

ALTERNATE POWERS IN SERRIN'S SWIRLING VORTEX SOLUTIONS

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Introduction

Recent radar studies indicate the fractal nature of the tornado-related vorticity field with respect to the grid size, more specifically, natural log of the vorticity and natural log of the grid spacing have a linear relationship, with a constant ratio. In some cases of tornadic storms, the ratio was close to 0.6 ([1]). We conjecture that stretching of the vortex lines could lead to a fractalization of the vortex and, possibly, to a vortex breakdown. The possibility of fractalization in the process of vortex stretching was pointed out by Chorin in [2]. In 1972, J. Serrin discovered ([3]) a special class of swirling vortex solutions to the Navier-Stokes Equations. Depending on kinematic viscosity and the value of a “pressure” parameter, he described three cases:

- Down-draft core with radial outflow
- Downdraft core with a compensating radial inflow
- Updraft core with radial inflow (single-cell vortex)

The Main Question

Are there fractal Serrin type solutions to the Navier-Stokes Equations with velocity field proportional to $1/r^\beta$ where β is a non-integer?

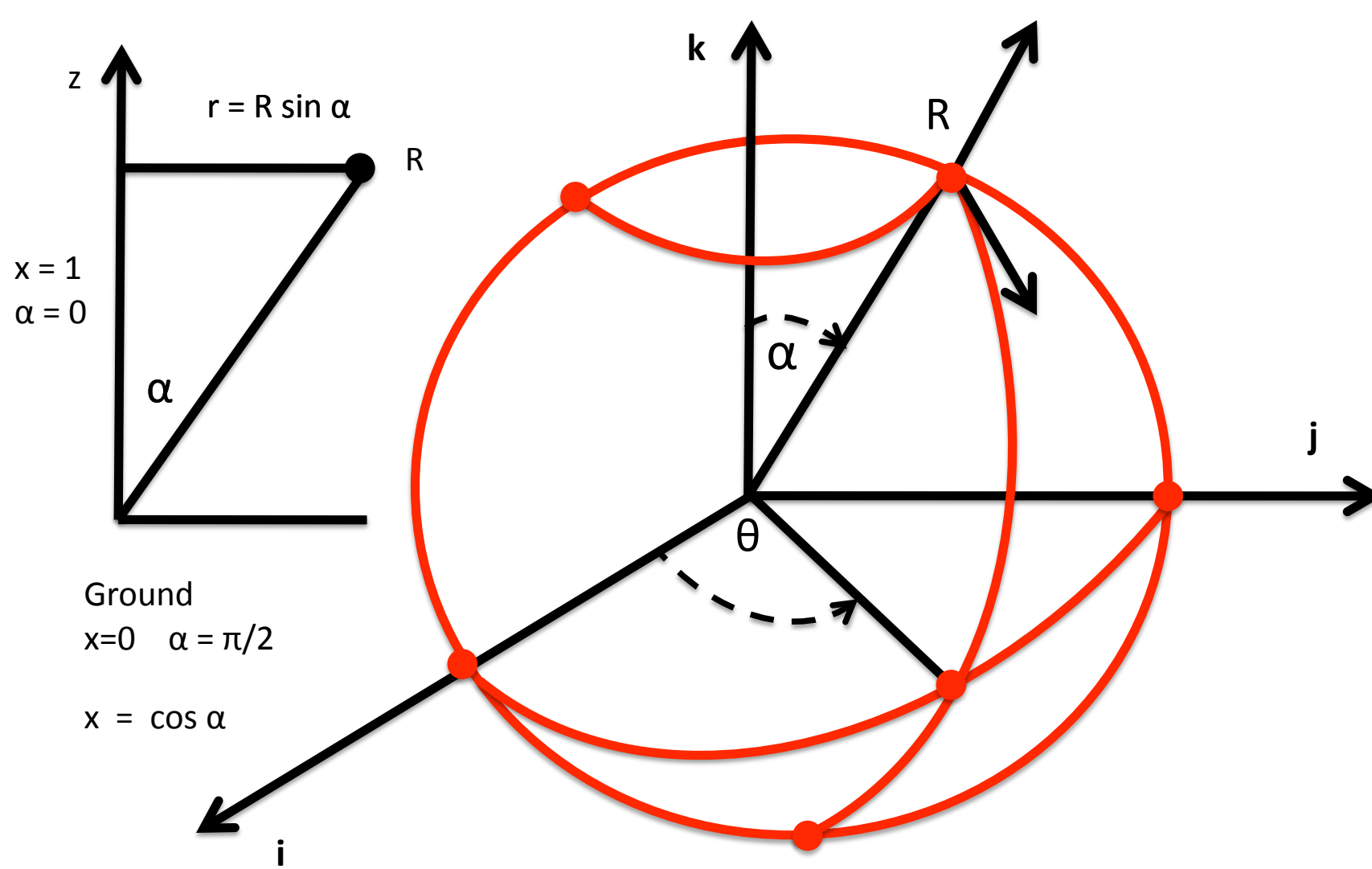
Outline of the Study

- Serrin solutions to Navier-Stokes with $\beta \neq 1$
- Shooting method & Numerical Results
- Comparison with flow in *Vortex Simulator*
- Implications for Tornadogenesis

Kinematic Quantities

Position vector $\mathbf{r}(R, \alpha, \theta) = R \cos \theta \sin \alpha \mathbf{i} + R \sin \theta \sin \alpha \mathbf{j} + R \cos \alpha \mathbf{k}$. Orthonormal basis $\mathbf{e}_R(\alpha, \theta) = \cos \theta \sin \alpha \mathbf{i} + \sin \theta \sin \alpha \mathbf{j} + \cos \alpha \mathbf{k}$, $\mathbf{e}_\alpha(\alpha, \theta) = \cos \theta \cos \alpha \mathbf{i} + \sin \theta \cos \alpha \mathbf{j} - \sin \alpha \mathbf{k}$, $\mathbf{e}_\theta(\alpha, \theta) = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} + 0 \mathbf{k}$. Velocity vector $\mathbf{v}(R, \alpha, \theta, t) = v_R(R, \alpha, \theta, t) \mathbf{e}_R(\alpha, \theta) + v_\alpha(R, \alpha, \theta, t) \mathbf{e}_\alpha(\alpha, \theta) + v_\theta(R, \alpha, \theta, t) \mathbf{e}_\theta(\alpha, \theta)$. Modified Serrin variables $r = R \sin \alpha$, $x = \cos \alpha$, $v_R(R, \alpha, \theta) = G(x)/r^\beta$, $v_\alpha(R, \alpha, \theta) = F(x)/r^\beta$, $v_\theta(R, \alpha, \theta) = \Omega(x)/r^\beta$.

Serrin Variables



Spherical Continuity Equation

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1/R^2 D[R^2 vR, R] + 1/(R Sin[a]) D[ve Sin[a], a] = 0 // Simplify
Solve[#, G[Cos[a]]] // Simplify
{G[Cos[a]] -> ((-1 + b) Cot[a] F[Cos[a]] - (-2 + b) G[Cos[a]] + Sin[a] F'[Cos[a]]) / (-2 + b)}
G[R_] := ((1 - b) x / Sqrt[1 - x^2] F[x] + Sqrt[1 - x^2] F'[x]) / (b - 2)
(1 - b) Cos[a] F[Cos[a]] + Sqrt[1 - Cos[a]^2] F'[Cos[a]] / (-2 + b)
    
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Reduction of the Spherical Navier-Stokes Equations

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eqn1 = VR D[VR, R] + ve RD[VR, a] + vt / (R Sin[a]) D[VR, t] - (ve^2 + vt^2) / R -
D[vr, R] -
nu (1/R^2 D[R^2 D[VR, R], R] - 1/(R^2 Sin[a]) D[Sin[a] D[VR, a], a] -
1/(R^2 Sin[a]^2) D[VR, t, t]) - 2 VR/R^2 - (2/R^2) D[ve, a] -
2 va Cot[a] / (R^2) - 2 / (R^2 Sin[a]^2) D[ve, t];
eqn1 // Simplify
-1/R (R Sin[a])^2 b F[Cos[a]]^2 - b G[Cos[a]]^2 -
Om[Cos[a]]^2 F[Cos[a]] (b Cot[a] G[Cos[a]] - Sin[a] G'[Cos[a]]) -
1/R^2 (R Sin[a])^2 (2 (-1 + b) nu Cot[a] F[Cos[a]] + nu (-1 + b^2 - Cos[2 a]) Csc[a]^2 G[Cos[a]] -
2 nu Sin[a] F'[Cos[a]] - 2 nu Cos[a] G'[Cos[a]] - 2 b nu Cos[a] G'[Cos[a]] -
nu Sin[a]^2 G''[Cos[a]] - R^2 (R Sin[a])^2 q^{(1+b)/2} [R, a, t]) = 0

eqn2 = VR D[VR, R] + ve RD[VR, a] + vt / (R Sin[a]) D[VR, t] - (ve VR) / R -
vt^2 Cos[a] / (R Sin[a]) -
-1/R D[vr, R] -
nu (1/R^2 D[R^2 D[VR, R], R] - 1/(R^2 Sin[a]) D[Sin[a] D[VR, a], a] -
1/(R^2 Sin[a]^2) D[VR, t, t]) - 2 VR/R^2 D[ve, a] - ve / (R^2 Sin[a]^2) -
2 Cos[a] / (R^2 Sin[a]^2) D[ve, t];
eqn2 // Simplify
1/R (R Sin[a])^2 (-b R Cot[a] F[Cos[a]]^2 - R Cot[a] Om[Cos[a]]^2 F[Cos[a]] -
((-1 + b) R G[Cos[a]] - (-1 + b^2) nu Csc[a]^2 (R Sin[a])^2 - R Sin[a] F'[Cos[a]]) +
(R Sin[a])^2 (2 b nu Cot[a] G[Cos[a]] - 2 (-1 + b) nu Cos[a] F'[Cos[a]] - 2 nu
Sin[a] G'[Cos[a]] - nu Sin[a]^2 F''[Cos[a]] - R (R Sin[a])^2 q^{(1+b)/2} [R, a, t]) = 0

eqn3 = VR D[VR, R] + ve RD[VR, a] + vt / (R Sin[a]) D[VR, t] - ve VR/R -
ve ve Cos[a] / (R Sin[a]) -
-1/R D[vr, R] -
nu (1/R^2 D[R^2 D[VR, R], R] - 1/(R^2 Sin[a]) D[Sin[a] D[VR, a], a] -
1/(R^2 Sin[a]^2) D[VR, t, t]) - ve / (R^2 Sin[a]^2) - 2 / (R^2 Sin[a]^2) D[ve, t] -
2 Cos[a] / (R^2 Sin[a]^2) D[ve, t];
eqn3 // Simplify
1/R (R Sin[a])^2 ((-1 + b) R G[Cos[a]] Om[Cos[a]] +
R F[Cos[a]] ((-1 + b) Cot[a] Om[Cos[a]] - Sin[a] Om'[Cos[a]]) -
nu (R Sin[a])^2 ((b - b^2) Cos[a]^2 - Cos[a]^2) Om[Cos[a]] -
2 (-1 + b) Cos[a] Om'[Cos[a]] - Sin[a]^2 Om''[Cos[a]]) = 0

(*Eliminate Pressure Term*)
aa = q^{(1+b)/2} [R, a, t] /. Solve[eqn1, q^{(1+b)/2} [R, a, t]];
bb = q^{(1+b)/2} [R, a, t] /. Solve[eqn2, q^{(1+b)/2} [R, a, t]];
(*Squaring mixed partials*)
eqn4 = D[aa, a][[1]] - D[bb, a][[1]];
eqn4 // Simplify
1/R (R Sin[a])^2 (-2 (-1 + b) b R Cot[a] F[Cos[a]]^2 + 2 b^2 R Cot[a] G[Cos[a]]^2 -
2 nu Cos[a] (R Sin[a])^2 F'[Cos[a]] - 4 b nu Cos[a] (R Sin[a])^2 F'[Cos[a]] -
2 b^2 nu Cos[a] (R Sin[a])^2 F'[Cos[a]] - 2 b nu Cos[a] Cot[a] (R Sin[a])^2 G'[Cos[a]] -
3 b^2 nu Cos[a] Cot[a] (R Sin[a])^2 G'[Cos[a]]) -
2 nu Sin[a] (R Sin[a])^2 G'[Cos[a]] - 4 b nu Sin[a] (R Sin[a])^2 G'[Cos[a]] -
b^2 nu Sin[a] (R Sin[a])^2 G'[Cos[a]] - 2 b^2 nu Cot[a]^2 (R Sin[a])^2 F'[Cos[a]] -
b^2 nu Cot[a]^2 (R Sin[a])^2 G'[Cos[a]] - b^2 nu Cot[a]^2 (R Sin[a])^2 -
b G[Cos[a]] (b nu (3 - b - Cos[2 a]) Cot[a] Csc[a]^2 (R Sin[a])^2 -
R Cos[a] F'[Cos[a]] - 2 R Sin[a] G'[Cos[a]]) -
2 R Om[Cos[a]] Sin[a] Om'[Cos[a]] - nu Sin[a] (R Sin[a])^2 F'[Cos[a]] -
b nu Sin[a]^2 (R Sin[a])^2 F'[Cos[a]] - 4 nu Cos[a] Sin[a] (R Sin[a])^2 G'[Cos[a]] +
3 b nu Cos[a] Sin[a] (R Sin[a])^2 G'[Cos[a]] - F[Cos[a]] -
(b R (2 - (-1 + 2 b) Cos[2 a]) Csc[a]^2 G[Cos[a]] - 2 nu (R Sin[a])^2 - 2 b nu (R Sin[a])^2 -
2 nu Cot[a]^2 (R Sin[a])^2 - 2 b^2 nu Cot[a]^2 (R Sin[a])^2 - nu Cot[a]^2 (R Sin[a])^2 -
b nu Csc[a]^2 (R Sin[a])^2 - b^2 nu Csc[a]^2 (R Sin[a])^2 - b^2 nu Csc[a]^2 (R Sin[a])^2 -
2 (-1 + b) R Sin[a] F'[Cos[a]] - R Cos[a] G'[Cos[a]] - 3 b R Cos[a] G'[Cos[a]] -
R Sin[a]^2 G''[Cos[a]]) - nu Sin[a]^2 G''[Cos[a]] = 0

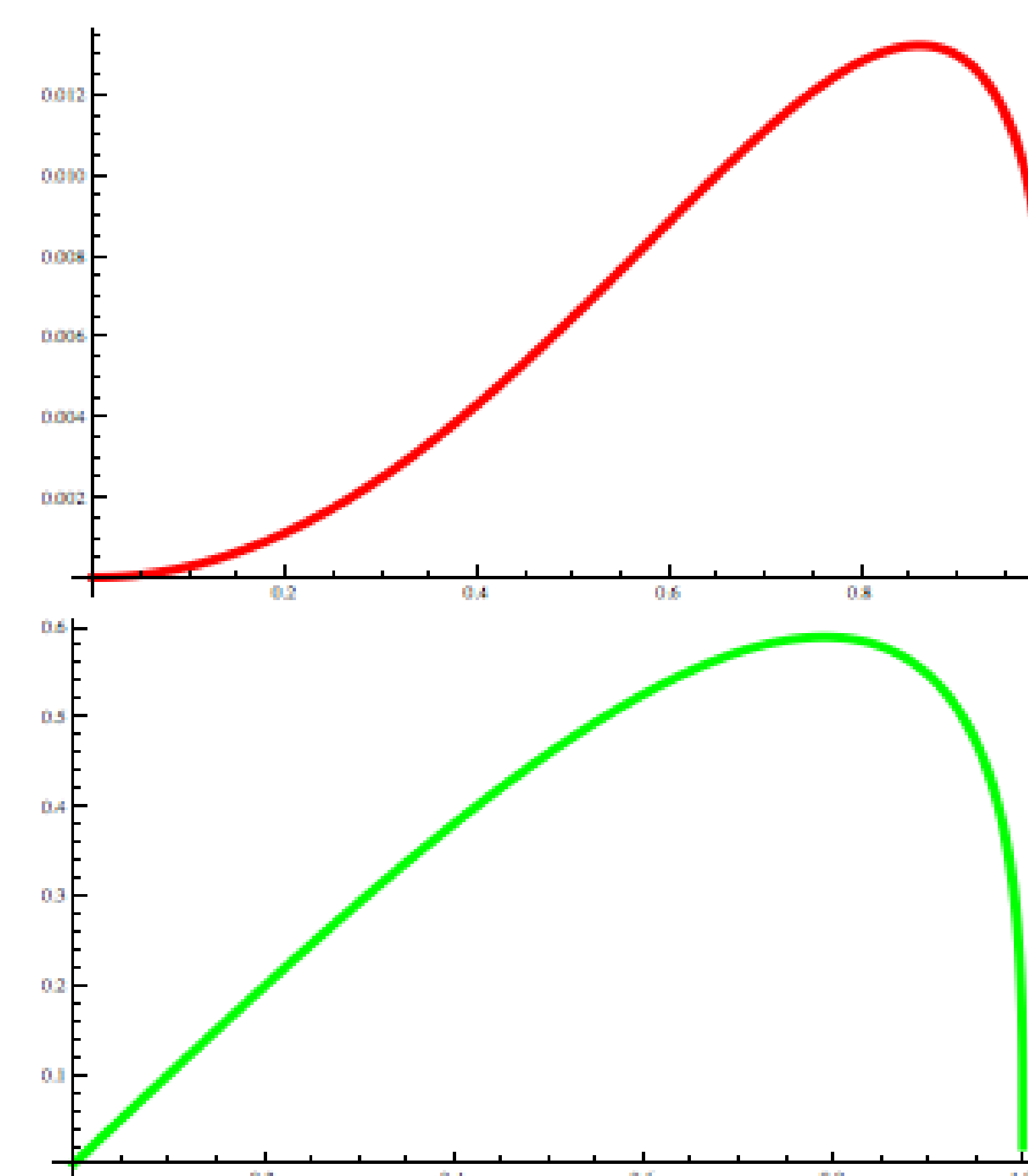
(*There is no R dependence for the Euler Equations*)
%/. nu -> 0 // Simplify
(R Sin[a])^2 (-2 (-1 + b) b Cot[a] F[Cos[a]]^2 -
2 b^2 Cot[a] G[Cos[a]]^2 - b G[Cos[a]] (Cos[a] F'[Cos[a]] + 2 Sin[a] G'[Cos[a]]) -
Sin[a] (R Sin[a] F'[Cos[a]] G'[Cos[a]] + 2 Om[Cos[a]] Om'[Cos[a]]) -
F[Cos[a]] (b (2 - (-1 + 2 b) Cos[2 a]) Csc[a]^2 G[Cos[a]] - 2 (-1 + b) Sin[a] F'[Cos[a]] -
Cos[a] G'[Cos[a]] - 3 b Cos[a] G'[Cos[a]] - Sin[a]^2 G''[Cos[a]]) = 0
    
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ODE System & Dependence on R

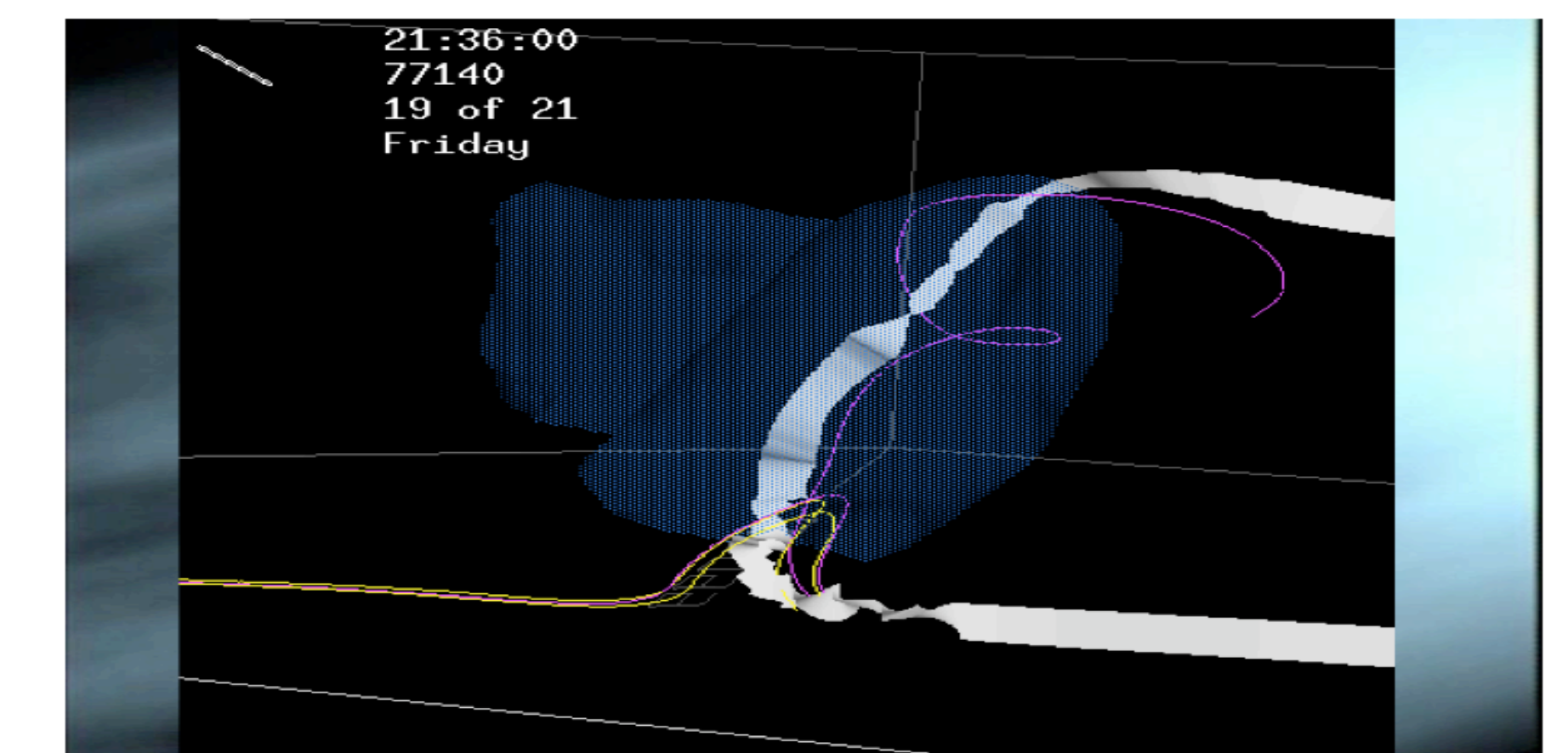
$$\begin{aligned} \{F^{(1)}[x] \rightarrow & \frac{1}{2 \nu (1-x^2)^2} R \left(\frac{4(-1+b) R x (R \sqrt{1-x^2})^{-1-b} (2+b-b(-1+2x^2)) F[x]^2}{(-2+b)(1-x^2)} \right. \\ & 4(-2+b+b^2) R x (R \sqrt{1-x^2})^{-1-b} (1-x^2) F[x]^2 - \\ & 4(-2+b) R (R \sqrt{1-x^2})^{-1-b} (1-x^2) Om[x] Om'[x] - \\ & \left. 4(-1+b) nu (2b+(-3+b)(-1+2x^2)) F'[x] - \frac{1}{-2+b} R \sqrt{1-x^2} \right)^{-1-b} \\ F[x] \left(\frac{2b(2-3b+b^2) nu x (R \sqrt{1-x^2})^b (-4-2b+2x^2)}{\sqrt{1-x^2}} \right. \\ & (2+b) R (1-x^2)^2 F''[x] - \frac{8(-2+b) nu x (1-x^2) F^{(1)}[x]}{R} - \frac{1}{-2+b} \\ & \left. (R \sqrt{1-x^2})^{-1-b} F[x] (-2(-1+b) R (2+5b+(2+3b)(-1+2x^2)) F'[x] + \right. \\ & \left. 4(4-5b+b^2) R x (1-x^2) F''[x] + (-2+b) \left(\frac{1}{(1-x^2)^{3/2}} (-1+b) \right. \right. \\ & \left. \left. nu (R \sqrt{1-x^2})^b (6+7b+6b^2+2b^3+4b(2+b)(-1+2x^2) \right. \right. \\ & \left. \left. - b(-1+2(-1+2x^2)^2)) + 2R(1-x^2)^2 F^{(1)}[x] \right) \right) \} \\ \{Om'[x] \rightarrow & -\frac{1}{nu(1-x^2)} \\ & R^2 (R \sqrt{1-x^2})^b \left(\frac{(-1+b) nu (R \sqrt{1-x^2})^{-b} (2+b-b(-1+2x^2)) Om[x]}{2R^2(1-x^2)} \right. \\ & \left. \frac{2(-1+b) nu x (R \sqrt{1-x^2})^{-b} Om'[x]}{x^2} - \frac{1}{(-2+b) R} \right. \\ & \left. (R \sqrt{1-x^2})^{-2b} (-1+b) \sqrt{1-x^2} Om[x] F'[x] + \right. \\ & \left. F[x] \left(\frac{(-1+b) x Om[x]}{\sqrt{1-x^2}} - (-2+b) \sqrt{1-x^2} Om'[x] \right) \right) \} \end{aligned}$$

Dependence on R in the right-hand side contradicts the original assumption which shows that Navier-Stokes in this case does not have solutions of the form $F(x)/r^\beta$ unless $\beta = 1$ or $\beta = 2$. However, the solutions of this form exist for Euler Equations for all $\beta > 0$. Also, despite the dependence on R in the right-hand side of the above nonlinear system, the solutions do not change in a significant way within a wide range of R values due to a special set of initial data used in a **Shooting Method**. Presumably, the solution curves stay “close” to the corresponding velocity field.

Latitudinal and Longitudinal Velocities



Storm Merger and Tornadogenesis



The color lines are the arching vortex lines, white ribbon is a streamline. The blue isosurface is 20 m/s updraft in a supercell thunderstorm (see [4] and [5]).

Cai [1] Results

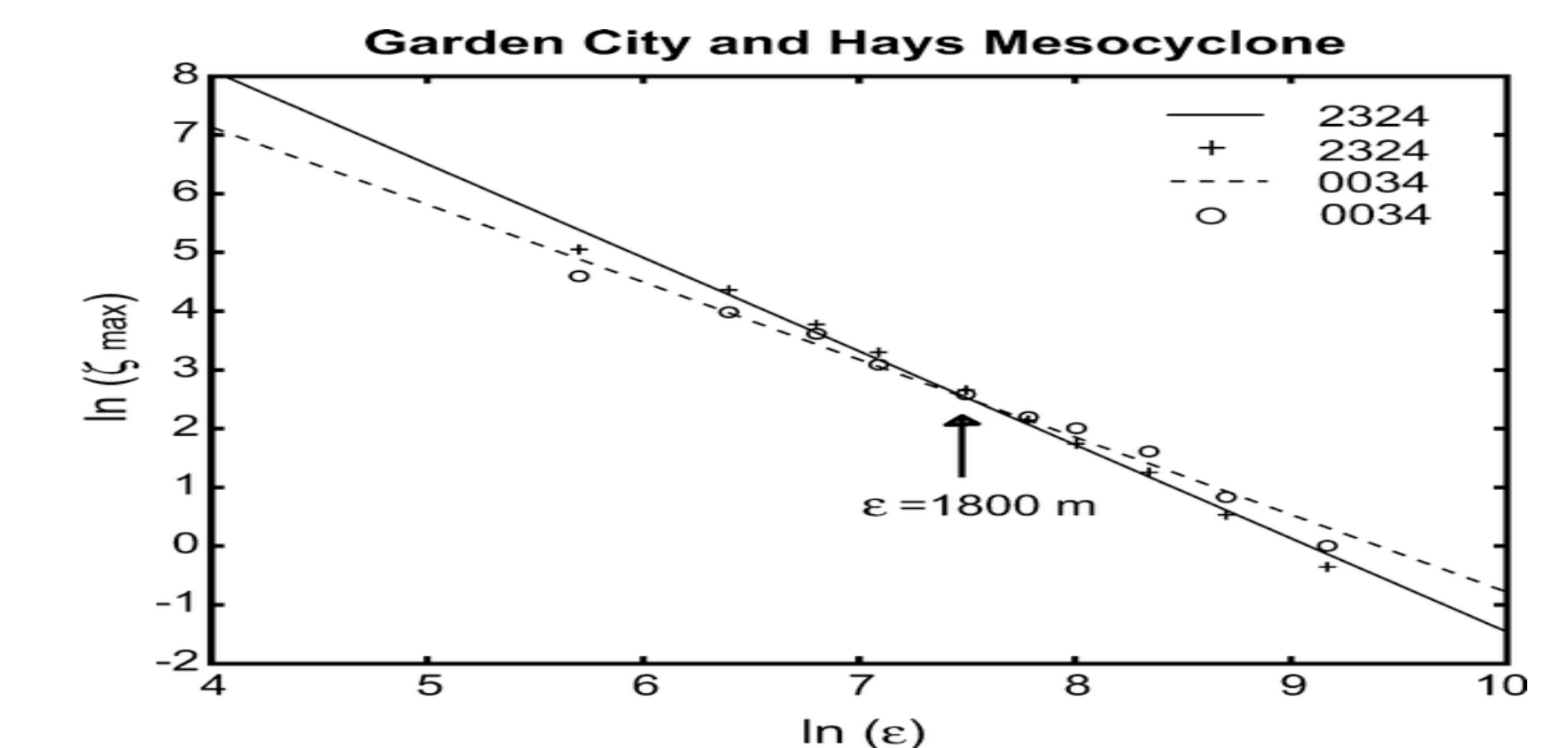


Fig. 2. Comparison between the vorticity lines of the tornadic Garden City and nontornadic Hays mesocyclones. A one-step Lense filter is applied to the wind field. Solid black line represents the Garden City mesocyclone at 0034 UTC just before tornadogenesis; dashed black line represents the Hays mesocyclone at 0034 UTC when it reached its peak intensity.

Vortex Simulator



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- [4] J. M. Straka, E. N. Rasmussen, R. P. Davies-Jones, P. M. Markowski, E-Journal of Severe Storms Meteorology **2** (2007), 1–22.
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