Classical Solutions of 2D Rotating Shallow Water Equations

Chunjing Xie

University of Michigan

cjxie@umich.edu

1. Introduction

The two dimensional rotating shallow water (RSW) equations are

$$\partial_t h + \nabla\cdot(h\mathbf{u}) = 0,$$
$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla h + \mathbf{u}^\perp = 0,$$

where h and $\mathbf{u} = (u_1, u_2)^T$ denote the total height and velocity of the fluids, respectively, and $\mathbf{u}^\perp := (-u_2, u_1)^T$ corresponds to the rotational force.
The perturbations $(\rho, \mathbf{u}) := (h-1, \mathbf{u})$ are governed by

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) + \nabla\cdot\mathbf{u} = 0, \quad (1)$$
$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\rho + \mathbf{u}^\perp = 0. \quad (2)$$

An important feature of the RSW system is that the relative vorticity $\theta := \nabla\times\mathbf{u} - \rho = (\partial_1 u_2 - \partial_2 u_1) - \rho$ is convected by $\mathbf{u}$,

$$\partial_t\theta + \nabla\cdot(\theta\mathbf{u}) = 0. \quad (3)$$

The equation (3) then suggests that $\theta \equiv 0$ be an invariant with respect to time (as long as $\mathbf{u} \in C^1$), i.e.

$$\theta_0 \equiv 0 \iff \theta(t,\cdot) \equiv 0 \iff \nabla\times\mathbf{u} \equiv \rho.$$

2. Main Results

Define the weighted space $H^{l,s}$ with the norm

$$\|v\|_{H^{l,s}} := \|(1+|x|^2)^{s/2}(1-\Delta)^{l/2}v\|_{L^2}.$$

The standard Sobolev space $H^l := H^{l,0}$.

Theorem 1 *Let $k \geq 52$, $\mathbf{u}_0 = (u_{1,0}, u_{2,0})^T \in H^{k+2,k}$ and*

$$\rho_0 = \partial_1 u_{2,0} - \partial_2 u_{1,0}.$$

If the initial data satisfies $\|\mathbf{u}_0\|_{H^{k+2,k}} = \delta < \delta_0$, then there exists a unique classical solution of RSW. Moreover, there exists $\mathbf{v}$ which is a solution of linear homogeneous Klein-Gordon equations such that

$$\sum_{j=0}^{1}\|\partial_t^j(\mathbf{u}(t) - \mathbf{v}(t))\|_{H^{k-15-j}} \leq C(1+t)^{-1}.$$

Theorem 2 *Let $k \geq 52$, $\|\mathbf{u}_0\|_{H^{k+2,k}} = \delta \leq \delta_0$, then there exist uniform constants C_1, C_2 and $\varepsilon_0 > 0$ which depend only on the structure of RSW system such that for any $\varepsilon \leq \varepsilon_0$ there exists a solution of RSW on $[0, C_1\varepsilon^{-\frac{1}{1+C_2\delta}}]$, if*

$$\|\rho_0 - (\partial_1 u_{2,0} - \partial_2 u_{1,0})\|_{H^s} < \varepsilon$$

for $s \geq 10$.

3. Reformulation of the Problem

Introduce a symmetrizer $m := 2\left(\sqrt{1+\rho} - 1\right)$ such that $\rho = m + \frac{1}{4}m^2$, then (1), (2) are transformed into a symmetric hyperbolic PDE system,

$$\partial_t m + \mathbf{u}\cdot\nabla m + \frac{1}{2}m\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0, \quad (4)$$
$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}m\nabla m + \nabla m + \mathbf{u}^\perp = 0. \quad (5)$$

Rewrite (4), (5) into a matrix-vector form,

$$\partial_t\mathbf{U} + \sum_{a=1,2} B_a\partial_a\mathbf{U} = \mathcal{L}(\mathbf{U}), \quad (6)$$

where $B_a = u_a I + \frac{1}{2}mJ_a$, and

$$\mathbf{U} = \begin{pmatrix} m \\ \mathbf{u}\end{pmatrix}, \quad J_1 := \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad J_2 := \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad \text{and } \mathcal{L}(\mathbf{U}) := -\begin{pmatrix}\nabla\cdot\mathbf{u}\\ \nabla m + \mathbf{u}^\perp\end{pmatrix}.$$

Lemma 1 *The flow with zero relative vorticity satisfies the following symmetric system of quasilinear Klein-Gordon equations for $\mathbf{U} := (m, \mathbf{u})^T$,*

$$\partial_{tt}\mathbf{U} - \Delta\mathbf{U} + \mathbf{U} = \sum_{i,j=1}^{2} A_{ij}(\mathbf{U})\partial_{ij}\mathbf{U} + \sum_{j=1}^{2} A_{0j}(\mathbf{U})\partial_{0j}\mathbf{U} + R(\tilde{\mathbf{U}}\otimes\tilde{\mathbf{U}}), \quad (7)$$

where linear functions A_{ij} and A_{0j} map R^3 vectors to symmetric 3×3 *matrices and satisfy $A_{ij} = A_{ji}$. The remainder term R depends linearly on the tensor product $\tilde{\mathbf{U}}\otimes\tilde{\mathbf{U}}$ with $\tilde{\mathbf{U}} := (\mathbf{U}^T, \partial_t\mathbf{U}^T, \partial_1\mathbf{U}^T, \partial_2\mathbf{U}^T)$.*

The key ideas of poof of Lemma; Differentiating (6) with respect to t, then we have

$$\partial_{tt}\mathbf{U} + N(\mathbf{U}) = \mathcal{L}^2(\mathbf{U}),$$

where the nonlinear term

$$N(\mathbf{U}) = \partial_t\sum_{a=1,2}(u_a I + \frac{1}{2}mJ_a)\partial_a\mathbf{U} + \mathcal{L}\left(\sum_{a=1,2}(u_a I + \frac{1}{2}mJ_a)\partial_a\mathbf{U}\right)$$

- The $\mathcal{L}^2$ term: using the calculus identity

$$\nabla(\nabla\cdot\mathbf{u}) - \nabla^\perp(\nabla\times\mathbf{u}) = \Delta\mathbf{u}$$

and

$$\partial_2 u_1 - \partial_1 u_2 = \rho = m + \frac{1}{4}m^2,$$

then

$$\mathcal{L}^2(\mathbf{U}) = (\Delta - 1)\mathbf{U} + \text{quadratic terms}$$

4. Global A Priori Estimates

Inspired by the ideas in [2–4], we need only to estimate the following functional

$$X(t) := \sup_{0\leq s\leq t}\Big(|\mathbf{U}|_{k-25,-1}(s) + \|\mathbf{U}\|_{H_\Gamma^{k-9}}(s) + \|\partial\mathbf{U}\|_{H_\Gamma^{k-9}}(s) + (1+s)^{-\sigma}\|\mathbf{U}\|_{H_\Gamma^k}(s) + (1+s)^{-\sigma}\|\partial\mathbf{U}\|_{H_\Gamma^k}(s)\Big), \quad (8)$$

here, pick any fixed $\sigma \in (0, 1/2)$, and

$$|\mathbf{U}|_{l,d}(t) := \sup_{0\leq s\leq t}\sum_{|\alpha|\leq l}|(1+s+|y|)^{-d}\Gamma^\alpha\mathbf{U}(s,y)|_{L_y^\infty},$$
$$\|\mathbf{U}\|_{H_\Gamma^l}(t) := \sup_{0\leq s\leq t}\sum_{|\alpha|\leq l}\|\Gamma^\alpha\mathbf{U}(s,y)\|_{L_y^2}.$$

The notations involved in above definitions are as follows

$$\Gamma := \{\Gamma_j\}_{j=1}^6 = \{\partial_0, \partial_1, \partial_2, L_1, L_2, \Omega_{12}\}, \ ,\partial^\alpha = \partial_t^{\alpha_1}\partial_1^{\alpha_2}\partial_2^{\alpha_3}, \ \Gamma^\beta = \Gamma_1^{\beta_1}\cdots\Gamma_6^{\beta_6};$$
$$L_j := x_j\partial_t + t\partial_j, \ j = 1,2; \quad \Omega_{12} := x_1\partial_2 - x_2\partial_1.$$

- Decompose $\mathbf{U} = \mathbf{V} + \mathbf{W}$. $\mathbf{W}$ is quadratic in $\mathbf{U}$ and has the form

$$\mathbf{W} = \left[\begin{pmatrix}\mathbf{U}\\ \partial_t\mathbf{U}\end{pmatrix}, Q, \begin{pmatrix}\mathbf{U}\\ \partial_t\mathbf{U}\end{pmatrix}\right],$$

where

$$[G,Q,H](x) := \int_{\mathbb{R}^2\times\mathbb{R}^2} G^T(y)Q(x-y, x-z)H(z)\,dydz.$$

After choosing Q appropriately, $\mathbf{V}$ satisfies a Klein-Gordon system with cubic and quartic nonlinearity,

$$(\partial_{tt} - \Delta + 1)\mathbf{V} = S. \quad (9)$$

- L^∞ estimate: L^∞ estimate for $\mathbf{V}$ is obtained by dispersive estimate for 2D Klein-Gordon equations; the bilinear estimate and interpolation inequalities yield the L^∞ estimate for $\mathbf{W}$.
- Intermediate L^2 estimate: L^2 estimate for $\mathbf{V}$ is obtained by Duhamel principle and a priori assumption on highest order L^2 bounds; The bilinear estimate and interpolation inequalities also yield the L^∞ estimate for $\mathbf{W}$.
- The highest order L^2 estimate: Energy estimates using the symmetric structure of system.

5. Lifespan for the solution with nonzero relative vorticity

When the initial data has non-zero relative vorticity, we decompose the solution as follows

$$U = U^K + U^Q + V,$$

where (ρ^Q, u^Q) satisfies the quasigeostrophic equations

$$\begin{cases}\mathbf{u}^Q = \nabla^\perp\rho^Q,\\ (\frac{\partial}{\partial t} + \mathbf{u}^Q\cdot\nabla)(\Delta\rho^Q - \rho^Q) = 0,\\ \rho^Q(0,\cdot) = \rho_0^Q,\end{cases}$$

and U^k satisfies the RSW with initial data $(\rho_0^K, \mathbf{u}_0^K)$. Here the initial data can be decomposed as follows

$$\rho_0 = \rho_0^K + \rho_0^Q, \ \mathbf{u}_0 = \mathbf{u}_0^K + \mathbf{u}_0^Q,$$
$$\rho_0^K = \partial_1 u_{2,0}^K - \partial_2 u_{1,0}^K, \ u_{1,0}^Q = -\partial_2\rho_0^Q, \ u_{2,0}^Q = \partial_1\rho_0^Q.$$

Indeed, ρ_0^Q satisfies

$$(\Delta - 1)\rho_0^Q = \partial_1 u_{2,0} - \partial_2 u_{1,0} - \rho_0.$$

Solve the problem for V, one can get its lifespan.

References

[1] Bin Cheng and Chunjing Xie, *On the Classical Solutions of Two Dimensional Inviscid Rotating Shallow Water System* (2009). Preprint.

[2] Sergiu Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions*, Comm. Pure Appl. Math. **38** (1985), no. 5, 631–641.

[3] Tohru Ozawa, Kimitoshi Tsutaya, and Yoshio Tsutsumi, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*, Math. Z. **222** (1996), no. 3, 341–362.

[4] Jalal Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38** (1985), no. 5, 685–696.