The quintic NLS as the mean field limit of a Boson gas with three-body interactions

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Outline of the talk

1. Introduction (a brief review of the derivation of the cubic NLS)
2. Statement of our results (the derivation of the quintic NLS)
3. Uniqueness of solutions to the GP hierarchy
4. Revisiting the GP hierarchy

1 Introduction

The mathematical analysis of interacting Bose gases is a research area that has seen remarkable progress in recent years.

1. One direction of research in this area is related to **proving, in a mathematically rigorous way, Bose-Einstein condensation for such systems** (*Lieb, Seiringer, Yngvason, and their collaborators*).

2. A related direction of research concerns **the derivation of the cubic nonlinear Schrödinger equation as the mean field limit of a boson gas with two-body interactions** (*Elgart-Erdös-Schlein-Yau, Adami-Bardos-Golse-Teta, Klainerman-Machedon, Kirkpatrick-Schlein-Staffilani*).

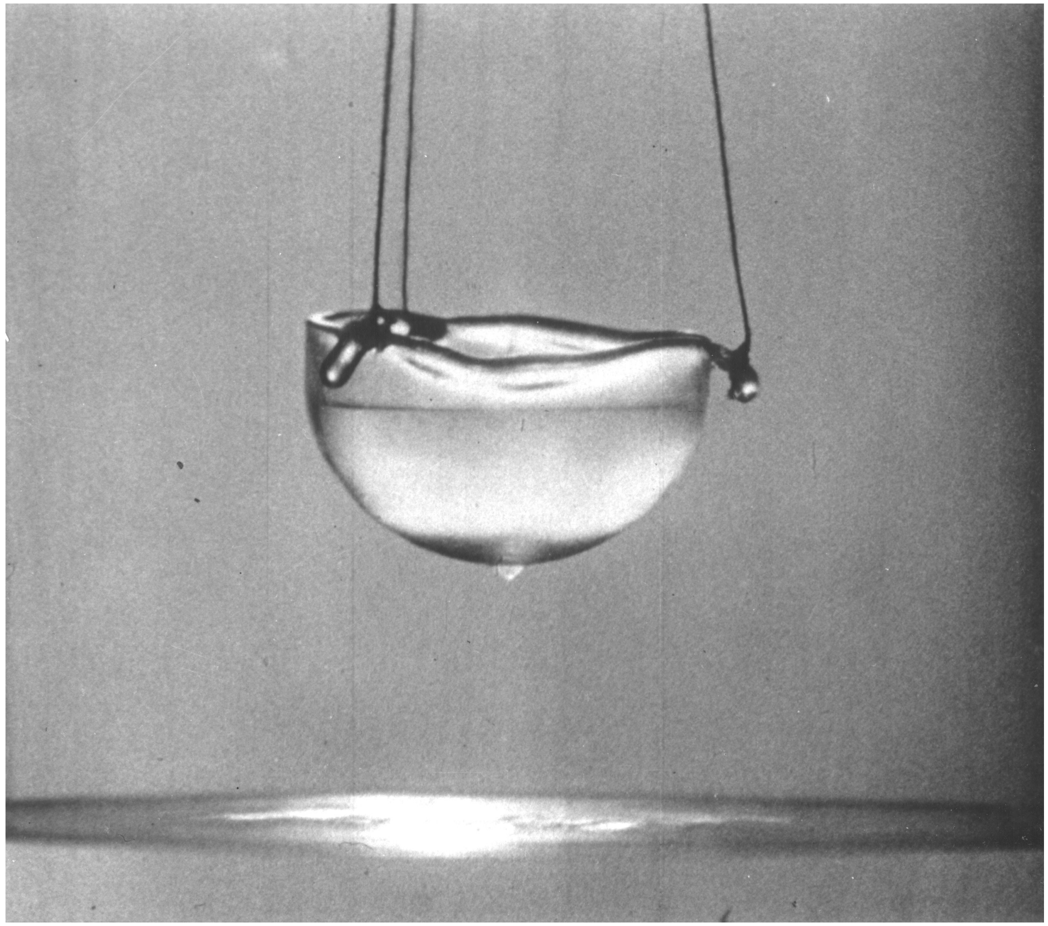

Figure 1: The liquid helium in the superfluid phase; Photo is a courtesy of Wikipedia

A brief review of the work of Erdös-Schlein-Yau on the derivation of the cubic NLS

In the series of works *Erdös, Schlein, and Yau* [2006-07] developed a method to obtain the mean field limit of the manybody quantum dynamics of interacting Bose gases.

The starting point of this work is **a system of N bosons whose dynamics is generated by the Hamiltonian**

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\leq i<j\leq N} N^{d\beta}V(N^{\beta}(x_i - x_j)), \tag{1.1}$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$, whose elements $\Psi(x_1, \ldots, x_N)$ are fully symmetric with respect to permutations of the arguments x_j.

The solutions of the Schrödinger equation

$$i\partial_t \Psi_{N,t} \;=\; H_N \, \Psi_{N,t}$$

with initial condition $\Psi_N \in \mathcal{H}_N$ determine:

- **the N-particle density matrix**

$$\gamma_N(t;\underline{x}_N;\underline{x}'_N) \;=\; \overline{\Psi_{N,t}(\underline{x}_N)}\Psi_{N,t}(\underline{x}'_N),$$

- **and its k-particle marginals**

$$\gamma^{(k)}_{N,t}(\underline{x}_k;\underline{x}'_k) \;=\; \int d\underline{x}_{N-k}\gamma_N(t;\underline{x}_k,\underline{x}_{N-k};\underline{x}'_k,\underline{x}_{N-k})\,,$$

  for $k = 1,\dots,N$.

Here

$$\underline{x}_k = (x_1,\dots,x_k),$$
$$\underline{x}_{N-k} = (x_{k+1},\dots,x_N).$$

The BBGKY hierarchy is given by

$$i\partial_t \gamma_{N,t}^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_{N,t}^{(k)}] + \frac{1}{N} \sum_{i<j}^{k} [V_N(x_i - x_j), \gamma_{N,t}^{(k)}]$$

$$+ \frac{(N-k)}{N} \sum_{j=1}^{k} \mathrm{Tr}_{k+1}[V_N(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}]$$

where

$$V_N(x) := N^{d\beta} V(N^\beta x) .$$

In the limit $N \to \infty$, the sums weighted by combinatorial factors have the following size:

- In the first interaction term on the rhs, $\frac{k^2}{N} \to 0$ for every fixed k and sufficiently small β.
- For the second term $\frac{N-k}{N} \to 1$, for every fixed k and $V_N(x_i - x_j) \to b_0 \delta(x_i - x_j)$, with $b_0 = \int dx\, V(x)$.

Accordingly, **in the limit $N \to \infty$, one obtains the infinite GP hierarchy**

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_{\infty,t}^{(k)}] + b_0 \sum_{j=1}^{k} B_{j;k+1} \gamma_{\infty,t}^{(k+1)} \qquad (1.2)$$

where the **"contraction operator"** is given via

$$\begin{aligned} & B_{j;k+1} \gamma_{\infty,t}^{(k+1)}(x_1, \dots, x_k; x_1', \dots, x_k') \qquad (1.3) \\ & := \int dx_{k+1} dx_{k+1}' \\ & \quad \big[\delta(x_j - x_{k+1}) \delta(x_j - x_{k+1}') - \delta(x_j' - x_{k+1}) \delta(x_j' - x_{k+1}') \big] \\ & \quad \gamma_{\infty,t}^{(k+1)}(x_1, \dots, x_j, \dots, x_k, x_{k+1}; x_1', \dots, x_k', x_{k+1}') \, . \end{aligned}$$

On the other hand, it is easy to see that

$$\gamma^{(k)}_{\infty,t} = \big|\,\phi_t\,\big\rangle\big\langle\,\phi_t\,\big|^{\otimes k} := \prod_{j=1}^{k} \overline{\phi_t(x_j)}\,\phi_t(x'_j)$$

is a solution of (1.2) iff ϕ_t satisfies the cubic NLS

$$i\partial_t\phi_t + \Delta_x\phi_t - b_0\,|\phi_t|^2\,\phi_t = 0$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Roughly speaking, the method of *Erdös, Schlein, and Yau* for deriving the cubic NLS justifies the heuristic explained above and it consists of the following two steps:

(i) **Deriving the Gross-Pitaevskii (GP) hierarchy as the limit as N $\to \infty$ of the BBGKY hierarchy of marginal density matrices for particle number N**, for a scaling where the particle interaction potential tends to a delta distribution.

(ii) **Proving uniqueness of solutions for the GP hierarchy,** which implies that for factorized initial data, the solutions of the GP hierarchy are determined by a cubic NLS.

The second step is the most involved one.

Why is it difficult to prove uniqueness?

Let us fix a positive integer r. One can express the solution $\gamma^{(r)}$ to the GP hierarchy as:

$$\begin{aligned}
\gamma^{(r)}(t_r,\cdot) &= \int_0^{t_r} e^{i(t_r-t_{r+1})\Delta_\pm^{(r)}} B_{r+1}(\gamma^{(r+1)}(t_{r+1}))\,dt_{r+1} \\
&= \int_0^{t_r}\int_0^{t_{r+1}} e^{i(t_r-t_{r+1})\Delta_\pm^{(r)}} B_{r+1} e^{i(t_{r+1}-t_{r+2})\Delta_\pm^{(r+1)}} B_{r+2}(\gamma^{(r+2)}(t_{r+2}))\,dt_{r+1}\,dt_{r+2} \\
&= \ldots \\
&= \int_0^{t_r}\ldots\int_0^{t_{r+n}} J^r(\underline{t}_{r+n})\,dt_{r+1}\ldots dt_{r+n},
\end{aligned} \tag{1.4}$$

where

$$\begin{aligned}
B_{r+1} &= \sum_{j=1}^{r} B_{j;r+1}, \\
\underline{t}_{r+n} &= (t_r, t_{r+1}, \ldots, t_{r+n}), \\
J^r(\underline{t}_{r+n}) &= e^{i(t_r-t_{r+1})\Delta_\pm^{(r)}} B_{r+1}\ldots e^{i(t_{r+(n-1)}-t_{r+n})\Delta_\pm^{(r+(n-1))}} B_{r+n}(\gamma^{(r+n)}(t_{r+n})).
\end{aligned}$$

Since the interaction term involves the sum, the iterated Duhamel's formula has $r(r+1)\ldots(r+n-1)$ terms.

Methods for proving uniqueness of solutions to the GP:

1. *Erdös, Schlein and Yau* developed very sophisticated techniques based on **Feynman graph expansions**.

2. In 2008 *Klainerman and Machedon* introduced an alternative method for proving uniqueness based on:

   - the **"board game argument"** and
   - the use of certain **space-time norms**.

We note that many **interesting results on related topics** have been obtained recently, e.g. the works of

- *Fröhlich-Graffi-Schwarz, Rodnianski-Schlein, Grillakis-Machedon-Margetis, Anapolitanos-Sigal, Abou Salem*
- *Grillakis-Margetis*
- *Fröhlich-Knowles*
- *Pickl*

Also we recall the **classical papers** of:

- *Hepp, Spohn* and *Ginibre-Velo*

It is natural to ask whether a quintic nonlinear Schrödinger equation can be derived as the mean field limit of the manybody quantum dynamics of interacting Bose gases.

With *Thomas Chen* we answered the above question in space dimensions $d = 1, 2$. Now we will discuss some details of **the derivation of a quintic nonlinear Schrödinger equation as the mean field limit of a Bose gas with 3-particle interactions**.

2 Statement of our result

We consider **a system of N bosons whose dynamics is generated by the Hamiltonian**

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N^2}\sum_{1\leq i<j<k\leq N} N^{2d\beta}V(N^\beta(x_i-x_j),N^\beta(x_i-x_k))\,,\tag{2.1}$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$, whose elements $\Psi(x_1,\ldots,x_N)$ are fully symmetric with respect to permutations of the arguments x_j.

We assume that the translation-invariant three-body potential V has the properties

$$V \geq 0 \quad , \quad V(x,y) = V(y,x) \quad , \quad V \in W^{3,p}(\mathbb{R}^{2d})$$

for $2d < p \leq \infty$.

Since

$$\begin{aligned} U(x_1-x_2,x_2-x_3,x_1-x_3) &= U(x_1-x_2,-(x_1-x_2)+(x_1-x_3),x_1-x_3) \\ &\equiv V(x_1-x_2,x_1-x_3)\,, \end{aligned}\tag{2.2}$$

every translation invariant three-body interaction potential U can be written as (2.2).

The solutions of the Schrödinger equation

$$i\partial_t \Psi_{N,t} \;=\; H_N \, \Psi_{N,t}$$

with initial condition $\Psi_N \in \mathcal{H}_N$ determine:

- **the N-particle density matrix**

$$\gamma_N(t;\underline{x}_N;\underline{x}'_N) \;=\; \overline{\Psi_{N,t}(\underline{x}_N)}\Psi_{N,t}(\underline{x}'_N),$$

- **and its k-particle marginals**

$$\gamma^{(k)}_{N,t}(\underline{x}_k;\underline{x}'_k) \;=\; \int d\underline{x}_{N-k}\gamma_N(t;\underline{x}_k,\underline{x}_{N-k};\underline{x}'_k,\underline{x}_{N-k})\,,$$

  for $k = 1,\ldots,N$.

Here

$$\underline{x}_k = (x_1,\ldots,x_k),$$
$$\underline{x}_{N-k} = (x_{k+1},\ldots,x_N).$$

The BBGKY hierarchy is given by

$$\begin{aligned} i\partial_t \gamma_{N,t}^{(k)} &= \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_{N,t}^{(k)}] + \frac{1}{N^2} \sum_{1\le i<j<\ell\le k} [V_N(x_i - x_j, x_i - x_\ell), \gamma_{N,t}^{(k)}] \\ &\quad + \frac{(N-k)}{N^2} \sum_{1\le i<j\le k} \mathrm{Tr}_{k+1} [V_N(x_i - x_j, x_i - x_{k+1}), \gamma_{N,t}^{(k+1)}] \\ &\quad + \frac{(N-k)(N-k-1)}{N^2} \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\mathrm{Tr}_{k+2} [V_N(x_j - x_{k+1}, x_j - x_{k+2}), \gamma_{N,t}^{(k+2)}] \end{aligned} \tag{2.3}$$

where

$$V_N(x,y) := N^{2d\beta} V(N^\beta x, N^\beta y) .$$

In the limit $N \to \infty$, the sums weighted by combinatorial factors have the following size:

- In the first interaction term on the rhs, $\frac{k^3}{N^2} \to 0$ for every fixed k and sufficiently small β.
- For the second term $\frac{(N-k)k^2}{N^2} \to 0$.
- For the third interaction term, $\frac{(N-k)(N-k-1)}{N^2} \to 1$ for every fixed k.

Accordingly, **in the limit $N \to \infty$, one obtains the infinite GP hierarchy**

$$i\partial_t \gamma^{(k)}_{\infty,t} \;=\; \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}_{\infty,t}] \,+\, b_0 \sum_{j=1}^{k} B_{j;k+1,k+2} \gamma^{(k+2)}_{\infty,t} \tag{2.4}$$

where the **coupling constant** b_0 is given via

$$b_0 \;=\; \int dx_1\, dx_2\, V(x_1, x_2)$$

and the **"contraction operator"** is given via

$$\begin{aligned} & B_{j;k+1,k+2} \gamma^{(k+2)}_{\infty,t}(x_1, \dots, x_k; x_1', \dots, x_k') \\ & \quad := \int dx_{k+1} dx'_{k+1} dx_{k+2} dx'_{k+2} \\ & \qquad \Big[\delta(x_j - x_{k+1}) \delta(x_j - x'_{k+1}) \delta(x_j - x_{k+2}) \delta(x_j - x'_{k+2}) \\ & \qquad\qquad - \delta(x_j' - x_{k+1}) \delta(x_j' - x'_{k+1}) \delta(x_j' - x_{k+2}) \delta(x_j' - x'_{k+2}) \Big] \\ & \qquad \gamma^{(k+2)}_{\infty,t}(x_1, \dots, x_j, \dots, x_k, x_{k+1}, x_{k+2}; x_1', \dots, x_k', x'_{k+1}, x'_{k+2}) \,. \end{aligned} \tag{2.5}$$

It is easy to see that

$$\gamma^{(k)}_{\infty,t} = \big|\, \phi_t \,\big\rangle\big\langle\, \phi_t \,\big|^{\otimes k}$$

is a solution of (2.5) iff ϕ_t satisfies the quintic NLS

$$i\partial_t \phi_t + \Delta_x \phi_t - b_0 \, |\phi_t|^4 \, \phi_t = 0$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Now we are ready to formulate our main result:

Theorem 2.1. *(Chen and P.)*
Assume that $d \in \{1, 2\}$, *and that* $V \in W^{3,p}$ *for* $p > 2d$, $V(x, x') = V(x', x)$, $V \geq 0$, *and* $0 < \beta < \frac{1}{2d+3}$. *Let* $\{\Psi_N\}_N$ *denote a family such that*

$$\frac{1}{N}\big\langle \Psi_N, H_N \Psi_N \big\rangle < \infty,$$

and there exists $\phi \in L^2(\mathbb{R}^d)$ *such that*

$$\mathrm{Tr}\Big|\, \gamma_N^{(1)} - |\phi\rangle\langle\phi|\, \Big| \to 0 \qquad \text{as } N \to \infty.$$

Then, it follows for the k-particle marginals $\gamma_{N,t}^{(k)}$ *associated to* $\Psi_{N,t} = e^{-itH_N}\Psi_N$ *that*

$$\mathrm{Tr}\Big|\, \gamma_{N,t}^{(k)} \,-\, |\,\phi_t\rangle\langle\phi_t\,|^{\otimes k}\, \Big| \,\to\, 0 \qquad (N \to \infty), \tag{2.6}$$

where ϕ_t *solves the defocusing quintic nonlinear Schrödinger equation*

$$i\partial_t\phi_t \,+\, \Delta\phi_t \,-\, b_0\,|\phi_t|^4\phi_t \,=\, 0 \tag{2.7}$$

with initial condition $\phi_0 = \phi$, *and with* $b_0 = \int_{\mathbb{R}^{2d}} dx\, dx'\, V(x, x')$.

All of the previous works considered Bose gases with pair interactions, which is natural in the absence of interactions with any external fields.

However, **once the interaction of the Bose gas with a background field of matter is included** in the model (for instance with phonons or photons), averaging over the latter will typically lead to a linear combination of effective n-particle interactions, $n = 2, 3, \ldots$.

For n-particle interactions with $n = 2, 3$, where the microscopic Hamiltonian would have the form

$$\begin{aligned} H_N \quad := \quad & \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N} \sum_{1 \le i < j \le N} N^{d\beta} V_2(N^\beta(x_i - x_j)) \\ & + \frac{1}{N^2} \sum_{1 \le i < j < k \le N} N^{2d\beta} V_3(N^\beta(x_i - x_j), N^\beta(x_i - x_k)) \,, \end{aligned} \tag{2.8}$$

a combination of the previous analysis with the one that we present here will straightforwardly produce a mean field limit described by the defocusing NLS

$$i\partial_t \phi_t + \Delta \phi_t - \lambda_2 |\phi_t|^2 \phi_t - \lambda_3 |\phi_t|^4 \phi_t \; = \; 0 \tag{2.9}$$

in $d = 1, 2$, where $\lambda_2 = \int dx V_2(x) \ge 0$ and $\lambda_3 = \int dx dx' V_3(x, x') \ge 0$ account for the mean-field strength of the 2- and 3-body interactions.

A brief description of the approach that we follow

(i) We prove the convergence of the BBGKY hierarchy to the GP hierarchy by straightforwardly adapting the arguments from the work of *Erdös-Schlein-Yau*.

(ii) In order to prove the uniqueness of the limiting hierarchy, we expand the approach introduced by *Klainerman-Machedon* and subsequently used by *Kirkpatrick-Schlein-Staffilani*.

3 Uniqueness of solutions to the GP hierarchy

The method of *Klainerman and Machedon* for proving uniqueness of the GP hierarchy is based on:

1. Introducing an elegant "board game", whose purpose is to organize the combinatorics related to expressing solutions of the GP hierarchy using iterated Duhamel formulas.

2. Establishing space-time bounds for solutions to the homogeneous GP hierarchy.

3. Combining the above steps to prove uniqueness of solutions **under the assumption that**

$$\int_0^T \|B_{j,k+1}\gamma^{(k+1)}(t,\cdot)\|_{\dot{H}^1_x(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,dt \lesssim C^k, \tag{3.1}$$

for some $C > 0$ and all $1 \leq j < k$.

The relevant space-time bound (3.1) **was assumed to hold in the work of** *Klainerman and Machedon*. **Subsequently,** *Kirkpatrick, Schlein and Staffilani* **proved the relevant space-time bound in** $d = 2$ **case.**

In order to prove the uniqueness of the limiting hierarchy in the quintic case, we expand the approach introduced by *Klainerman and Machedon*. More precisely, when $d = 2$,

1. We **introduce a different board game** to suit the new operators $B_{j;k+1,k+2}$ that appear in our limiting hierarchy. This new board game helps us organize the Duhamel's expansions in a similar manner as in the work of *Klainerman and Machedon*.

2. We **establish the space-time $\mathbf{L^2_t H^\alpha_x}$ bound for the homogeneous GP hierarchy,** which shall be used iteratively along the nested Duhamel's expansions.

3. Finally, we **prove a spatial a-priori $\mathbf{H^\alpha_x}$ bound for the full GP hierarchy,** which immediately implies an analogue of the bound (3.1). **Hence we can combine the steps 1. and 2. to conclude the unconditional uniqueness of solutions to the GP hierarchy.**

On the board game

Let us fix a positive integer r. We express the solution $\gamma^{(r)}$ to the GP hierarchy in terms of the iterates $\gamma^{(r+2)}, \gamma^{(r+4)}, ..., \gamma^{(r+2n)}$ as follows:

$$\begin{aligned}\gamma^{(r)}(t_r, \cdot) &= \int_0^{t_r} e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2}(\gamma^{(r+2)}(t_{r+2}))\, dt_{r+2} \\ &= \int_0^{t_r} \int_0^{t_{r+2}} e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2} e^{i(t_{r+2} - t_{r+4})\Delta_{\pm}^{(r+2)}} B_{r+4}(\gamma^{(r+4)}(t_{r+4}))\, dt_{r+2}\, dt_{r+4} \\ &= ... \\ &= \int_0^{t_r} ... \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n})\, dt_{r+2} ... dt_{r+2n},\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\underline{t}_{r+2n} &= (t_r, t_{r+2}, ..., t_{r+2n}), \\ J^r(\underline{t}_{r+2n}) &= e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2} ... e^{i(t_{r+2(n-1)} - t_{r+2n})\Delta_{\pm}^{(r+2(n-1))}} B_{r+2n}(\gamma^{(r+2n)}(t_{r+2n})).\end{aligned}$$

However since the interaction term involves the sum, the iterated Duhamel's formula has $r(r+2)...(r+2n-2)$ terms.

Recalling that $B_{k+2} = \sum_{j=1}^{k} B_{j;k+1,k+2}$ we can rewrite $J^r(\underline{t}_{r+2n})$ as

$$J^r(\underline{t}_{r+2n}) = \sum_{\mu \in M} J^r(\underline{t}_{r+2n};\mu), \tag{3.3}$$

where

$$\begin{aligned} J^r(\underline{t}_{r+2n};\mu) =& e^{i(t_r - t_{r+2})\Delta_\pm^{(r)}} B_{\mu(r+1);r+1,r+2} e^{i(t_{r+2}-t_{r+4})\Delta_\pm^{(r+2)}} \dots \\ & e^{i(t_{r+2(n-1)}-t_{r+2n})\Delta_\pm^{(r+2(n-1))}} B_{\mu(r+2n-1);r+2n-1,r+2n}(\gamma^{(r+2n)}(t_{r+2n})), \end{aligned}$$

and μ is a map from

$$\{r+1, r+2, ..., r+2n-1\}$$

to

$$\{1, 2, ..., r+2n-2\}$$

such that $\mu(2) = 1$ and $\mu(j) < j$ for all j.

Here M denotes the set of all such mappings μ.

We observe that such a mapping μ can be represented by highlighting one nonzero entry in each column of the $(r+2n-2)\times n$ matrix:

$$\begin{bmatrix} \mathbf{B_{1;r+1,r+2}} & B_{1;r+3,r+4} & \cdots & \mathbf{B_{1;r+2n-1,r+2n}} \\ \cdots & \mathbf{B_{2;r+3,r+4}} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ B_{r;r+1,r+2} & B_{r;r+3,r+4} & \cdots & \cdots \\ 0 & B_{r+1;r+3,r+4} & \cdots & \cdots \\ 0 & B_{r+2;r+3,r+4} & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{r+2n-2;r+2n-1,r+2n} \end{bmatrix}. \tag{3.4}$$

Then we can rewrite $\gamma^{(r)}$ as

$$\gamma^{(r)}(t_r,\cdot) = \int_0^{t_r} ... \int_0^{t_{r+2n}} \sum_{\mu \in M} J^r(\underline{t}_{r+2n},\mu)\, dt_{r+2}...dt_{r+2n}. \tag{3.5}$$

Also the integrals of the following type are of interest to us:

$$I(\mu,\sigma) = \int_{t_r \geq t_{\sigma(r+2)} \geq \cdots \geq t_{\sigma(r+2n)}} J^r(\underline{t}_{r+2n},\mu)\, dt_{r+2}...dt_{r+2n}, \tag{3.6}$$

where σ is a permutation of $\{r+2, r+4, ..., r+2n\}$. **We would like to associate to such an integral a matrix, which will help us visualize $\mathbf{B}_{\mu(\mathbf{r+2j-1});\mathbf{r+2j-1},\mathbf{r+2j}}$'s as well as σ at the same time.**

More precisely, to $I(\mu, \sigma)$ we associate the matrix

$$\begin{bmatrix} t_{\sigma^{-1}(r+2)} & t_{\sigma^{-1}(r+4)} & \cdots & t_{\sigma^{-1}(r+2n)} \\ B_{1;r+1,r+2} & \mathbf{B_{1;r+3,r+4}} & \cdots & \mathbf{B_{1;r+2n-1,r+2n}} \\ \mathbf{B_{2;r+1,r+2}} & B_{2;r+3,r+4} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ B_{r;r+1,r+2} & B_{r;r+3,r+4} & \cdots & \cdots \\ 0 & B_{r+1;r+3,r+4} & \cdots & \cdots \\ 0 & B_{r+2;r+3,r+4} & \cdots & \cdots \\ \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{r+2n-2;r+2n-1,r+2n} \end{bmatrix}$$

whose columns are labeled 1 through n and whose rows are labeled $0, 1, \ldots r + 2n - 2$.

We introduce a board game on the set of such matrices. In particular, the following move shall be called an "acceptable move": If $\mu(r + 2j + 1) < \mu(r + 2j - 1)$, the player as allowed to do the following four changes at the same time:

- exchange the highlights in columns j and $j + 1$,
- exchange the highlights in rows $r + 2j - 1$ and $r + 2j + 1$,
- exchange the highlights in rows $r + 2j$ and $r + 2j + 2$,
- exchange $t_{\sigma^{-1}(r+2j)}$ and $t_{\sigma^{-1}(r+2j+2)}$.

As in the work of *Klainerman and Machedon*, the importance of this game is visible from the following lemma:

Lemma 3.1. *If* (μ, σ) *is transformed into* (μ', σ') *by an acceptable move, then* $I(\mu, \sigma) = I(\mu', \sigma')$.

Upper echelon matrices

We say that a matrix of the type (3.4) is in upper echelon form if each highlighted entry in a row is to the left of each highlighted entry in a lower row. For example, the following matrix is in upper echelon form (with $r = 1$ and $n = 3$):

$$\begin{bmatrix} \mathbf{B_{1;2,3}} & B_{1;4,5} & B_{1;6,7} \\ 0 & \mathbf{B_{2;4,5}} & B_{2;6,7} \\ 0 & B_{3;4,5} & B_{3;6,7} \\ 0 & 0 & \mathbf{B_{4;6,7}} \\ 0 & 0 & B_{5;6,7} \end{bmatrix}.$$

As in the work of *Klainerman and Machedon* we have:

Lemma 3.2. *For each matrix in M there is a finite number of acceptable moves that transforms the matrix into upper echelon form.*

Let $C_{r,n}$ denote the number of upper echelon matrices of the size $(r + 2n - 2) \times n$. We prove the following lemma which gives an upper bound on $C_{r,n}$.

Lemma 3.3. *The following holds:*

$$C_{r,n} \leq 2^{r+3n-2}.$$

Class of equivalence on the set of our matrices

Let us denote by N the set of those matrices in M which are in "upper echelon" form. Let μ_{es} be a matrix in N. We write $\mu \sim \mu_{es}$ if μ can be transformed into μ_{es} in finitely many acceptable moves.

It can be seen that:

Theorem 3.4. *Suppose $\mu_{es} \in N$. Then there exists a subset of $[0, t_r]^n$, denoted by D, such that*

$$\sum_{\mu \sim \mu_{es}} \int_0^{t_r} \ldots \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} \ldots dt_{r+2n} = \int_D J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} \ldots dt_{r+2n}.$$

The uniqueness of the GP hierarchy follows by combining:

(A) The representation obtained in (3.5):

$$\gamma^{(r)}(t_r,\cdot)=\int_0^{t_r}\ldots\int_0^{t_{r+2n}}\sum_{\mu\in M}J^r(\underline{t}_{r+2n},\mu)\,dt_{r+2}\ldots dt_{r+2n}$$

(B) The upper bound on the number $C_{r,n}$ of upper echelon matrices.

(C) Theorem 3.4 which for a fixed $\mu_{es}\in N$, provides the existence of a subset D of $[0,t_r]^n$, such that

$$\sum_{\mu\sim\mu_{es}}\int_0^{t_r}\ldots\int_0^{t_{r+2n}}J^r(\underline{t}_{r+2n},\mu)\,dt_{r+2}\ldots dt_{r+2n}=\int_D J^r(\underline{t}_{r+2n},\mu)\,dt_{r+2}\ldots dt_{r+2n}.$$

(D)

Theorem 3.5. *Assume that $d\in\{1,2\}$ and $t_r\in[0,T]$. The estimate*

$$\left\|\int_D J^r(\underline{t}_{r+2n},\mu)\,dt_{r+2}\ldots dt_{r+2n}\right\|_{H^\alpha(\mathbb{R}^{dr}\times\mathbb{R}^{dr})}<C^r\,(C_0T)^n \tag{3.7}$$

holds for a constant C_0 independent of r and T.

On the other hand, in order to prove Theorem 3.5 we obtain:

- A space-time $L^2_t H^\alpha_x$ bound for the homogeneous GP hierarchy:

$$\|B_{j;k+1,k+2}\, e^{it\Delta_{\pm}^{(k+2)}} \gamma^{(k+2)}\|_{L^2_t H^\alpha(\mathbb{R}\times\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \lesssim \|\gamma^{(k+2)}\|_{H^\alpha(\mathbb{R}^{d(k+2)}\times\mathbb{R}^{d(k+2)})}.$$

- A spatial a-priori H^α_x bound for the full GP hierarchy which implies:

$$\|B_{j;k+1,k+2}\gamma^{(k+2)}\|_{L^1_{t\in[0,T]} H^\alpha(\mathbb{R}\times\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \leq C^k, \tag{3.8}$$

  for all $k \geq 1$.

We recall that in the case of **the cubic GP hierarchy**,

Klainerman and Machedon, **considered the case $d = 3$ and assumed that the analogue of (3.8) was satisfied.**

Subsequently, *Kirkpatrick, Schlein and Staffilani* **proved the relevant space-time bound in $d = 2$ case.**

4 Revisiting the uniqueness of solutions to the GP hierarchy

Recently, with *Thomas Chen*, we introduced **the new approach to look at the the Cauchy problem for focusing and defocusing GP hierarchy**, which allows us to prove local in time existence and uniqueness of solutions in generalized Sobolev spaces of sequences of marginal density matrices.

Another look at the cubic GP hierarchy

Recall,

$$\Delta_{\pm}^{(k)} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k}, \quad \text{with} \quad \Delta_{\underline{x}_k} = \sum_{j=1}^{k} \Delta_{x_j}.$$

We introduce the notation:

$$\Gamma = (\gamma^{(k)}(x_1, \dots, x_k; x'_1, \dots, x'_k))_{k \in \mathbb{N}},$$

$$\widehat{\Delta}_{\pm}\Gamma := (\Delta_{\pm}^{(k)}\gamma^{(k)})_{k \in \mathbb{N}},$$

$$\widehat{B}\Gamma := (B_{k+1}\gamma^{(k+1)})_{k \in \mathbb{N}}.$$

Then, the cubic GP hierarchy can be written as[a]

$$\mathbf{i}\partial_{\mathbf{t}}\mathbf{\Gamma} + \widehat{\mathbf{\Delta}}_{\pm}\mathbf{\Gamma} = \mu\widehat{\mathbf{B}}\mathbf{\Gamma}. \tag{4.1}$$

[a] Moreover, for $\mu = 1$ we refer to the GP hierarchy as defocusing, and for $\mu = -1$ as focusing.

As a crucial ingredient of our arguments, we introduce Banach spaces $\mathcal{H}_\xi^\alpha = \{ \Gamma \in \mathfrak{G} \,|\, \| \Gamma \|_{\mathcal{H}_\xi^\alpha} < \infty \}$ where

$$\mathfrak{G} = \{ \Gamma = (\gamma^{(k)}(x_1, \ldots, x_k; x_1', \ldots, x_k'))_{k \in \mathbb{N}} \,|\, \mathrm{Tr} \gamma^{(k)} < \infty \}$$

is the space of sequences of k-particle density matrices, and

$$\| \Gamma \|_{\mathcal{H}_\xi^\alpha} := \sum_{k \in \mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} .$$

The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of a given Cauchy problem. If $\Gamma \in \mathcal{H}_\xi^\alpha$, then ξ^{-1} is the typical H^α-energy per particle.

The parameter α determines the regularity of the solution, and our results hold for $\alpha \in \mathfrak{A}(d,p)$ where

$$\mathfrak{A}(d,p) := \begin{cases} (\frac{1}{2}, \infty) & \text{if } d = 1 \\ (\frac{d}{2} - \frac{1}{2(p-1)}, \infty) & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \left[1, \infty\right) & \text{if } (d,p) = (3,2)\,, \end{cases}$$

in dimensions $d \geq 1$, and where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy.

In particular, we prove the following

4.1 Local existence and uniqueness (*Chen, P. '08, '09*)

We prove **local existence and uniqueness** of solutions for the cubic and quintic GP hierarchy with focusing or defocusing interactions, in $\mathcal{H}^\alpha_\xi$, for $\alpha \in \mathfrak{A}(d,p)$, which satisfy a spacetime bound

$$\|\widehat{B}\Gamma\|_{L^1_{t\in I}\mathcal{H}^\alpha_\xi} < \infty, \tag{4.2}$$

for some $\xi > 0$.

Note that the GP hierarchy can be formally written as a system of integral equations

$$\Gamma(t) = e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s) \tag{4.3}$$

$$\widehat{B}\Gamma(t) = \widehat{B}\, e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, \widehat{B}\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)\,, \tag{4.4}$$

where (4.4) is obtained by applying the operator $\widehat{B}$ on the linear non-homogeneous equation (4.3).

We prove the local well-posedness result by applying the fixed point argument in the following space:

$$\mathfrak{W}^\alpha_\xi(I) := \{\, \Gamma \in L^\infty_{t\in I}\mathcal{H}^\alpha_\xi \mid \widehat{B}\Gamma \in L^2_{t\in I}\mathcal{H}^\alpha_\xi \,\}, \tag{4.5}$$

where $I = [0, T]$.

4.2 Blow-up (*Chen, P., Tzirakis '09*)

In this work we

(A) **Identify an observable corresponding to the average energy per particle** and prove that it is conserved.

(B) Prove, on the L^2 critical and supercritical level, that solutions of focusing GP hierarchies with a negative average energy per particle **blow up in finite time.**

4.3 Global existence (*Chen, P. '09*)

We prove **the global existence and uniqueness of solutions:**

(A) To energy sub-critical defocusing GP hierarchies with $\Gamma_0 \in \mathcal{H}^1_\xi$, based on

- The conservation of higher order energy functionals
- Generalization of the Sobolev inequality

(B) To L^2 subcritical, focusing and defocusing GP hierarchies, based on

- Generalization of the Gagliardo-Nirenberg inequality