On helical flows with little regularity: global existence and vanishing viscosity

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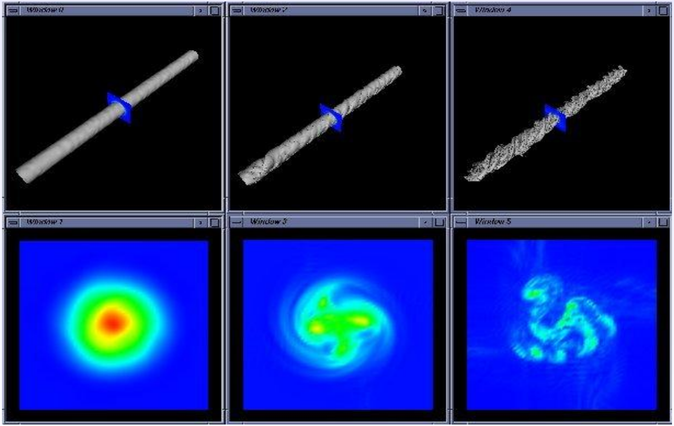


Time-varying visualization of vortex production.

Greg Blaisdell (Purdue Univ.)

Introduction

Let $\mathcal{D} \subset \mathbb{R}^3$ be open set with smooth boundary.

Navier-Stokes equations for incompressible fluid flow:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u, & \text{in } \mathcal{D} \times (0, \infty) \\ \operatorname{div} u = 0, & \text{in } \mathcal{D} \times [0, \infty). \end{cases} \tag{1}$$

Above, $u = (u_1, u_2, \ldots, u_N)$ is velocity, p is (scalar) pressure, and $(u \cdot \nabla) u$ means:

$$\sum_{i=1}^{N} u_i \partial_{x_i} u_j, \quad j = 1, \ldots, N.$$

$\nu = 0 \longleftrightarrow$ Euler equations of incompressible, ideal fluid flow.

Existence results (not comprehensive nor state-of-the-art):

$\nu > 0 \to$ global $\exists$ weak solutions, $u_0 \in L^2$;

$\nu \geq 0 \to$ local-in-time $\exists\,!$ strong solutions, $u_0 \in H^s$, $s > 5/2$.

Also, if $\mathcal{D} = \mathbb{R}^3$ then solution to ν-NS $\to$ solution of Euler as $\nu \to 0$, $u_0 \in H^3$. (Swann, 1971 and Kato, 1972.)

Flows with special symmetries

Axisymmetric flows: introduce rotation

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Say flow is axisymmetric if $u(R_\theta x) = R_\theta u(x)$ for every θ.

Relevant quantity: swirl,

$$u_\theta \equiv u \cdot \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \quad r = \sqrt{x^2 + y^2}.$$

If initial swirl vanishes then both Navier-Stokes and Euler evolution preserve vanishing swirl.

Axisymmetric no-swirl [$u_\theta = 0$]:

$\nu \geq 0$ global $\exists$, smooth data.
Like $2D$, no vortex stretching.

Axisymmetric with swirl:

have vortex stretching; global $\exists$ ($\nu > 0$) if fluid domain far from symmetry axis.

Flows wth helical symmetry

Investigated rigorously by a few authors: Dutrifoy 1999, Titi-Mahalov-Leibowich 1990, Ettinger-Titi, 2008. Follow Ettinger-Titi formulation.

Introduce helical translation

$$S_{\theta,\kappa}X = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \kappa\theta \end{bmatrix}.$$

Say u is helically symmetric if

$$u(S_{\theta,\kappa}x) = R_\theta u(x).$$

In particular, helical symmetry $\Longrightarrow$ flow is $2\pi\kappa$-periodic in z-variable.

Introduce special tangent vector to helices:

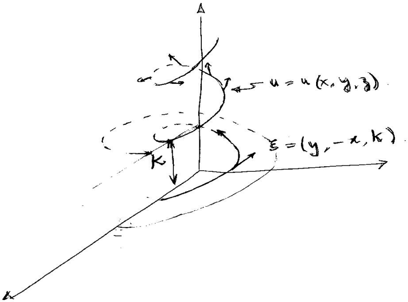
$$\xi = (y, -x, \kappa).$$

Some properties due to helical symmetry: f scalar function; say

f is helical if $f(S_{\theta,\kappa}x) = f(x) \Longleftrightarrow \frac{\partial f}{\partial \xi} = 0$;

u helically symmetric (vector field) $\Longleftrightarrow$

$$\begin{cases} \frac{\partial u_1}{\partial \xi} = u_2 \\ \\ \frac{\partial u_2}{\partial \xi} = -u_1 \\ \\ \frac{\partial u_3}{\partial \xi} = 0. \end{cases}$$



HELICAL FLOWS: $u(S_{\kappa,\theta}\, x) = R_\theta\, u(x)$

Lemma

If u is helically symmetric and $\eta \equiv u \cdot \xi$ *then*

$$\omega \equiv \; curl\, u = \omega_3 \frac{\xi}{\kappa} + \left(\frac{\partial \eta}{\partial y}, -\frac{\partial \eta}{\partial x}, 0 \right) \frac{1}{\kappa},$$

where $\omega_3 = \partial_x u_2 - \partial_y u_1$.

Call $\eta = u \cdot \xi$ helical swirl.

Vanishing helical swirl prevents vortex stretching. If ω is vorticity then

$$\frac{D\omega}{Dt} + \frac{1}{\kappa}\omega_3(u_2, -u_1, 0) + \frac{1}{\kappa}(\partial_x \eta \partial_y u - \partial_y \eta \partial_x u) = \nu \Delta \omega.$$

If $\eta = 0$ and $\nu = 0$ then

$$\frac{D\omega_3}{Dt} = 0.$$

$\exists$ results for helical flows:

no helical swirl: well-posedness for smooth data (Dutrifoy, 1999);

$\exists\,!$ weak solution, Euler, if curl $u = \omega \in L^\infty$ (Ettinger-Titi, 2008). Proof is Yudovich-type argument.

any helical swirl: also, $\exists\,!$ strong solution, ν-Navier-Stokes, bounded (helical) domain, no-slip at boundary, $u_0 \in H^1$, (Mahalov-Titi-Leibowich, 1990). Proof relies on key *a priori* estimate for helical vector fields:

$$\|u\|_{L^4} \leq C\kappa^{1/4}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2};$$

3D helical flow much more like 2D than axially symmetric (at least in case without swirl);

axial symmetry means $\kappa = 0$, 2D means $\kappa = \infty$, helical flow is interpolant;

easy to see if think of flow as point vortex dynamics (vortex line dynamics in 3D): helical flow is dynamics of vortex helices;

in axially symmetric (no swirl) flow vortex helices degenerate into circular vortex sheets, more complicated.

Some questions

1) Quantify order of vanishing of helical swirl so that, as $\nu \to 0$, $u^\nu \to u^E$?

2) How irregular can helical u_0^E be to get global weak solution for Euler?

First problem: study case where fluid domain $\mathcal{D}$ is the cylinder $\{x^2 + y^2 < 1\} \times \mathbb{R}$.

Avoid issues with boundary layer; choose Navier friction boundary conditions.

What follows: joint work with Milton Lopes Filho, Quansen Jiu and Dongjuan Niu.

Fix $\mathcal{D} = \{x^2 + y^2 < 1\} \times \mathbb{R}$ **and** $\kappa = 1$.

Let $u_0^\nu \in H^1_{\text{per}}(\mathcal{D})$ i.e.

$$u_0^\nu(x, y, z + 2\pi) = u_0^\nu(x, y, z).$$

Given: $\eta_0^\nu \equiv u_0^\nu \cdot \xi$, where $\xi = (y, -x, 1)$.

Suppose

$$u_0^\nu \to u_0^E \quad \text{strongly in } L^2 \text{ as } \nu \to 0.$$

Assume also $u_0^E \cdot \xi = 0$. [Hence $\eta_0^\nu \to 0$ sL^2 as $\nu \to 0$.]

Let $u^\nu = u^\nu(x, y, z, t)$ be ν-NS solution, initial data u_0^ν.

The η-equation

$$\partial_t \eta + u \cdot \nabla \eta = \nu \Delta \eta + 2\nu\omega_3, \qquad (2)$$

$\omega_3 = \partial_x u_2 - \partial_y u_1$.

Very nice – try equation for ω_3:

$$\partial_t \omega_3 + u \cdot \nabla \omega_3 + \partial_x \eta \partial_y u_3 - \partial_y \eta \partial_x u_3 = \nu \Delta \omega_3.$$

Cannot control red term... – vortex stretching taking place.

Let u be HELICAL and DIV-FREE.

Introduce

$$W \equiv \eta \frac{\xi}{|\xi|^2}, \quad \eta = u \cdot \xi;$$

$$V \equiv u - W.$$

Lemma

V and W are both HELICAL and DIV-FREE. Of course, $V \cdot \xi = 0$ and $V \cdot W = 0$.

Corollary

We have $\Omega \equiv curl\ V = \Omega_3 \xi$.

Navier friction boundary condition

$$[2(Du)_S\hat{n}]\cdot\hat{\tau}+\beta u\cdot\hat{\tau}=0,\quad \text{on } \{x^2+y^2=1\}\times\mathbb{R}.$$

For technical reasons treat ONLY $\beta = 1$. Analogous to J.-L. Lions' free-boundary condition in $2D$.

We will need ∂-conditions for η and for $\Omega_3 \equiv (\text{ curl } V)_3$:

$$\frac{\partial\eta}{\partial\hat{n}}+2(xu_2-yu_1)+\eta=0,\quad \text{on } \partial\mathcal{D}, \tag{3}$$

$$\Omega_3=0 \quad \text{on } \partial\mathcal{D}, \quad (\text{here is where } \beta=1 \text{ is needed}). \tag{4}$$

Obs. Can work with $\kappa \neq 1$ and with more general helical domains, but *need* $\beta = 1$.

Equation for V

$$\partial_t V + V \cdot \nabla V + \nabla p - \nu \Delta V$$

$$= -\frac{\eta}{|\xi|^2}\partial_\xi V - \left(V \cdot \nabla \left[\frac{\xi}{|\xi|^2}\right]\right)\eta - \frac{\eta^2}{|\xi|^2}\partial_\xi \left(\frac{\xi}{|\xi|^2}\right)$$

$$+2\nu\nabla\eta\cdot\nabla\left(\frac{\xi}{|\xi|^2}\right) + \nu\eta\Delta\left[\frac{\xi}{|\xi|^2}\right] - 2\nu\Omega_3\frac{\xi}{|\xi|^2} - 2\nu\left[\text{curl }\left(\frac{\eta\xi}{|\xi|^2}\right)\right]_3\frac{\xi}{|\xi|^2}.$$

AWFUL LOOKING BEAST!

Equation for Ω_3

(Summarized version.)

$$\partial_t \Omega_3 + V \cdot \nabla \Omega_3 - \nu \Delta \Omega_3$$

$$= D_{x,y}((V_1, V_2)\, \eta\, g_1(\xi)) + D_z(|\eta|^2 g_2(\xi)) \tag{5}$$

$$+\nu \left[D^2_{x,y}(\eta g_3(\xi)) + D_{x,y}(\eta g_4(\xi)) + D_{x,y}(\Omega_3\, g_5(\xi)) \right].$$

From η-equation (2) get easily

$$\|\eta\|^2_{L^\infty(dt;L^2(dx))} + \nu\|\nabla\eta\|^2_{L^2(dt;L^2(dx))} \leq \|\eta_0\|^2_{L^2} + \nu T\|u_0\|^2_{L^2},$$

for any $T > 0$.

We will first take η_0^ν such that

$$\|\eta_0^\nu\|_{L^2} \leq C\sqrt{\nu}.$$

Then, $\|\nabla\eta\|_{L^2(dt;L^2(dx))}$ will be bounded wrt ν and $\|\eta^\nu\|_{L^\infty(dt;L^2(dx))} = \mathcal{O}(\sqrt{\nu})$.

Introduce $A = A(t) \equiv \int_0^t \|\Omega_3\|_{L^2}^2 \, ds$.

Not so easily, from η-equation get

$$\|\eta\|^2_{L^\infty(dt;L^4(dx))} \leq C(\|\eta_0\|^2_{L^4} + \nu\sqrt{A(t)}\sqrt{t})e^T,$$

for any $T > 0$.

Ingredients: u bounded $L^\infty(L^2)$; $u = V + W$,

$\|V\|^2_{L^4} \leq \|\nabla V\|_{L^2}\|V\|_{L^2}$, elliptic regularity.

Assume further that $\|\eta_0^\nu\|_{L^4} \leq C\sqrt{\nu}$.

Use the two previous estimates, the Ω_3-equation (5) and the more restrictive hypothesis on η_0^ν to obtain

$$\|\Omega_3\|^2_{L^\infty(dt;L^2(dx))} \leq (\|\Omega_{3,0}\|^2_{L^2} + \|\eta\|^2_{L^2(dt;H^1)} + C(T))e^{CT}.$$

Additional ingredients: elliptic regularity and "helical estimate" for $\eta\xi/|\xi|^2$. Get differential inequality for A'' estimated by A'. Integrate in time, use Gronwall, recall $A' = \|\Omega_3\|^2_{L^2}$.

Obs. In Ω_3-equation, red terms are easy, because small (ν). Blue are harder to estimate.

Summary of available estimates for u^ν

$\|u^\nu\|_{L^\infty(dt;L^2(dx))} \leq C$, independent of ν;

curl $u^\nu =$ curl $V^\nu +$ curl $W^\nu = \Omega_3^\nu \xi +$ curl $\eta^\nu \left(\frac{\xi}{|\xi|^2}\right)$;

$\|\Omega_3^\nu\|_{L^\infty(dt;L^2(dx))} \leq C$, independent of ν;

$\|\eta^\nu\|_{L^2(dt;H^1(dx))} \leq C$, independent of ν.

Navier-Stokes equation gives temporal estimates for equicontinuity.

First theorem

Theorem

Let $u_0^\nu \in H^1_{per}(\mathcal{D})$ *such that*

- $\|\eta_0^\nu\|_{L^4} \leq C\sqrt{\nu}$;
- $u_0^\nu \rightharpoonup u_0$ *weakly in* H^1_{per}.

Let u^ν *be* ν*-NS solution initially* u_0^ν*. Then, passing to subsequences as needed,*

$$u^\nu \to u$$

strongly in $C(dt; L^2)$; $u \in L^\infty(dt; L^2) \cap L^2(dt; H^1_{per})$ *is a weak solution of 3D Euler with helical symmetry;* $u \cdot \hat{n} = 0$ *on boundary;* $u \cdot \xi = 0$ *a.e.*

What happens in “full space” $\mathbb{R}^2 \times (0, 2\pi\kappa)$, periodic in z? Work in progress.

Obs1. We used ξ bounded; no longer so. We used ellptic regularity to compare ∇V with Ω_3. Should be $\Omega_3\xi$. Need to understand *decay properties* of helical flow.

Obs2. Too much like 2D flow. Can show: if $\int \Omega_3 dx = 0$ then $|V| = \mathcal{O}(|x^2 + y^2|^{-1})$ at infinity. Otherwise, must deal with stationary flow with no decay.

Obs3. If $\varphi \in C_c^\infty(0,+\infty)$ then the vector field

$$\Lambda \equiv \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, \frac{1}{\kappa} \right) \int_0^{\sqrt{x^2+y^2}} r\varphi(r)\, dr$$

is DIV-FREE, HELICAL, has VANISHING HELICAL SWIRL, its CURL is $\varphi(\sqrt{x^2+y^2})\xi$ and it is a STATIONARY Euler solution. For large x, y third component is constant.

To deal with full space, subtract stationary flow and do energy estimates for what is left.

Much like DiPerna-Majda decompostion of 2D flow. We think it works... to get analogous theorem.

Alternative path: study critical regularity of u_0 for global existence weak 3D Euler solution. Consider full space case.

Assume $u_0 \cdot \xi = 0$, u_0 helical. Take smooth approximations $u_0^n \to u_0$ strongly in (some) L^r, some r.

(Substract stationary part if needed.) Have curl $u = \omega_3 \xi / \kappa$. Assume $\omega_{3,0} \in L^p$ some p. Which p?

"Easy" result: $u \in L^q$, compact if $q < 3p/(3-p)$. Can take $q = p' = p/(p-1)$ as long as $p > 3/2$. Then u compact in $L^{p'}$ so can pass to limit in nonlinear term.

Gives existence of weak solution if $\omega_3 \in L^p$, $p > 3/2$.

Alternative: use Delort-type symmetrization. Work in progress, joint with Anne Bronzi and Milton Lopes Filho.

Write: $u = K * (\omega_3 \xi)$.

Estimate $K = K(x)$ using Bessel-Fourier series.

Get $|K(x)| \leq C(|x|^{-2} + |\tilde{x}|^{-1})$.

Symmetrize nonlinear term in weak form:

$$\int \nabla\psi(x) \int K(x-y) \times \xi(y)\omega_3(y)\, dy \omega_3(x)\xi(x)\, dx$$

$$= \int\int \mathcal{H}_\psi(x,y)\omega_3(x)\omega_3(y)\, dxdy,$$

where

$$\mathcal{H}_\psi = \frac{1}{2\kappa} K(x-y)\cdot(\xi(y)\times(\nabla\psi(x)-\nabla\psi(y))-(\xi(x)-\xi(y))\times\nabla\psi(y)).$$

With this get $|\mathcal{H}_\psi(x,y)| \leq C(|x-y|^{-1} + |\tilde{x}-\tilde{y}|^{-1})$.

Critical regularity becomes like 2D: $\omega \in L^p$, $p > 4/3$.

1. What is the analogue of Delort's result for axisymmetric flow, regarding concentration of kinetic energy, for helical flow with vortex sheet regularity initial data?

2. Existence with non-vanishing helical swirl?

3. What is the "rate of confinement" for helical, no-helical swirl, flow with non-negative ω_3?

Thank you!