# ALTERNATE POWERS IN SERRIN'S SWIRLING VORTEX SOLUTIONS

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## Introduction

Recent radar studies indicate the fractal nature of the tornado-related vorticity field with respect to the grid size, more specifically, natural log of the vorticity and natural log of the grid spacing have a linear relationship, with a constant ratio. In some cases of tornadic storms, the ratio was close to 0.6 ([1]). We conjecture that stretching of the vortex lines could lead to a fractalization of the vortex and, possibly, to a vortex breakdown. The possibility of fractalization in the process of vortex stretching was pointed out by Chorin in [2]. In 1972, J. Serrin discovered ([3]) a special class of swirling vortex solutions to the Navier-Stokes Equations. Depending on kinematic viscosity and the value of a "pressure" parameter, he described three cases:

- Down-draft core with radial outflow
- Downdraft core with a compensating radial inflow
- Updraft core with radial inflow (single-cell vortex)

## The Main Question

Are there fractal Serrin type solutions to the Navier-Stokes Equations with velocity field proportional to $\mathbf{1/r^{\beta}}$ where $\boldsymbol{\beta}$ is a non-integer?

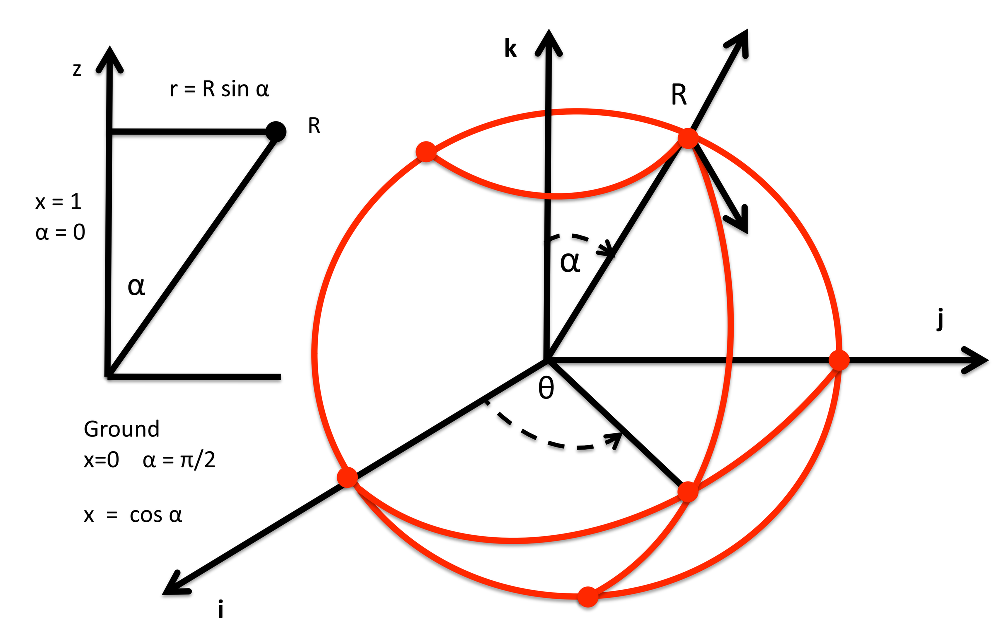
## Outline of the Study

- Serrin solutions to Navier-Stokes with $\beta \neq 1$
- Shooting method & Numerical Results
- Comparison with flow in *Vortex Simulator*
- Implications for Tornadogenesis

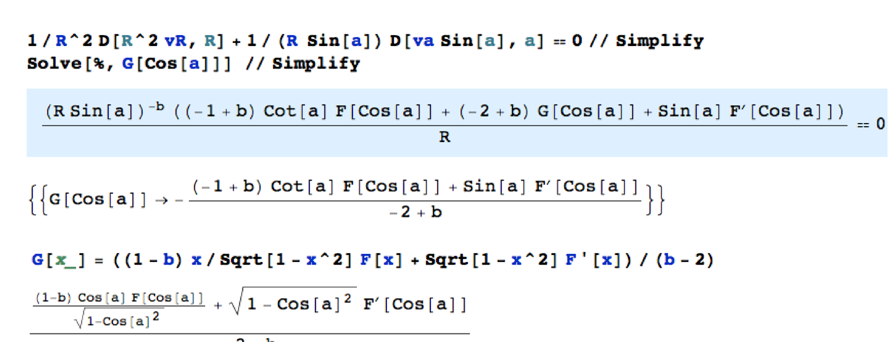
## Kinematic Quantities

Position vector $\mathbf{r}(R,\alpha,\theta) = R\cos\theta\sin\alpha\,\mathbf{i} + R\sin\theta\sin\alpha\,\mathbf{i} + R\cos\alpha\,\mathbf{k}$. Orthonormal basis $\mathbf{e}_R(\alpha,\theta) = \cos\theta\sin\alpha\,\mathbf{i} + \sin\theta\sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{k}$, $\mathbf{e}_\alpha(\alpha,\theta) = \cos\theta\cos\alpha\,\mathbf{i} + \sin\theta\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{k}$, $\mathbf{e}_\theta(\alpha,\theta) = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + 0\,\mathbf{k}$. Velocity vector $\mathbf{v}(R,\alpha,\theta,t) = v_R(R,\alpha,\theta,t)\,\mathbf{e}_R(\alpha,\theta) + v_\alpha(R,\alpha,\theta,t)\,\mathbf{e}_\alpha(\alpha,\theta) + v_\theta(R,\alpha,\theta,t)\mathbf{e}_\theta(\alpha,\theta)$. Modified Serrin variables $r = R\sin\alpha$, $x = \cos\alpha$, $v_R(R,\alpha,\theta) = G(x)/r^{\beta}$, $v_\alpha(R,\alpha,\theta) = F(x)/r^{\beta}$, $v_\theta(R,\alpha,\theta) = \Omega(x)/r^{\beta}$.

## Serrin Variables

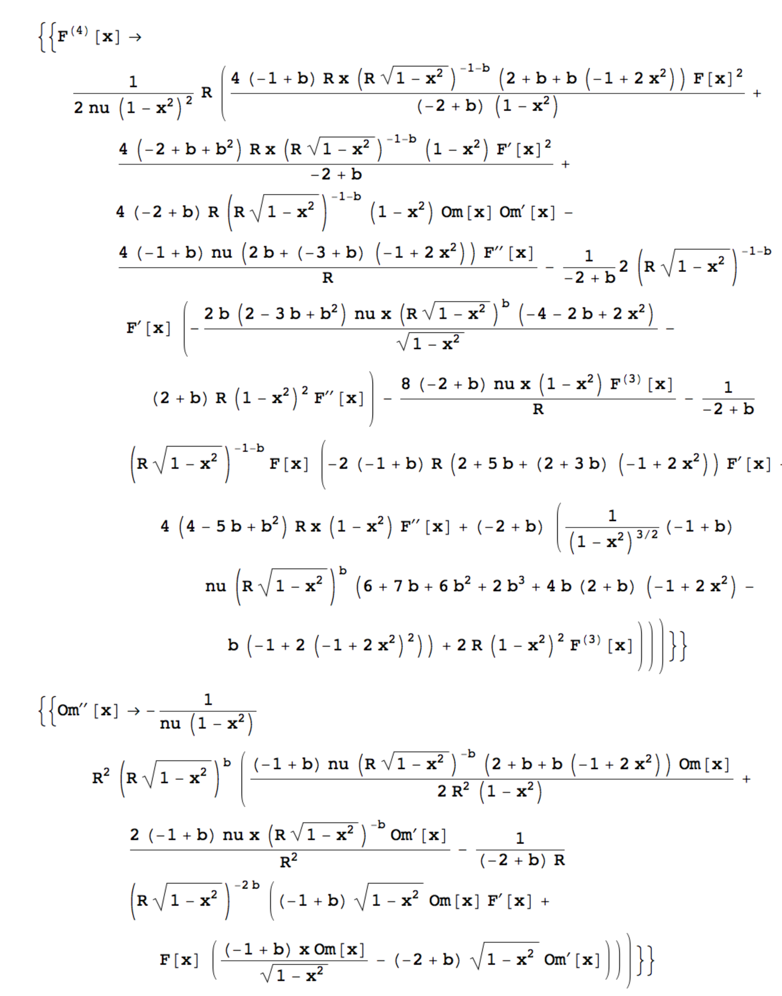



## Spherical Continuity Equation



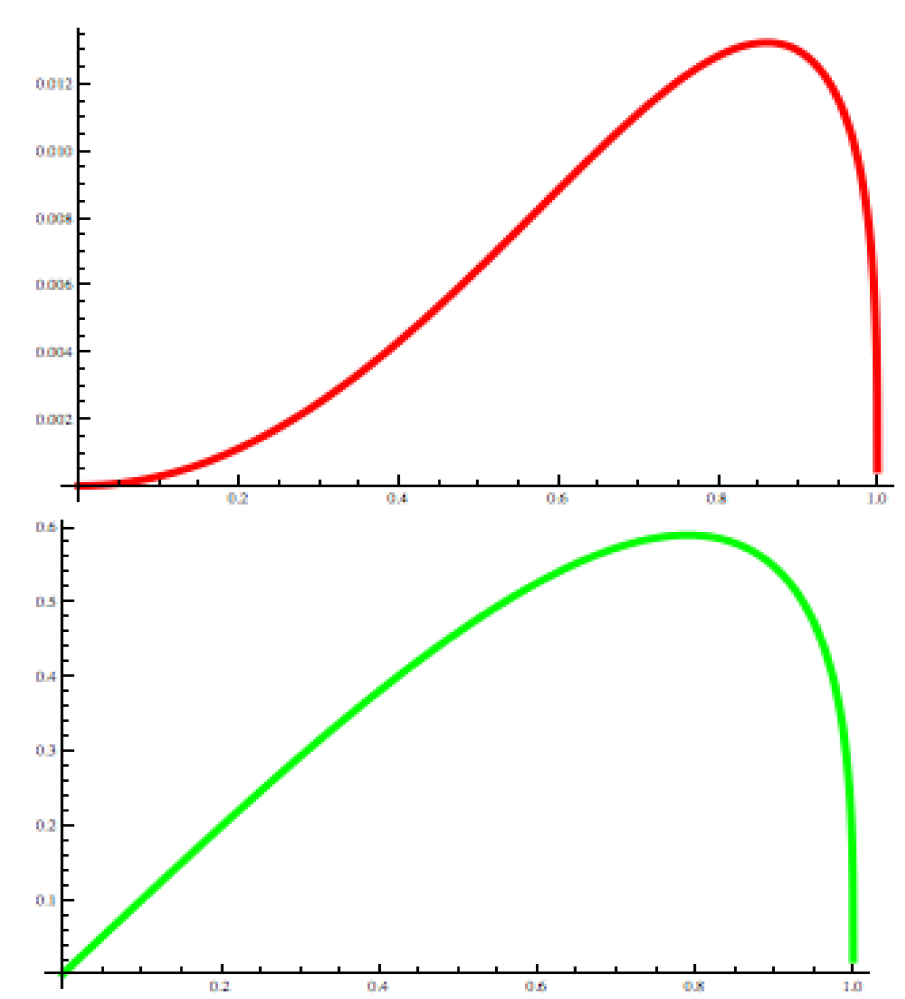
## Reduction of the Spherical Navier-Stokes Equations

## ODE System & Dependence on $R$

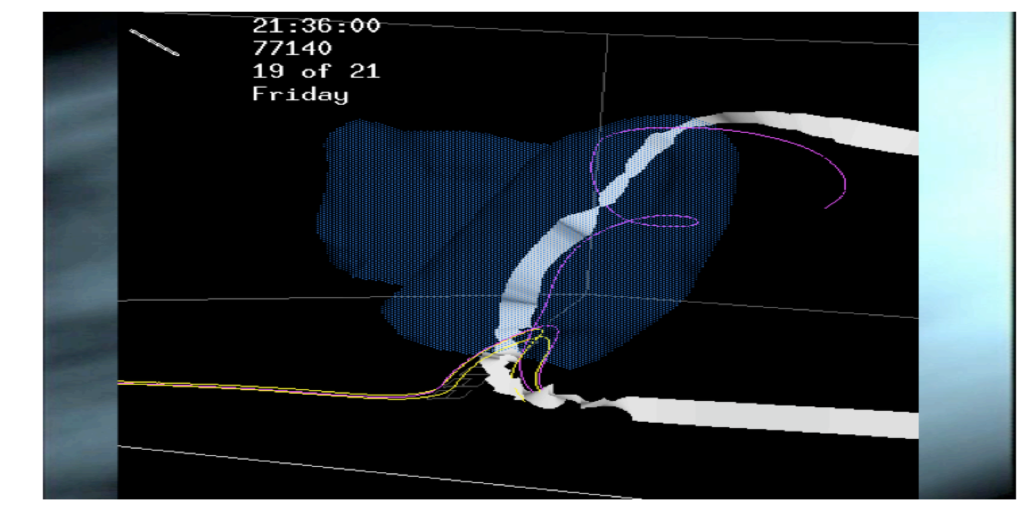


Dependence on $R$ in the right-hand side contradicts the original assumption which shows that Navier-Stokes in this case does not have solutions of the form $F(x)/r^{\beta}$ unless $\beta = 1$ or $\beta = 2$. However, the solutions of this form exist for Euler Equations for all $\beta > 0$. Also, despite the dependence on $R$ in the right-hand side of the above nonlinear system, the solutions do not change in a significant way within a wide range of $R$ values due to a special set of initial data used in a **Shooting Method**. Presumably, the solution curves stay "close" to the corresponding velocity field.

## Latitudinal and Longitudinal Velocities



## Storm Merger and Tornadogenesis




The color lines are the arching vortex lines, white ribbon is a streamline. The blue isosurface is 20 m/s updraft in a supercell thunderstorm (see [4] and [5]).

## Cai [1] Results

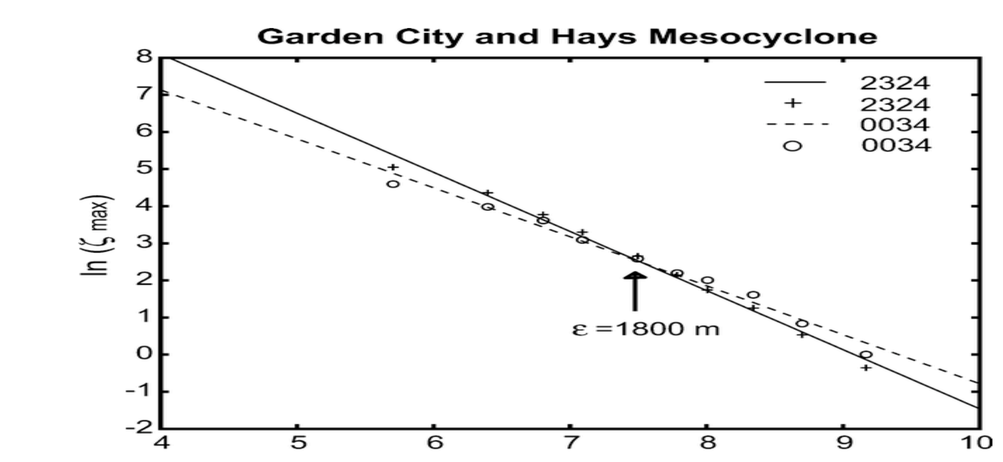



Fig. 2. Comparison between the vorticity lines of the tornadic Garden City and nontornadic Hays mesocyclones. A one-step Leise filter is applied to the wind field. Solid black line represents the Garden City mesocyclone at 2324 UTC just before tornadogenesis; dashed black line represents the Hays mesocyclone at 0034 UTC when it reached its peak intensity.

## Vortex Simulator


[1] H. Cai, Monthly Weather Review **133** (2005), 2535–2551

[2] A. J. Chorin, Vorticity and Turbulence, Springer, New York (1994)

[3] J. Serrin, Phil. Trans. Roy. Soc. London, Series A, Math & Phys. Sci. **271** (1972), 325–360

[4] J. M. Straka, E. N. Rasmussen, R. P. Davies-Jones, P. M. Markowski, E-Journal of Severe Storms Meteorology **2** (2007), 1–22.

[5] D. P. Dokken, K. Scholz, P. Shanahan, L. Edholm, CSUMS Report (2009)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