-1}C_L C_{ns} < \nu$$

where $C_L > 0$ *(independent of* ν, δ, h*)*

Estimating with Brinkman

Consistency, with respect to NSE

Theorem 6. *Let* u^δ *be a solution of the Brinkman problem in* Ω*. Let* $u : \Omega_f \to \mathbb{R}^3$ *be a solution of the equilibrium NSE with no slip boundary conditions enforced on* $\partial\Omega_f$*. Extending to* Ω_s *by* $u|_{\Omega_s} \equiv 0$*, we can show that, under the same constraint for existence of solutions,*

$$||u^\delta - u||_1 \leq \mathcal{O}(\nu^{-1}\delta)$$

A similar result holds for the discrete case.

Convergence of FEM

Theorem 7. *Suppose that the small data condition for uniqueness is satisfied. Then,*

$$\begin{aligned} &||\nabla(u^\delta - u^{\delta,h})|| \leq \mathcal{O}\left(\nu^{-1}\left[h^{s+1} + \nu^{-1}(\nu^2 + \sqrt{\delta})h^k\right]\right), \\ &||u^\delta - u^{\delta,h}||_{H^1(\Omega_s)} \leq \mathcal{O}\left(\nu^{-1}\delta\right) \end{aligned}$$

for integers s, k*, smooth enough* u^δ*, and characteristic mesh width* $h > 0$*.*

Convergence of FEM to NSE solution

Theorem 8. *Furthermore, if* u *is a solution of the equilibrium Navier-Stokes problem in* Ω_f *extended to* 0 *in* Ω_s*, then*

$$||u^{\delta,h} - u||_1 \leq \mathcal{O}\left(\nu^{-1}\left[\delta + h^{s+1} + \nu^{-1}(\nu^2 + \sqrt{\delta})h^k\right]\right)$$

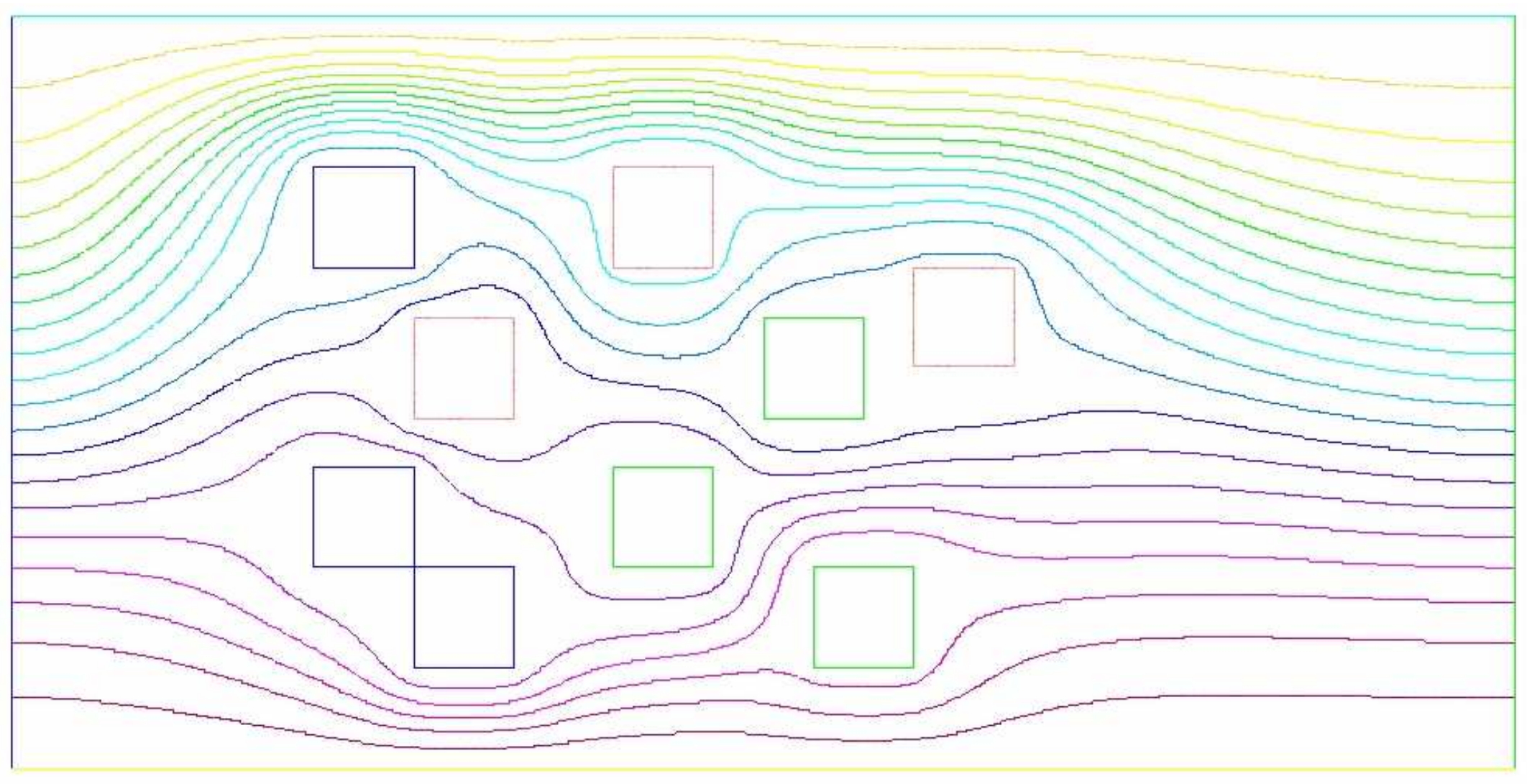
Experiment

Compute $u^{\delta,h}$ - a finite element solution of the Brinkman equation (linear, equilibrium) obtained via Taylor-Hood finite elements (P1-P2) with Dirichlet boundary conditions:

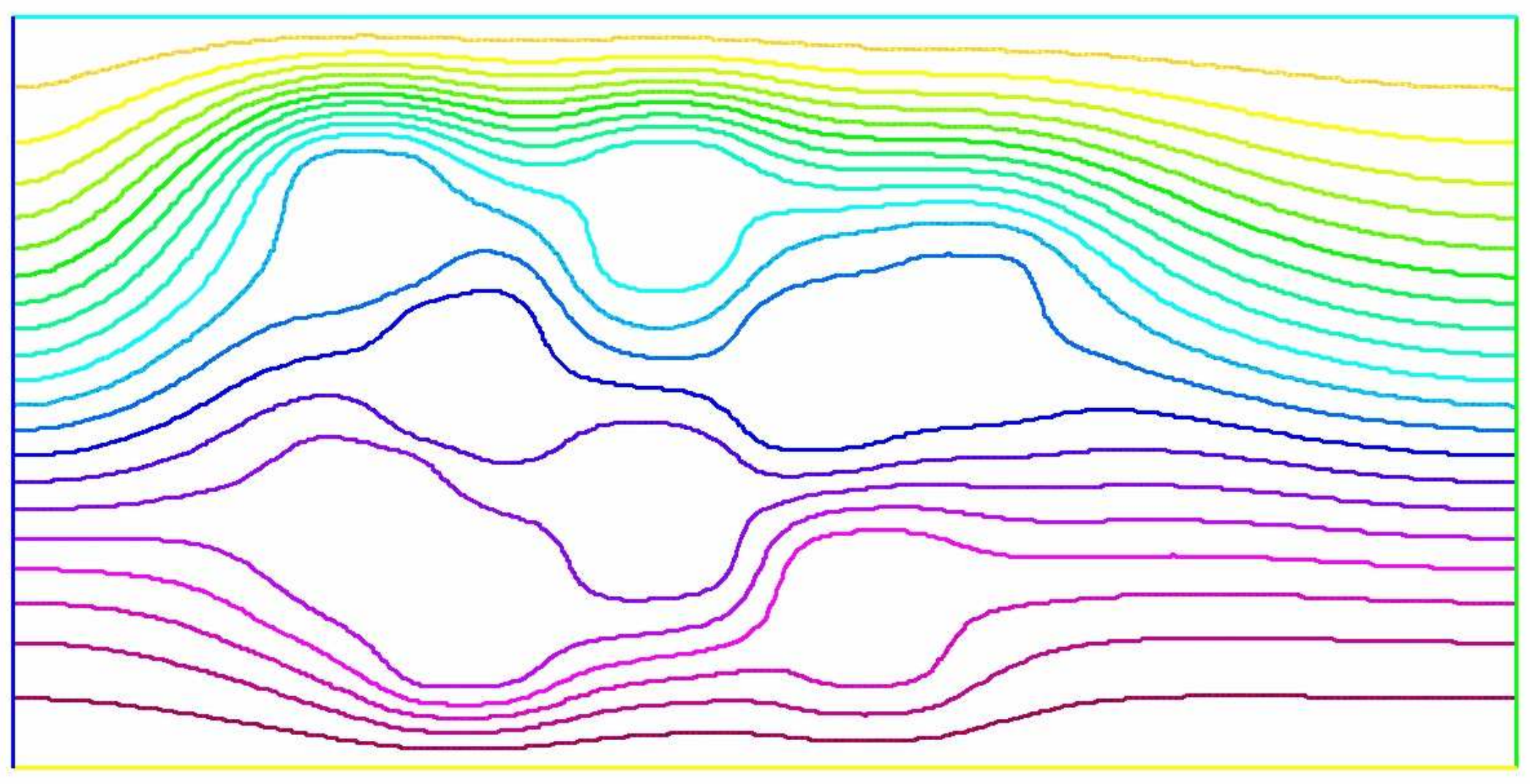
$$u|_{x=0} = y(1-y), \qquad u|_{x=2} = y(1-y), \qquad u|_{y=0,1} = 0$$

Let u be a Stokes velocity obtained via Taylor-Hood mixed finite elements on a fine mesh, $h_{max} = 0.018760$. No-slip boundary conditions are enforced on the solid obstacles in computing u. The mesh is constructed by FreeFem++ based on the Delaunay triangulation.

Stokes streamlines



Brinkman streamlines



Convergence data (in Ω)

	h	$\|\|u^{\delta,h} - u\|\|$	Rate(L^2)	$\|u^{\delta,h} - u\|\|_1$	Rate(H^1)
$\delta = 10^{-5}$	0.09428	2.6968*e*-2	—	1.4324*e*-0	—
	0.04714	1.1757*e*-2	1.2	8.6499*e*-1	0.73
	0.02357	7.0509*e*-3	0.74	4.9880*e*-1	0.79
$\delta = 10^{-10}$	0.09428	2.4180*e*-2	—	1.4413*e*-0	—
	0.04714	7.9136*e*-3	1.6	8.8644*e*-1	0.70
	0.02357	2.0529*e*-3	1.9	4.8989*e*-1	0.86
$\delta = 10^{-15}$	0.09428	2.4180*e*-2	—	1.4413*e*-0	—
	0.04714	7.9154*e*-3	1.6	8.6437*e*-1	0.74
	0.02357	2.0529*e*-3	1.9	4.8989*e*-1	0.82

Convergence data (in solid obstacles Ω_s)

	$\delta = 10^{-5}$	$\delta = 10^{-10}$	$\delta = 10^{-15}$
$\|\|u^{\delta,h}\|\|_{L^2(\Omega_s)} =$	2.0353*e*-3	2.0983*e*-08	2.0983*e*-13
$\|u^{\delta,h}\|_{H^1(\Omega_s)} =$	7.8208*e*-5	8.0846*e*-10	8.0846*e*-15

Conclusions

The Brinkman model for fluid flow is simple to implement and integrate into existing computing platforms. The uniform stability of Brinkman velocities as $\delta \to 0$ suggests that the finite element Brinkman approximations are dependably accurate representations of Stokes and Navier-Stokes flows, but avoids the cumbersome and often times infeasible task of enforcing no-slip boundary conditions on all interior solid obstacle boundaries.

Motivated by the ambitious task of accurately modeling the flow of fluids in gas-cooled, pebble-bed nuclear reactors, we are interested in extending the Brinkman model to the case of compressible fluids and coupling Brinkman flow with the equations of convective and radiative heat transfer. Our preliminary finite element analysis for the steady NS-Brinkman provides encouragement for these advances.