Pressure boundary conditions, projection methods, and finite element computations

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with

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- Well-posed dynamics with *boundary conditions for pressure* (Based on estimates for the Laplace-Leray commutator $\Delta\mathcal{P} - \mathcal{P}\Delta$)
- Improvements in stability and accuracy for numerics:
  - ⋆ Time-discrete projection methods up to 3rd-order (4th?)
  - ⋆ Flexible choice of finite elements for velocity & pressure (e.g., piecewise linear for both)

Navier-Stokes equations for incompressible flow

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \nu\Delta\boldsymbol{u} + \boldsymbol{f} && \text{in } \Omega && \text{momentum} \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega && \text{incompressibility} \\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega && \text{no slip} \end{aligned}$$

Ways to regard and treat pressure:

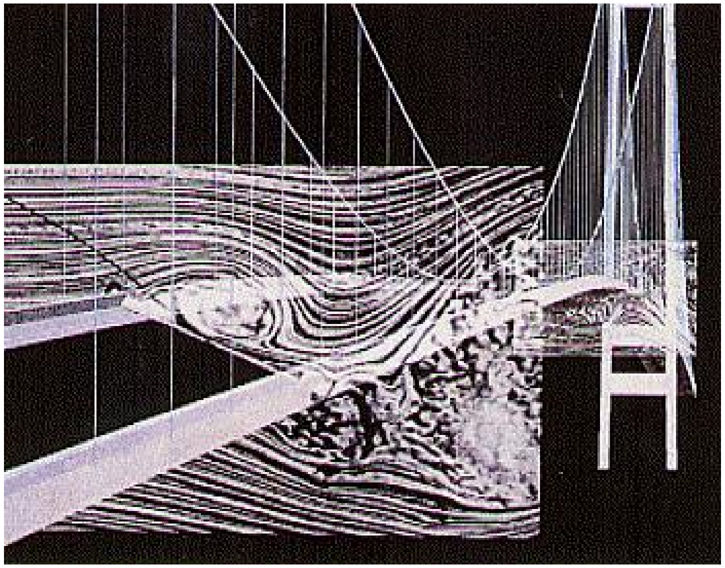
(i) Pressure is like a Lagrange multiplier to enforce incompressibility

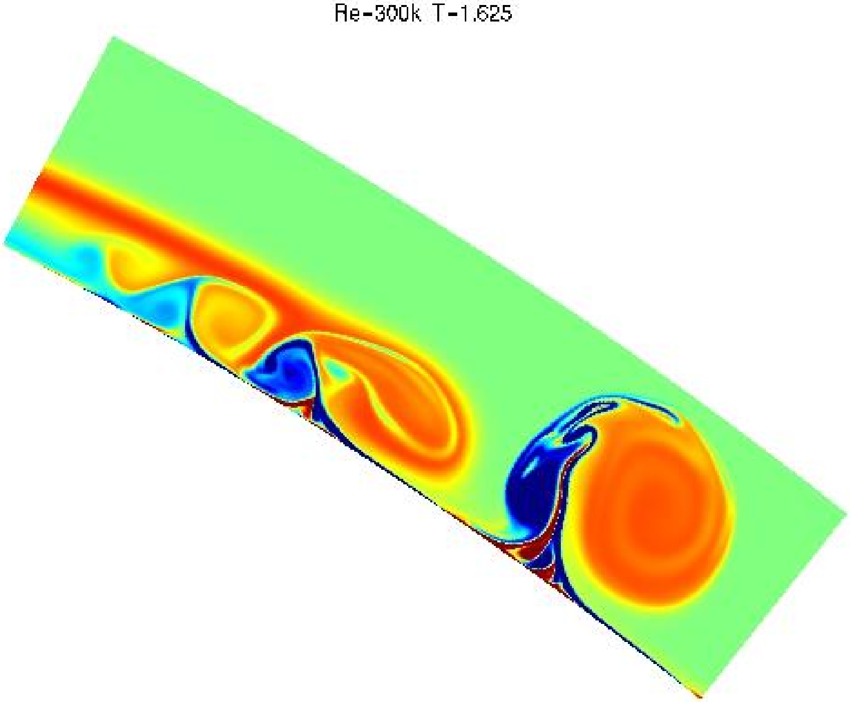
(ii) Pressure can be ‘eliminated’ by projection on divergence-free fields

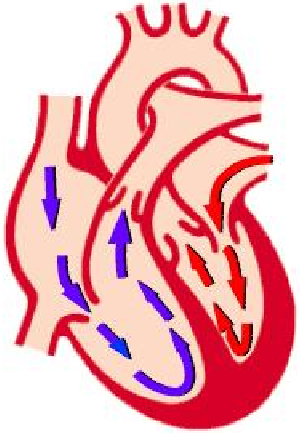
(iii) Pressure can be found from $\boldsymbol{u}$ and $\boldsymbol{f}$ by solving Poisson equations

Flow phenomena modeled by incompressible NSE

Lift, viscous drag, boundary layers & separation, vortex shedding, . . .







Leray-Helmholtz projection $\mathcal{P}$ onto divergence-free fields

$$L^2(\Omega, \mathbb{R}^N) = \nabla H^1(\Omega) \oplus \mathcal{P} L^2(\Omega, \mathbb{R}^N)$$

$$\boldsymbol{v} = \nabla\phi + \boldsymbol{w}, \qquad \langle \nabla\phi, \boldsymbol{w} \rangle = 0.$$

$$\nabla \cdot \boldsymbol{w} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{n} \cdot \boldsymbol{w} = 0 \quad \text{on } \partial\Omega.$$

Write $\quad \boldsymbol{w} = \mathcal{P}\boldsymbol{v}, \qquad \phi = \mathcal{Q}\boldsymbol{v}$

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Commutator identities: For all $\boldsymbol{v} \in H^2(\Omega,\mathbb{R}^N)$ we have

$$\nabla\nabla\cdot\boldsymbol{v} = \nabla\Delta\phi = \Delta\nabla\phi = \Delta(I-\mathcal{P})\boldsymbol{v},$$

$$\Delta\mathcal{P}\boldsymbol{v} = (\Delta - \nabla\nabla\cdot)\boldsymbol{v},$$

$$\mathcal{P}\Delta\boldsymbol{v} = \mathcal{P}(\Delta - \nabla\nabla\cdot)\boldsymbol{v},$$

$$(\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{v} = (I-\mathcal{P})(\Delta - \nabla\nabla\cdot)\boldsymbol{v}.$$

The Laplace-Leray commutator

$$(\Delta\mathcal{P} - \mathcal{P}\Delta)\boldsymbol{v} = (I - \mathcal{P})(\Delta - \nabla\nabla\cdot)\boldsymbol{v} \quad =: \nabla p_{\mathsf{s}}(\boldsymbol{v}).$$

This defines the *Stokes pressure* for any $\boldsymbol{u} \in H^2(\Omega, \mathbb{R}^N)$:

$$p_{\mathsf{s}}(\boldsymbol{u}) = \mathcal{Q}(\Delta - \nabla\nabla\cdot)\boldsymbol{u} \qquad = \mathcal{Q}(-\nabla \times \nabla \times \boldsymbol{u}).$$

- Note $(\Delta - \nabla\nabla\cdot)\boldsymbol{u} \in H(\mathrm{div};\Omega)$, hence p_{s} satsifies the BVP

$$\Delta p_{\mathsf{s}} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{n}\cdot\nabla p_{\mathsf{s}} = \boldsymbol{n}\cdot(\Delta - \nabla\nabla\cdot)\boldsymbol{u} \quad \text{in } H^{-1/2}(\partial\Omega)$$

- p_{s} arises from *circulation of vorticity at the boundary*:

3D weak form: $$\int_\Omega \nabla p_{\mathsf{s}}\cdot\nabla q = \int_{\partial\Omega} (\nabla\times\boldsymbol{u})\cdot(\boldsymbol{n}\times\nabla q) \quad \forall q \in H^1(\Omega)$$

Formula for the Navier-Stokes pressure

Suppose that $\boldsymbol{u} = \mathcal{P}\boldsymbol{u}$ is a strong solution of NSE:

$$\begin{aligned} \boldsymbol{u}_t + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{f} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} \qquad & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 \qquad & \text{in } \Omega \\ \boldsymbol{u} = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Apply $\mathcal{P}$ and note $\mathcal{P}\nabla p = 0$, $\mathcal{P}\Delta \boldsymbol{u} = \Delta \mathcal{P}\boldsymbol{u} - \nabla p_{\mathsf{s}}(\boldsymbol{u})$:

$$\boldsymbol{u}_t + \nu \nabla p_{\mathsf{s}}(\boldsymbol{u}) = \nu \Delta \boldsymbol{u} + \mathcal{P}(\boldsymbol{f} - \boldsymbol{u} \cdot \nabla \boldsymbol{u})$$

Subtracting, we find that (up to spatial constant) *necessarily*

$$p = \nu p_{\mathsf{s}}(\boldsymbol{u}) + \mathcal{Q}(\boldsymbol{f} - \boldsymbol{u} \cdot \nabla \boldsymbol{u}).$$

Pressure formula with inflow/outflow

Suppose we require: $\boldsymbol{u} = \boldsymbol{g}$ on $\partial\Omega$,

where to conserve volume, $\displaystyle\int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{g} = 0 \quad (t \geq 0)$.

Let $\mathcal{R}(\boldsymbol{g})$ solve $\Delta\mathcal{R}(\boldsymbol{g}) = 0$ in Ω, $\boldsymbol{n} \cdot \nabla\mathcal{R}(\boldsymbol{g}) = \boldsymbol{n} \cdot \boldsymbol{g}$ on $\partial\Omega$.

If $\nabla \cdot \boldsymbol{u} = 0$ then $\boldsymbol{u} - \nabla\mathcal{R}(\boldsymbol{g}) = \mathcal{P}(\boldsymbol{u} - \nabla\mathcal{R}(\boldsymbol{g})) = \mathcal{P}\boldsymbol{u}$.

Applying $\mathcal{P}$ to NSE we now find

$$\partial_t(\boldsymbol{u} - \nabla\mathcal{R}(\boldsymbol{g})) + \nu\nabla p_{\mathsf{s}}(\boldsymbol{u}) = \nu\Delta\boldsymbol{u} + \mathcal{P}(\boldsymbol{f} - \boldsymbol{u} \cdot \nabla\boldsymbol{u}),$$

$$p = \nu p_{\mathsf{s}}(\boldsymbol{u}) - \mathcal{R}(\partial_t\boldsymbol{g}) + \mathcal{Q}(\boldsymbol{f} - \boldsymbol{u} \cdot \nabla\boldsymbol{u}).$$

What's a formula good for?

1. Analysis: simple proof of local-time existence and uniqueness for strong solutions, extending Navier-Stokes dynamics to $\nabla \cdot \boldsymbol{u} \neq 0$
2. Numerical analysis: proof of stability and convergence for C^1 FEM *without inf-sup compatibility* for velocity/pressure approximations
3. Numerical methods: formal analysis and improvement of accurate time-stepping schemes, suitable for C^0 *finite element* discretization.

Message: An accurate velocity delivers an accurate pressure.

Local-time well posedness for extended NSE dynamics

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}, \qquad \boldsymbol{u}|_{\partial\Omega} = 0$$

$$p = \nu p_{\mathsf{s}}(\boldsymbol{u}) + \mathcal{Q}(\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u})$$

or: $$\boldsymbol{u}_t + \nu\nabla p_{\mathsf{s}}(\boldsymbol{u}) = \nu\Delta\boldsymbol{u} + \mathcal{P}(\boldsymbol{f} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}), \qquad \boldsymbol{u}|_{\partial\Omega} = 0.$$

- $\boldsymbol{u}_{\mathrm{in}} \in H_0^1 \Rightarrow \exists!$ strong solution $\boldsymbol{u} \in L^2(0,T^*;H^2) \cap H^1(0,T^*;L^2)$
- The divergence $w = \nabla\cdot\boldsymbol{u}$ is smooth and satisfies

$$w_t = \nu\Delta w \quad \text{in } \Omega, \qquad \boldsymbol{n}\cdot\nabla w = 0 \quad \text{on } \partial\Omega.$$

- $w \equiv 0$ implies $\boldsymbol{u}$ solves NSE (cf. Grubb-Solonnikov 1991)

Estimate for the Laplace-Leray commutator

Theorem (LLP '07) Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded C^3 domain. Then $\forall \varepsilon > 0 \;\; \exists C \geq 0$ such that for all $\boldsymbol{u} \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N)$,

$$\int_\Omega |\nabla p_{\mathsf{s}}(\boldsymbol{u})|^2 \leq \left(\frac{1}{2} + \varepsilon\right) \int_\Omega |\Delta \boldsymbol{u}|^2 + C \int_\Omega |\nabla \boldsymbol{u}|^2$$

Basic ingredients in the (elementary) proof:

(i) A parallel-normal decomposition $\boldsymbol{u} = \boldsymbol{u}_{||} + \boldsymbol{u}_{\perp}$

on $\partial\Omega$: $\qquad \nabla \cdot \boldsymbol{u}_{||} = 0, \quad \nabla \times \boldsymbol{u}_{\perp} = 0$

(ii) An orthogonality identity: $\|\Delta \boldsymbol{u}_{||}\|^2 = \|\nabla p_{\mathsf{s}}\|^2 + \|\nabla p_{\mathsf{s}} - \Delta \boldsymbol{u}_{||}\|^2$

(iii) Sharp N2D estimates on tubes: $\|\nabla p_{\mathsf{s}_{||}}\|^2_{\Gamma(r)} \sim \|\nabla p_{\mathsf{s}_{\perp}}\|^2_{\Gamma(r)}$

Planar domains with corners – a negative result

Theorem (E. Cozzi & P.) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a locally straight corner. Given *any* $\beta < 1$ and $C > 0$, there exists $\boldsymbol{u} \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega |\nabla p_{\mathrm{s}}(\boldsymbol{u})|^2 > \beta \int_\Omega |\Delta \boldsymbol{u}|^2 + C \int_\Omega |\nabla \boldsymbol{u}|^2$$

Main ideas of the proof:

(i) Calculate that the optimal $\beta = 1$ for an infinite wedge.

(ii) Scale and localize.

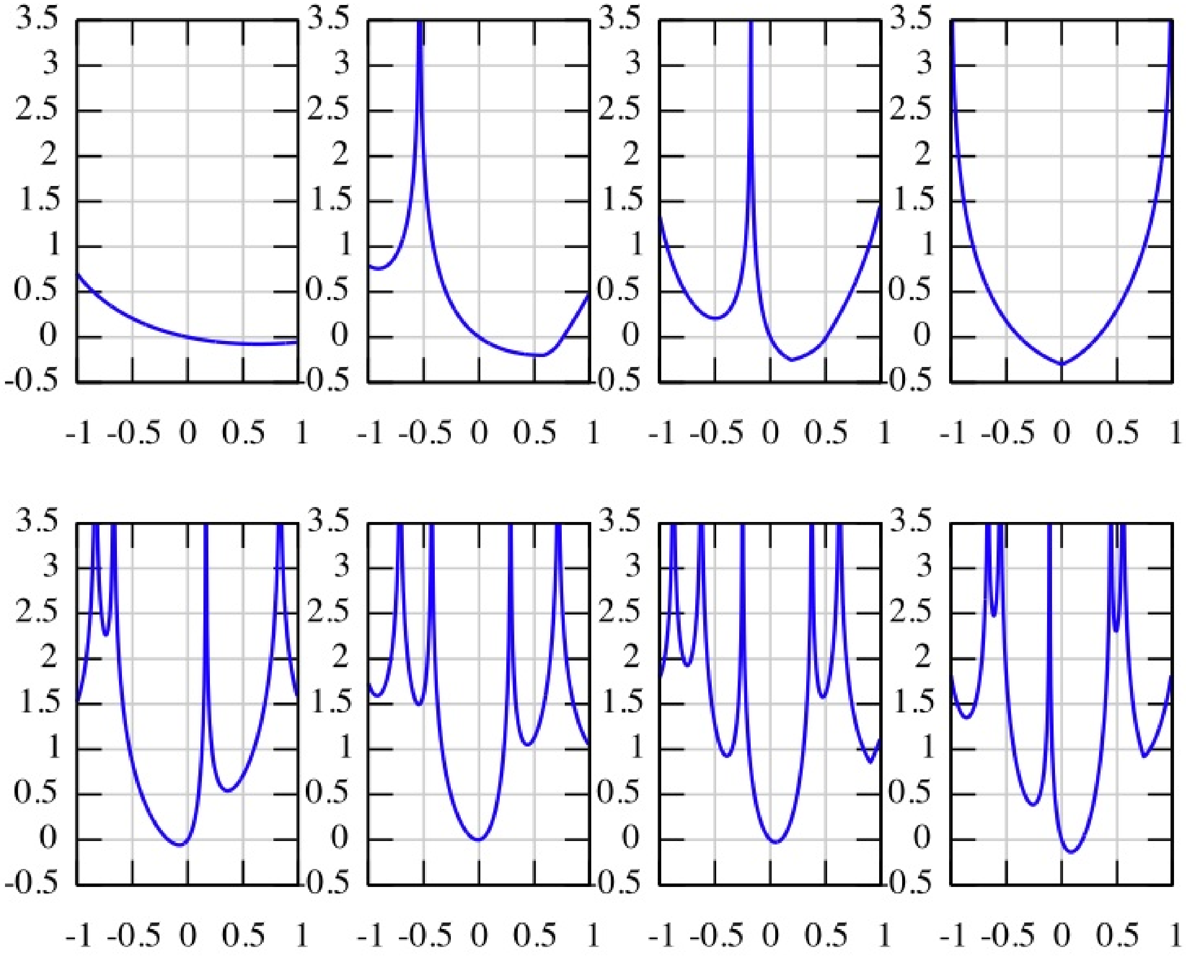
Open Q: Is extended Navier-Stokes dynamics well-posed in Lipschitz (or polygonal) domains?

Optimal estimate in wedge $0 < \theta < \sigma$ with weight $r^{2\alpha}$

$$\int_{\Omega(\sigma)} r^{2\alpha} |\nabla p_{\mathsf{S}}(\boldsymbol{u})|^2 \le \beta \int_{\Omega(\sigma)} r^{2\alpha} |\Delta \boldsymbol{u}|^2,$$

$$\beta = \max_{k,\pm} \operatorname{Re} \left\{ \frac{k^2 + \alpha^2 (1 - e^{-2k\sigma})(1 - \alpha + ik)(1 \pm e^{-(k + i\alpha - 2i)\sigma})}{2k^2 (1 - \alpha)(1 \pm e^{-(k - i\alpha)\sigma})(1 - e^{-2(k + i\alpha - i)\sigma})} \right\}.$$

$\log_{10}\beta$ **vs** α **for** $\sigma = \pi \cdot [0.4, 0.65, 0.85, 1,\ \ 1.2, 1.4, 1.6, 1.8]$



Critical value: $\sigma \approx 1.43\pi$, where $\sigma = \tan\sigma$: $\beta \geq 1\ \forall\alpha$

Conjecture: For convex domains, $\beta < 1$ for $0 < \alpha < \alpha_0$.

• Classic inf-sup condition for steady Stokes flow FEM

Find $\boldsymbol{u} \in X \subset H_0^1(\Omega, \mathbb{R}^N)$, $p \in Y \subset L^2(\Omega)/\mathbb{R}$ so that

$$\begin{aligned} \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle + \langle \nabla p, \boldsymbol{v} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in X, \\ \langle \boldsymbol{u}, \nabla q \rangle &= 0 \qquad \forall q \in Y. \end{aligned}$$

$$\inf_{p \in Y} \sup_{\boldsymbol{v} \in X} \frac{\langle p, \nabla \cdot \boldsymbol{v} \rangle}{\|p\| \|\nabla \boldsymbol{v}\|} \geq c > 0. \qquad (1)$$

Well-known:

- Condition (1) is *necessary and sufficient* for stability of $(p, \boldsymbol{u})$ in $L^2 \times H_0^1$ (the weak NS solution space).
- Many simple finite elements *fail* this condition (e.g., equal-order Lagrange)

Steady Stokes flow in terms of Stokes pressure

Find $\boldsymbol{u} \in X \subset H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$, $p \in Y \subset H^1(\Omega)/\mathbb{R}$ so that

$$\begin{aligned} \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle + \langle \nabla p, \boldsymbol{v} \rangle &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in X, \\ \langle \nabla p, \nabla q \rangle - \langle \nabla \times \boldsymbol{u}, \boldsymbol{n} \times \nabla q \rangle_\Gamma &= \langle f, \nabla q \rangle \quad \forall q \in Y. \end{aligned}$$

Stability of $(p, \boldsymbol{u}) \in H^1 \times (H^2 \cap H_0^1(\Omega, \mathbb{R}^N))$ (strong solution space) is not subject to standard inf-sup velocity-pressure compatibility.

Still a mixed formulation computationally for steady flow

Possible gain in simplicity of discretization with C^0 elements (e.g. for complex fluid-coupled systems) — nonconforming for H^2

• Time discretization: Higher-order projection methods

Chorin-Temam method: 1st-order in time, boundary layers for p

Brown, Cortez, Minion '01: Explain 2nd-order time accuracy

Guermond, Minev, Shen '06: Review many projection methods, FEM

- use higher-order backward time difference formulas, and
- improve boundary-layer accuracy using various strategies, e.g.:
  - ⋆ Pressure approximation (Karniadakis et al '91): extrapolate curl-curl BCs to approximate ∇p^{n+1} in a RHS $\boldsymbol{F}^{n+1}$
  - ⋆ Pressure update (van Kan '86, BCG '89): predict then correct pressure, avoiding curl-curl BCs
  - ⋆ Slip correction (Kim-Moin '86): extrapolate to improve BCs for $\boldsymbol{U}^{n+1}$

A stable (3,2) slip-corrected projection method

Use BD3 formula $$\mathcal{D}_3\boldsymbol{u}^{n+1} = \frac{1}{\Delta t}\Big(\alpha_0\boldsymbol{u}^{n+1} + \sum_{k\geq 1}\alpha_k\boldsymbol{u}^{n+1-k}\Big)$$

Write $\boldsymbol{H}^j = \boldsymbol{U}^j\cdot\nabla\boldsymbol{U}^j$ & extrapolate:

$$\mathcal{E}_3\boldsymbol{H}^{n+1} = 3\boldsymbol{H}^n - 3\boldsymbol{H}^{n-1} + \boldsymbol{H}^{n-2}, \qquad \mathcal{E}_2\phi^{n+1} = 2\phi^n - \phi^{n-1}.$$

1. Given $\boldsymbol{U}^j \approx \boldsymbol{u}(j\Delta t)$ $(j\leq n)$, solve (without any pressure!)

$$\begin{aligned} \frac{1}{\Delta t}\Big(\alpha_0\boldsymbol{u}_*^{n+1} + \sum_{k\geq 1}\alpha_k\boldsymbol{U}^{n+1-k}\Big) &= \nu\Delta\boldsymbol{u}_*^{n+1} + \boldsymbol{f}^{n+1} - \mathcal{E}_3\boldsymbol{H}^{n+1}, \\ \boldsymbol{u}_*^{n+1} &= \boldsymbol{g}^{n+1} + \nabla\mathcal{E}_2\phi^{n+1} \quad \text{on } \Gamma. \end{aligned}$$

2. $\boldsymbol{U}^{n+1} = \boldsymbol{u}_*^{n+1} - \nabla\phi^{n+1}$, $\Delta\phi^{n+1} = \nabla\cdot\boldsymbol{u}^{n+1}$, $\boldsymbol{n}\cdot\nabla\phi^{n+1} = 0.$

Formal accuracy: Apply $\mathcal{P}$ and recall

$$\mathcal{P}\boldsymbol{u}_*^j = \mathcal{P}\boldsymbol{U}^j = \boldsymbol{U}^j - \nabla\mathcal{R}(\boldsymbol{g}^j),$$

$$\mathcal{P}\Delta\boldsymbol{u}_*^{n+1} = \Delta\mathcal{P}\boldsymbol{u}_*^{n+1} - \nabla p_{\mathsf{s}}(\boldsymbol{u}_*^{n+1}) = \Delta\boldsymbol{U}^{n+1} - \nabla p_{\mathsf{s}}(\boldsymbol{U}^{n+1}) :$$

$$\mathcal{D}_3\boldsymbol{U}^{n+1} + \nabla\bar{p}^{n+1} = \nu\Delta\boldsymbol{U}^{n+1} + \boldsymbol{f}^{n+1} - \mathcal{E}_3\boldsymbol{H}^{n+1},$$

$$\bar{p}^{n+1} = \nu p_{\mathsf{s}}(\boldsymbol{U}^{n+1}) - \mathcal{R}(\mathcal{D}_3\boldsymbol{g}^{n+1}) + \mathcal{Q}\boldsymbol{f}^{n+1} - \mathcal{Q}\mathcal{E}_3\boldsymbol{H}^{n+1},$$

$$\boldsymbol{U}^{n+1} = \boldsymbol{g}^{n+1} - \nabla(\phi^{n+1} - \mathcal{E}_2\phi^{n+1}) \quad \text{on } \Gamma.$$

$$\left(\frac{\alpha_0}{\Delta t} - \nu\Delta\right)(\phi^{n+1} - \mathcal{E}_2\phi^{n+1}) = \bar{p}^{n+1} - 2\bar{p}^n + \bar{p}^{n-1} = O(\Delta t^2),$$

hence $\boldsymbol{U}^{n+1} - \boldsymbol{g}^{n+1} = O(\Delta t^3)$ formally and the scheme is formally 3rd-order accurate overall.

A stable (3,2) pressure approximation method

Update intermediate velocity $\boldsymbol{u}_*^n$ via $(\boldsymbol{H}^n = \boldsymbol{u}_*^n \cdot \nabla \boldsymbol{u}_*^n)$

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{f}^{n+1} - \mathcal{E}_3 \boldsymbol{H}^{n+1} - \frac{1}{\Delta t} \sum_{j \geq 1} \alpha_j \boldsymbol{u}_*^{n+1-j}, \\ \bar{P} &= \nu p_{\mathrm{s}}(\mathcal{E}_2 \boldsymbol{u}_*^{n+1}) - \frac{\alpha_0}{\Delta t} \mathcal{R}(\boldsymbol{g}^{n+1}) + \mathcal{Q}\boldsymbol{F} \end{aligned}$$

$$\begin{aligned} \frac{\alpha_0}{\Delta t} \boldsymbol{u}_*^{n+1} - \nu \Delta \boldsymbol{u}_*^{n+1} &= -\nabla \bar{P} + \boldsymbol{F} \\ \boldsymbol{u}_*^{n+1} &= \boldsymbol{g}^{n+1} \quad \text{on } \Gamma \end{aligned}$$

The div-free velocity and consistent pressure are

$$\begin{aligned} \boldsymbol{U}^{n+1} &= \boldsymbol{u}_*^{n+1} - \nabla \phi^{n+1}, \ \Delta \phi^{n+1} = \nabla \cdot \boldsymbol{u}^{n+1}, \ \boldsymbol{n} \cdot \nabla \phi^{n+1} = 0, \\ p^{n+1} &= \bar{P} - \nu \nabla \cdot \boldsymbol{u}_*^{n+1}. \quad \text{(Improves over previous methods)} \end{aligned}$$

(3,3) pressure update method (cf. BCG '89, Ren et al '05)

1. Find $\boldsymbol{u}_*^{n+1}$ to solve

$$\frac{1}{\Delta t}\Big(\alpha_0 \boldsymbol{u}_*^{n+1} + \sum_{k\geq 1} \alpha_k \boldsymbol{U}^{n+1-k}\Big) + \nabla \mathcal{E}_3 p^{n+1}$$

$$= \ \nu \Delta \boldsymbol{u}_*^{n+1} + \boldsymbol{f}^{n+1} - \mathcal{E}_3 \boldsymbol{H}^{n+1},$$

$$\boldsymbol{u}_*^{n+1} = \boldsymbol{g}^{n+1} \quad \text{on } \Gamma.$$

2. $\boldsymbol{U}^{n+1} = \boldsymbol{u}_*^{n+1} - \nabla\phi^{n+1}$, $\Delta\phi^{n+1} = \nabla\cdot\boldsymbol{u}_*^{n+1}$, $\boldsymbol{n}\cdot\nabla\phi^{n+1} = 0.$

3. Update pressure via

$$p^{n+1} = \mathcal{E}_3 p^{n+1} + \left(\frac{\alpha_0}{\Delta t} - \nu\Delta\right)\phi^{n+1}$$

Single mode Stokes test problem in strip $-1 < x < 1$

$$\partial_t \boldsymbol{u} + \nabla p_{\mathsf{S}}(\boldsymbol{u}) = \boldsymbol{\Delta u}, \qquad \boldsymbol{u}|_{x=\pm 1} = 0$$

$$\boldsymbol{u}(t,x,y) = e^{i\xi y - \sigma t}(u(x,\xi), iv(x,\xi))$$

For the case of odd symmetry: $p = e^{i\xi y - \sigma t} \sinh \xi x$,

$$u(x) = A\left(\frac{\cosh \xi x}{\cosh \xi} - \frac{\cos \mu x}{\cos \mu}\right), \qquad A = \frac{\xi \cosh \xi}{\xi^2 + \mu^2},$$

$$v(x) = B\left(\frac{\sinh \xi x}{\sinh \xi} - \frac{\sin \mu x}{\sin \mu}\right), \qquad B = \frac{\xi \sinh \xi}{\xi^2 + \mu^2}.$$

$$\xi \tanh \xi + \mu \tan \mu = 0, \qquad \sigma = \xi^2 + \mu^2.$$

Usually take $\xi = 1$, $\mu \approx 2.883356$, $\sigma \approx 9.314$.

Single-mode time-discrete stability tests

$$A_0\boldsymbol{u}^{n+1} + A_1\boldsymbol{u}^{n} + \ldots + A_k\boldsymbol{u}^{n+1-k} = 0.$$

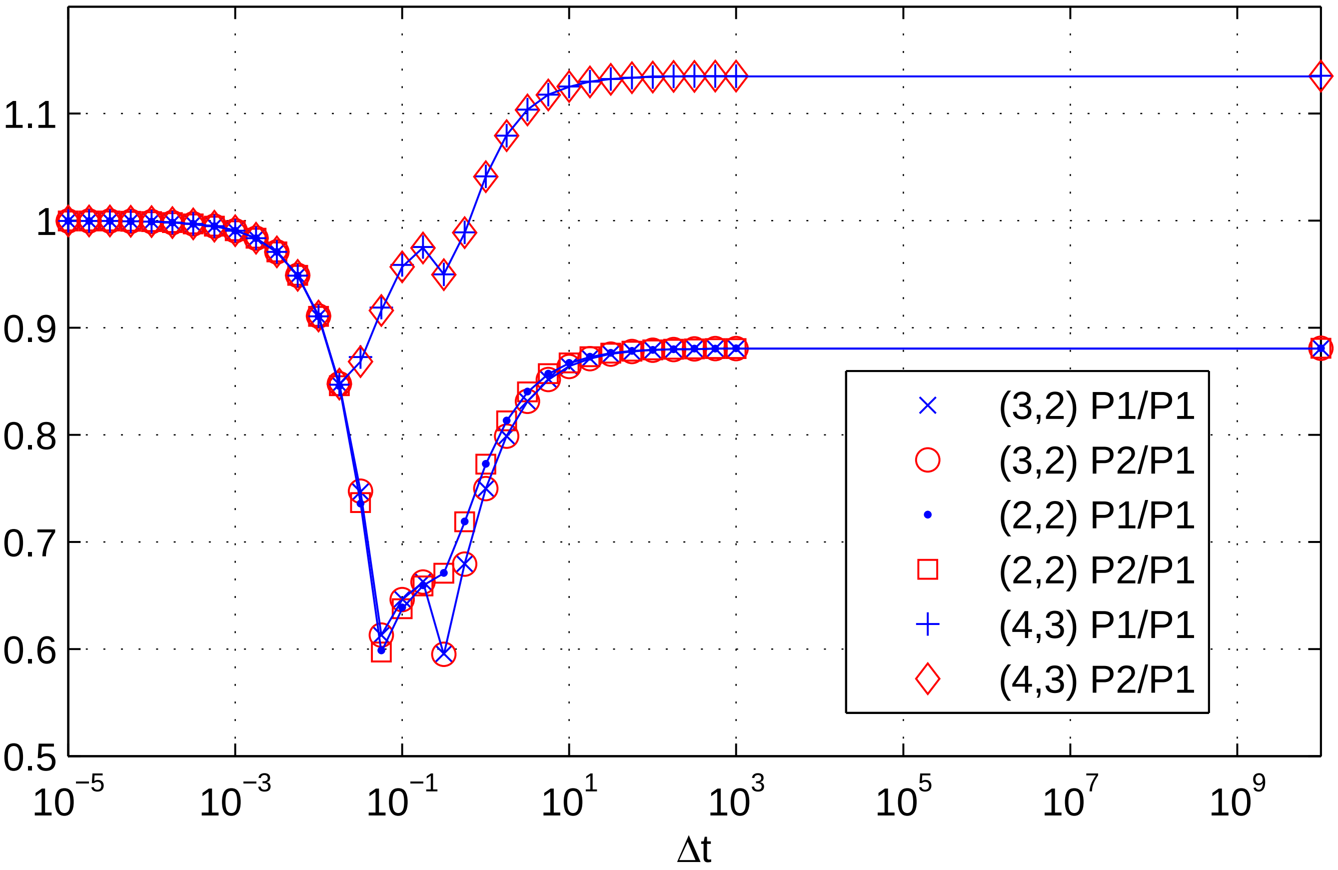
Look for $\boldsymbol{u}^n = \kappa^n\boldsymbol{u}$ with

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -A_3 & -A_2 & -A_1 \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \kappa\boldsymbol{u} \\ \kappa^2\boldsymbol{u} \end{pmatrix} = \kappa\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \kappa\boldsymbol{u} \\ \kappa^2\boldsymbol{u} \end{pmatrix}.$$

Solve for eigenvalues κ using Matlab eigs.

Plot max $|\kappa|$ vs Δt for various combinations of finite-element velocity/pressure pairs and orders of accuracy.

Slip-correction schemes, single-mode stability test

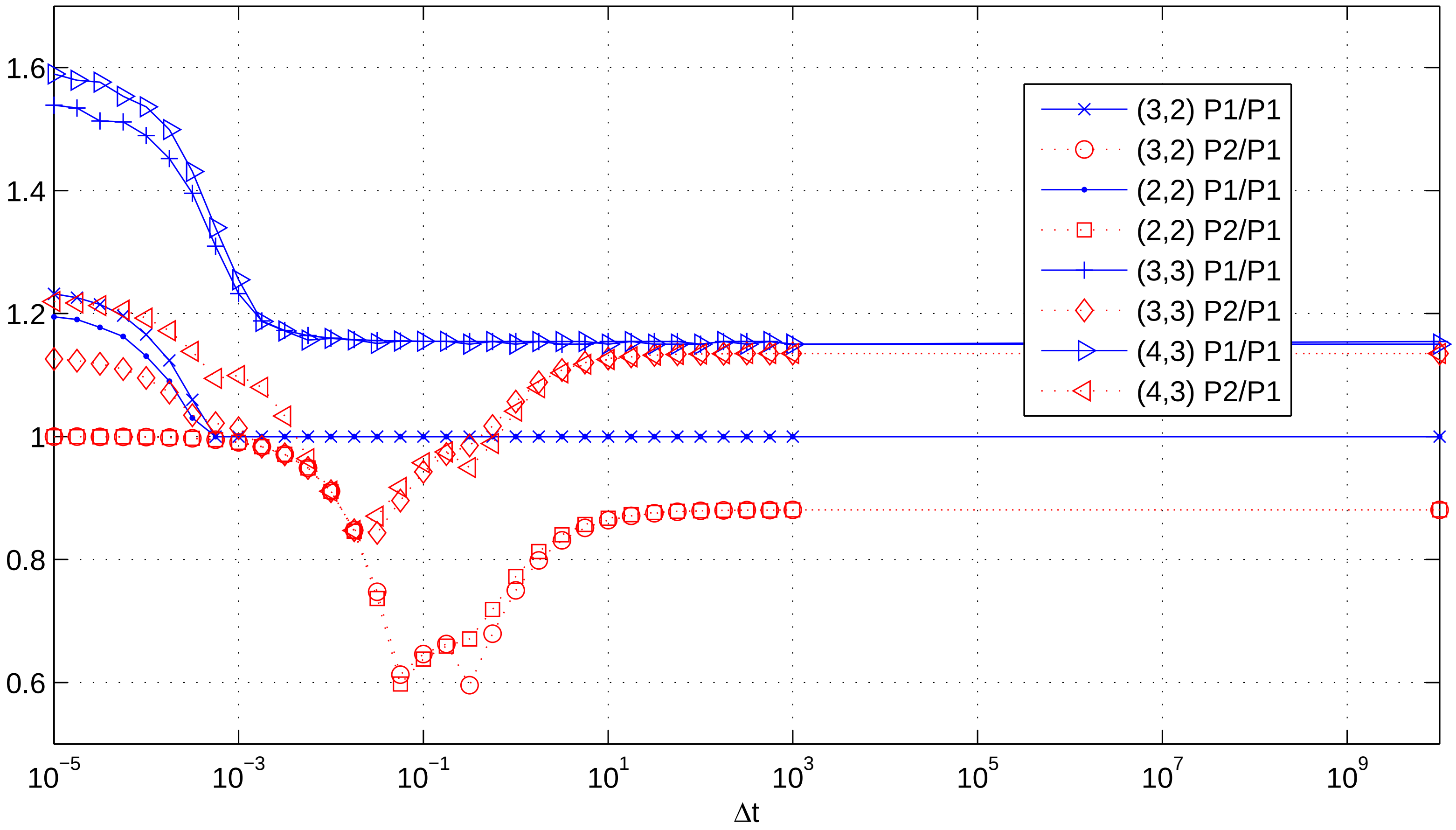


1: max $|\kappa|$ vs Δt. 30 elements for each var. Solid lines: space-continuous theory.

(2,2), (3,2): unconditional stability. (cf. Leriche et al '06 w/PA)

(3,3), (4,3): stability for $\Delta t < \Delta t_c$ independent of h, ξ. (smooth?)

Pressure update schemes, single-mode stability test



2: max $|\kappa|$ vs Δt. 30 elements for each var. Solid lines: interpolated.

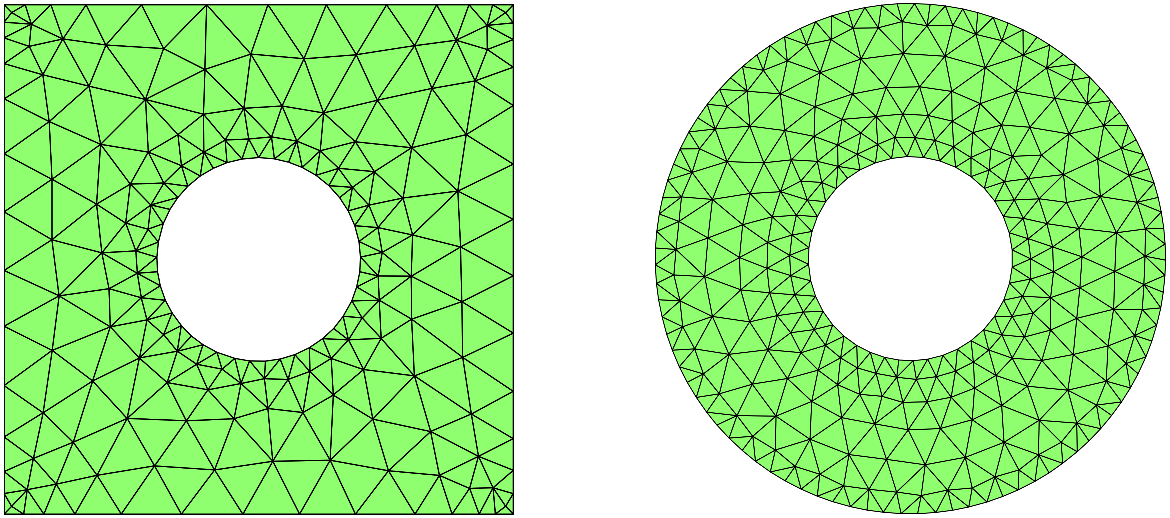
All P1/P1 PU schemes: unstable. P2/P1 velocity-pressure pairs: (2,2), (3,2): unconditional stability. (3,3),(4,3): stability window

2D test problem: stability & accuracy

$\Omega =$ square $[-1,1]^2-$ hole, or ring. $\nu = 2$, $t = 2$

$$\boldsymbol{u}(t) = g(t) \begin{pmatrix} \cos^2(\pi x/2)\sin(\pi y) \\ -\sin(\pi x)\cos^2(\pi y/2) \end{pmatrix},$$

$$p(t) = g(t)\cos(\pi x/2)\sin(\pi y/2).$$



2D test: stability in domains with & without corners

$\Delta t_c \quad \backslash \quad N$	0	1	2
(3,3) P1/P1	0.045	0.015	0.00067
(3,3) P2/P2	0.00027	0.000073	0.000018
(4,3) P1/P1	0.048	0.017	0.00074
(4,3) P2/P2	0.00029	0.000080	0.000020

Largest time step for linear stability of PA schemes in a square with hole.

$N = \#$ of refinements from grid in figure. Diffusive constraint on Δt.

$\Delta t_c \quad \backslash \quad N$	0	1	2
(3,3) P1/P1	0.022	0.014	0.010
(3,3) P2/P2	0.007	0.005	0.005
(4,3) P1/P1	0.024	0.017	0.014
(4,3) P2/P2	0.011	0.011	0.010

Largest time step for linear stability of PA schemes in annulus.

$N = \#$ of refinements from grid in figure. No diffusive constraint on Δt.

Accuracy: slip-corrected P4 FEM, square with hole

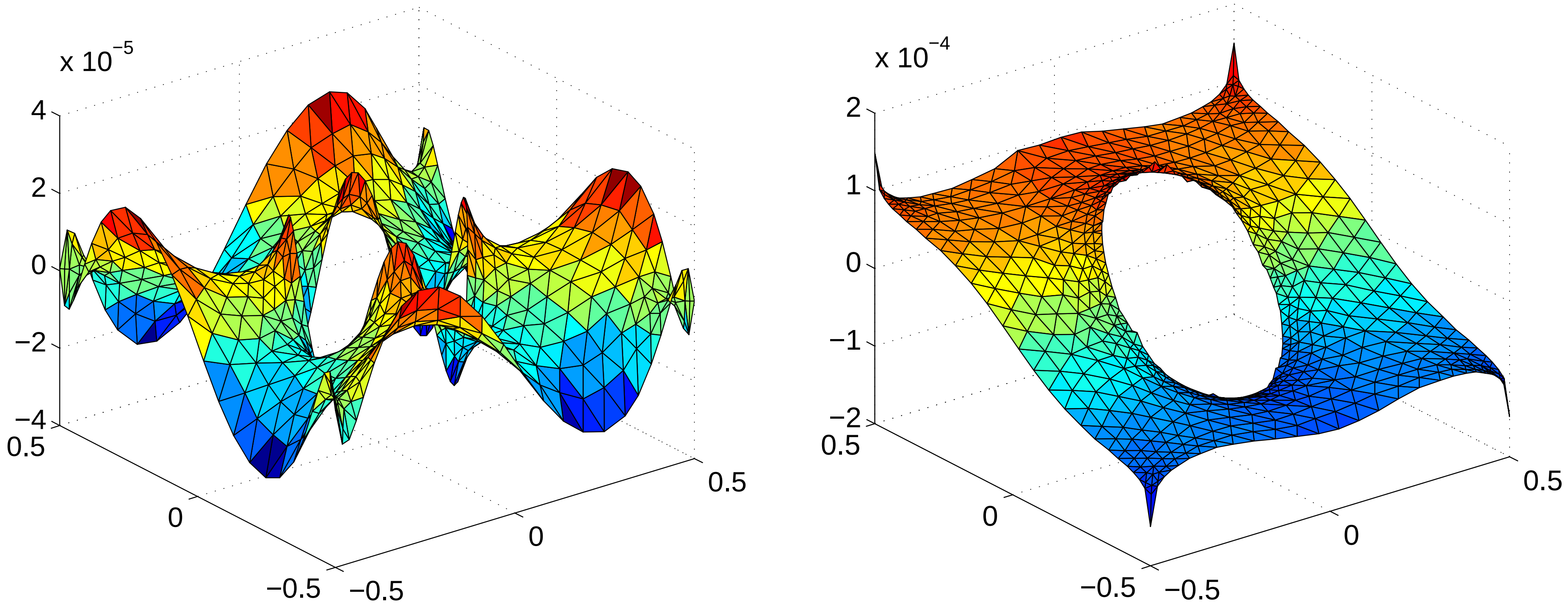
$-\log_{10} E$ (& local order α) vs. Δt

$E \quad \backslash \quad \Delta t$	0.02	0.01	0.005
$\|p-\hat{p}_h\|_{L^2}$	4.18 (2.71)	5.08 (2.98)	5.99 (3.02)
$\|\nabla(p-\hat{p}_h)\|_{L^2}$	3.47 (3.73)	4.39 (3.04)	5.17 (2.62)
$\|u-u_h\|_{L^\infty}$	5.75 (2.73)	6.65 (3.02)	7.57 (3.05)
$\|\nabla(u-u_h)\|_{L^\infty}$	3.94 (3.22)	4.95 (3.35)	5.83 (2.92)

Top: (3,2) SC scheme.   Bottom: (2,1) SC scheme.

$E \quad \backslash \quad \Delta t$	0.02	0.01	0.005
$\|p-\hat{p}_h\|_{L^2}$	2.04 (1.97)	2.64 (2.02)	3.26 (2.04)
$\|\nabla(p-\hat{p}_h)\|_{L^2}$	1.55 (1.96)	2.15 (2)	2.76 (2.02)
$\|u-u_h\|_{L^\infty}$	3.61 (1.96)	4.22 (2.02)	4.84 (2.05)
$\|\nabla(u-u_h)\|_{L^\infty}$	2.19 (1.97)	2.8 (2.01)	3.41 (2.04)

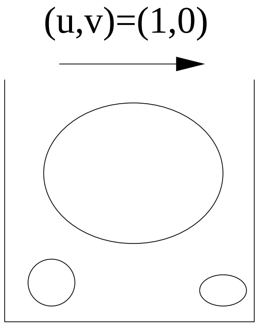
Pressure error for square with hole, (3,2) schemes

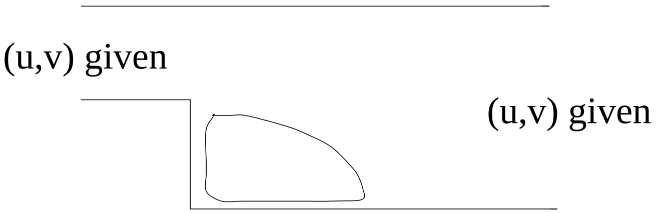


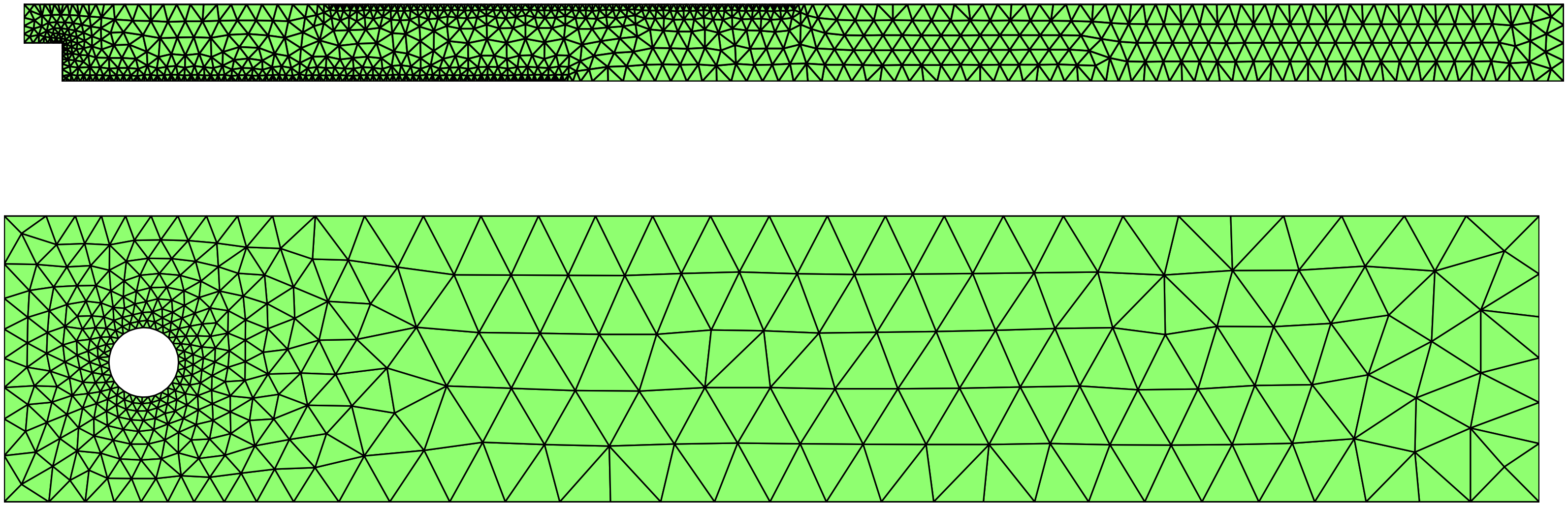
Pressure error for the (3,2) PA (left) and SC (right) schemes.

$\Delta t = 0.02$. 1296 P4 elements for each variable.
Only values at vertices of triangles are used in the plots.

Benchmark tests with equal-order C^0 finite elements

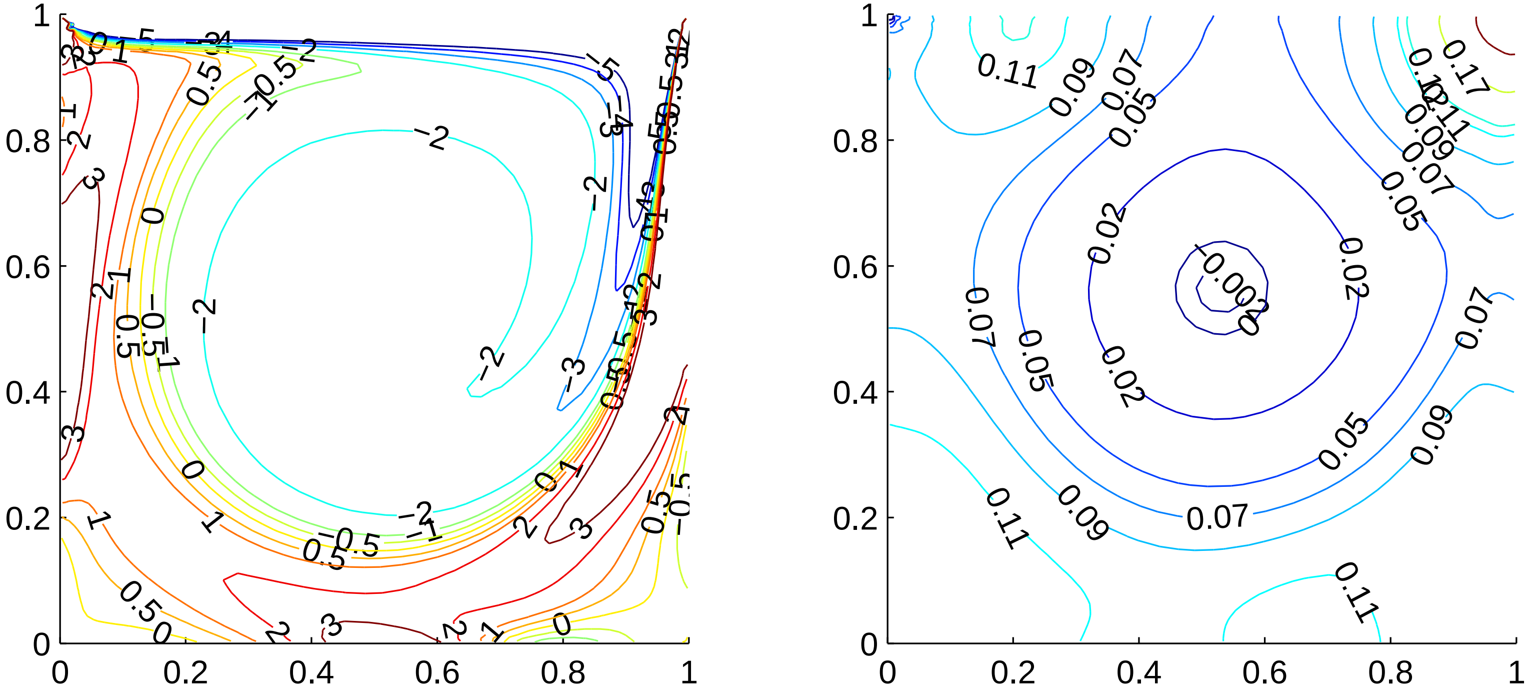






3: Mesh used for backward facing step flow when $\nu = 1/600$ and for flow past a cylinder when $\nu = 1/1000$.

Driven Cavity, Re=1000, t=50

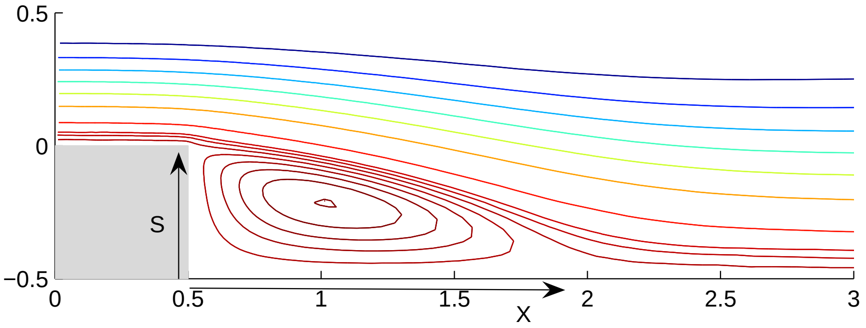


$2 \times 64 \times 64$ stretched rectangular grid
8192 piecewise linear C^0 elements for each variable.

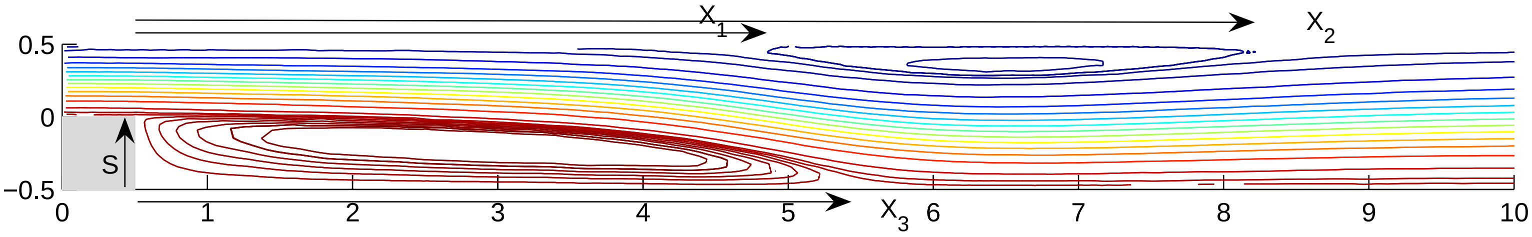
$h_{\min} = 0.00594$, $\Delta t = 0.006$, (3,2) slip-corrected scheme.

Left: vorticity contours. Right: pressure contours.

Backward-facing step

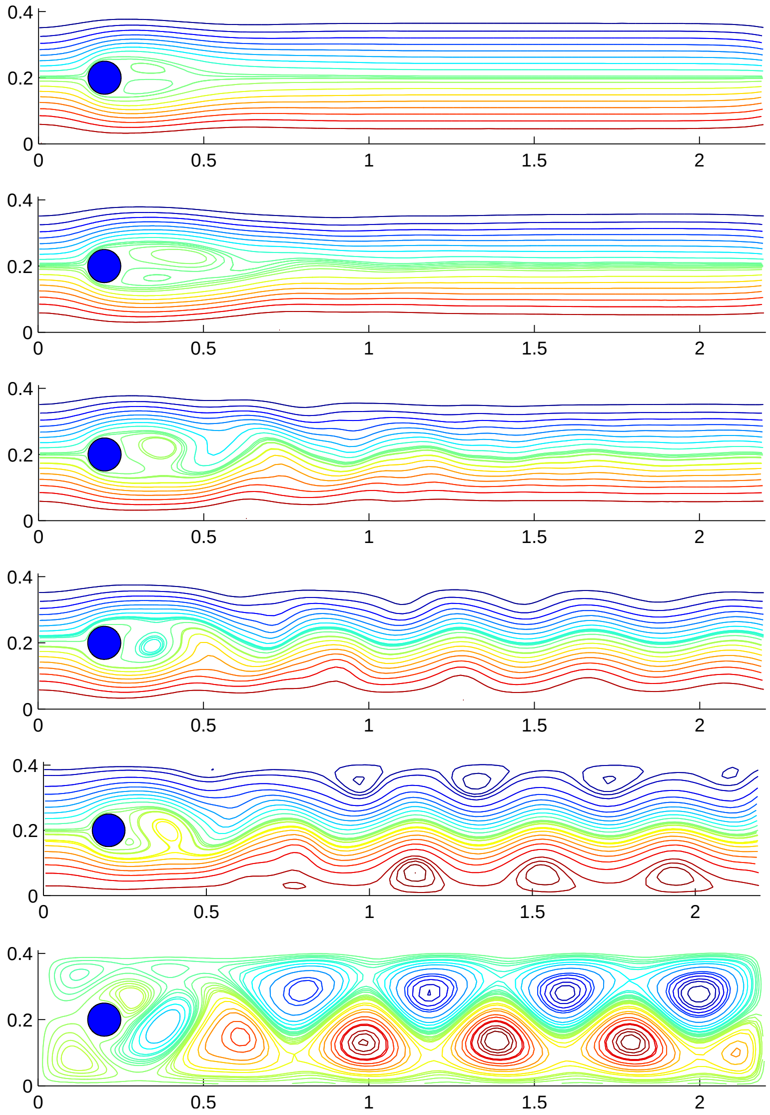


(3,2) SC scheme, 6640 P1 elements for each variable (dof=3487).
$\nu = 1/100,\ t = 20,\ h_{\min} = 0.00783,\ \Delta t = 0.006.$

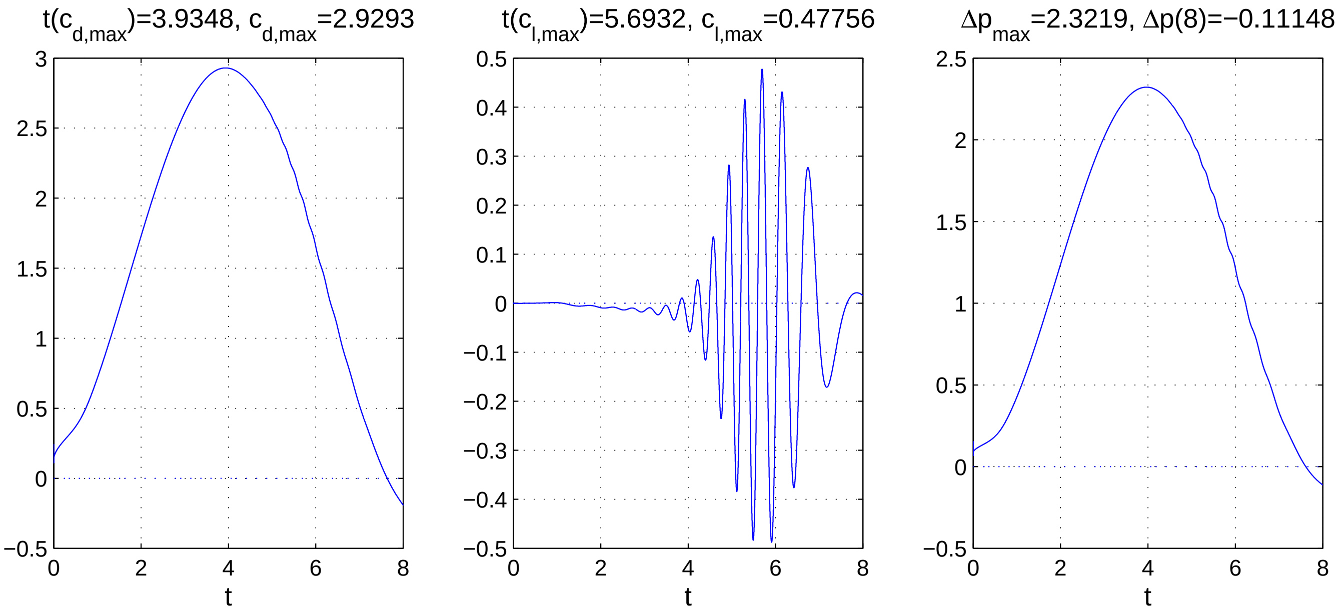


(3,2) SC scheme, 1700 P2 elements for each variable (dof=3925).
$\nu = 1/600,\ t = 120,\ h_{\min} = 0.0186,\ \Delta t = 0.003.$

Flow past cylinder, $\nu = 1/1000$



Drag, lift, pressure drop



(3,2) SC scheme

763 isoparametric P4 elements (dof=6322) for each variable.

Summary

- The Navier-Stokes pressure for strong solutions in C^3 domains is determined by the current velocity and data by

$$p = \nu p_{\mathrm{s}}(\boldsymbol{u}) - \mathcal{R}(\partial_t \boldsymbol{g}) + \mathcal{Q}(\boldsymbol{f} - \boldsymbol{u} \cdot \nabla \boldsymbol{u}).$$

- The Laplace-Leray commutator is $\nabla p_{\mathrm{s}}(u)$ and is strictly controlled by $\Delta \boldsymbol{u}$ at leading order, when $\boldsymbol{u} = 0$ on the boundary and Ω is C^3.
- Improved accuracy (esp. for pressure) in numerical computations
- Improved flexibility of finite-element methods with PA and SC (but *not* PU) projection methods — classical inf-sup condition may not need to be respected.
- Short-time wellposedness extends to $\nabla \cdot \boldsymbol{u} \neq 0$ with simple proof

Lots of open questions

- Well-posedness for domains with corners?
- Weaker solutions?
- Analysis of C^0 finite element methods?
- Improved handling of corners?
- Boundary layers/zero-viscosity limits?
- Other boundary conditions: free boundary, fluid/membrane...?
  ($\exists$ numerical work on open BCs by Jie Liu...)
- Complex fluids? ($\exists$ stability theorem for ideal MHD...)

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Thank you!