Steady flow of a second grade fluid past an obstacle

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Abstract

We study the steady flow of a second grade fluid past an obstacle in three space dimension. We prove the existence of solution in weighted Lebesgue spaces with anisotropic weights and thus the existence of the wake region behind the obstacle. We use the properties of the fundamental Oseen tensor together with the results achieved in [1] and properties of solutions to steady transport equations to get up to arbitrarily small ε the same decay as the Oseen tensor.

1. The model

STEADY flow of the second grade fluid is governed by the following system of equations

$$\left.\begin{aligned}\rho\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p=\operatorname{div}\mathbf{T}+\rho\mathbf{f}\\ \operatorname{div}\mathbf{v}=0\end{aligned}\right\}\text{ in }\Omega, \tag{1}$$

where v denotes the fluid velocity, p is the pressure, ρ is the constant density of the fluid, $\mathbf{f}$ stands for the external forces and $\mathbf{T}$ is the Cauchy stress tensor which for the second grade fluid is given by

$$\mathbf{T}=2\mu\mathbf{D}+2\alpha_1\mathbf{A}_1-4\alpha_1\mathbf{D}^2. \tag{2}$$

Here μ is a constant viscosity, $\mathbf{D}=\frac{1}{2}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)$ is the symmetric part of the velocity gradient, α_1 is the stress modul and $\mathbf{A}_1$ is given by

$$\mathbf{A}_1=\mathbf{v}\cdot\nabla\mathbf{D}+(\nabla\mathbf{v})^T\mathbf{D}+\mathbf{D}\nabla\mathbf{v}. \tag{3}$$

We consider a flow past an obstacle. Therefore Ω is the exterior domain $\mathbb{R}^3\setminus\mathcal{B}$ and we assume $B_{\kappa L}(\mathbf{0})\subset\mathcal{B}\subset B_L(\mathbf{0})$. We prescribe the velocity at infinity and use homogenous Dirichlet boundary condition at the obstacle

$$\begin{aligned}\mathbf{v}&=\mathbf{0} &&\text{on } \partial\Omega=\partial\mathcal{B}\\ \mathbf{v}&\to\mathbf{v}_\infty &&\text{as } |\mathbf{x}|\to\infty.\end{aligned} \tag{4}$$

We make the following series of steps

- rotate the coordinate system such that $\mathbf{v}_\infty=\beta\mathbf{e}_1=\beta(1,0,0)$, $\beta>0$
- denote $\mathbf{u}=\mathbf{v}-\mathbf{v}_\infty$
- renormalize = rewrite in dimensionless form
- denote $\mathcal{R}=\frac{\rho\beta L}{\mu}$ and $\mathcal{W}=\frac{\alpha_1\beta}{L\mu}$ and end up with

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$$\begin{aligned}&-\Delta\mathbf{u}-\mathcal{W}\mathbf{u}\cdot\nabla\Delta\mathbf{u}-\mathcal{W}\Delta\frac{\partial\mathbf{u}}{\partial x_1}+\mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1}+\mathcal{R}\nabla p=\\ &\quad=-\mathcal{R}\mathbf{u}\cdot\nabla\mathbf{u}+\mathcal{R}\mathbf{f}+\mathcal{W}\operatorname{div}\left[(\nabla\mathbf{u})^T(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)\right]\end{aligned} \tag{5}$$

with boundary conditions

- $\Omega=\mathbb{R}^3\setminus\mathcal{B}$, where $B_\kappa\subset\mathcal{B}\subset B_1(\mathbf{0})$
- $\mathbf{u}=-\mathbf{e}_1=(-1,0,0)$ on $\partial\Omega$
- $\mathbf{u}\to\mathbf{0}$ for $|\mathbf{x}|\to\infty$

2. Decomposition

We decompose this equation in the following way (introduced by Mogilevskii and Solonnikov)

$$\mathcal{R}p=q+\mathcal{W}((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)q$$

Oseen equation

$$\begin{aligned}-\Delta\mathbf{u}+\mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1}+\nabla q&=\mathbf{z}\\ \operatorname{div}\mathbf{u}&=0\end{aligned} \tag{6}$$

Steady transport equation

$$\begin{aligned}&\mathbf{z}+\mathcal{W}((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{z}=\mathcal{R}\mathbf{f}-\mathcal{R}\mathbf{u}\cdot\nabla\mathbf{u}+\mathcal{R}\mathcal{W}\frac{\partial^2\mathbf{u}}{\partial x_1^2}+\\ &\quad+\mathcal{W}\operatorname{div}\left[(\nabla\mathbf{u})^T(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)-(\nabla\mathbf{u})^Tq+\mathcal{R}\frac{\partial\mathbf{u}}{\partial x_1}\otimes\mathbf{u}\right]\end{aligned} \tag{7}$$

We study the properties of the solutions using mapping

$$\mathcal{M}:(\mathbf{w},s)\to\mathbf{z}\to(\mathbf{u},q), \tag{8}$$

where

- for given $(\mathbf{w},s)$ we compute $\mathbf{z}$ as the solution of the transport equation (7) with the right-hand side derived from $(\mathbf{w},s)$,
- for this $\mathbf{z}$ we compute $(\mathbf{u},q)$ as the solution of the Oseen equation (6) with the right-hand side $\mathbf{z}$.

We search for fixed point of $\mathcal{M}$ in suitable spaces.

3. Oseen kernel

Notation

- $s(\mathbf{x})=|\mathbf{x}|-x_1$
- $L^p(\Omega,w)=\left\{f;\|f\|_{p,(w)}=\|fw\|_p<\infty\right\}$ for $1\le p\le\infty$
- $\eta_B^A(\mathbf{x},\mathcal{R})=(1+|\mathcal{R}\mathbf{x}|)^A(1+s(\mathcal{R}\mathbf{x}))^B$
- $\mu_B^{A,\omega}(\mathbf{x},\mathcal{R})=|\mathbf{x}|^\omega(1+|\mathcal{R}\mathbf{x}|)^{A-\omega}(1+s(\mathcal{R}\mathbf{x}))^B$

Oseen kernel

There is a fundamental solution $(\boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R}),\mathbf{e}(\mathbf{x}))$ to the Oseen equation (6) with the following properties

- $\boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R})=\mathcal{R}\boldsymbol{\mathcal{O}}(\mathcal{R}\mathbf{x},1)$
- For $|\mathbf{x}|\to\infty$ we have $D^\alpha\boldsymbol{\mathcal{O}}(\mathbf{x},1)\sim|\mathbf{x}|^{-1-|\alpha|/2}(1+s(\mathbf{x}))^{-1-|\alpha|/2}$, $|\alpha|\ge0$
- $\mathbf{e}(\mathbf{x})=\nabla\mathcal{E}(\mathbf{x})$, where $\mathcal{E}$ is the fundamental solution to the Laplace equation

Representation formulas

There are representation formulas for the solutions of Oseen equation with the right hand side $\mathbf{z}$ in exterior domains

$$\begin{aligned}u_j(\mathbf{x})=&\int_\Omega\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R})z_i(\mathbf{y})\mathrm{d}\mathbf{y}+\\ &\int_{\partial\Omega}\left[-\mathcal{R}\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R})u_i(\mathbf{y})\delta_{1k}+\right.\\ &+u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}_{.j},e_j)(\mathbf{x}-\mathbf{y},\mathcal{R})+\\ &\left.\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y},\mathcal{R})T_{ik}(\mathbf{u},q)(\mathbf{y})\right]n_k(\mathbf{y})\mathrm{d}S,\end{aligned}$$

where

$$T_{ij}(\mathbf{u},q)=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-q\delta_{ij}.$$

Similar formulas hold for $\nabla\mathbf{u}$ and $\nabla^2\mathbf{u}$ and also for the pressure. Note that in the case of $\mathbf{z}=\operatorname{div}\mathbf{Z}$ we can integrate by parts in the volume integral and get convolution of $\nabla\boldsymbol{\mathcal{O}}$ with $\mathbf{Z}$ together with additional boundary integral.

4. Main Theorem

Theorem 1 *Let $\mathbf{f}=\operatorname{div}\mathbf{H}$, let $\mathbf{f},\mathbf{H}\in L^p(\Omega,\mu_{1-2/p}^{3/2-3/p,\omega}(\mathbf{x},\mathcal{R}))$ for all $p>6$. Moreover let $\mathbf{H}\in L^2(\Omega)$ and $\mathbf{f}\in W^{k,2}(\Omega)$ for $k\ge3$. Finally let $\mathcal{R},\mathcal{W}$ be sufficiently small. Then there exists a solution to our problem such that*

- $\mathbf{u}\in L^p(\Omega,\mu_{1-2/p}^{1-3/p,\omega}(\mathbf{x},\mathcal{R}))$,
- $\nabla\mathbf{u},\nabla^2\mathbf{u}\in L^p(\Omega,\mu_{1-2/p}^{3/2-3/p,\omega}(\mathbf{x},\mathcal{R}))$,
- $q,\nabla q\in L^p(\Omega,\mu_{1/2-2/p}^{3/2-3/p,\omega}(\mathbf{x},\mathcal{R}))$

for all $p>6$. In particular this implies

$$\mathbf{u}\in L^\infty(\Omega,\mu_{1-2/p}^{1-3/p,\omega}(\mathbf{x},\mathcal{R})) \tag{9}$$

for any $p>6$ and therefore almost the same asymptotic behavior as the Oseen kernel.

We present main steps in the proof.

Existence

We start with proving existence of solutions in Sobolev spaces by using Banach fixed point theorem for the mapping $\mathcal{M}$ in spaces

$$V_k=\{(\mathbf{u},q):\mathbf{u}\in L^4(\Omega),\nabla\mathbf{u},q\in W^{k,2}(\Omega)\}. \tag{10}$$

This is done in a standard way and one obtaines estimates

$$\begin{aligned}\mathcal{R}^{\frac14}\|\mathbf{u}_n\|_4+\|\nabla\mathbf{u}_n\|_{k,2}+\|q_n\|_{k,2}&\le C\\ \|\mathbf{C}(\mathbf{H},\mathbf{u}_n,q_n)\|_{k,2}&\le C,\end{aligned} \tag{11}$$

on the succesive approximations $(\mathbf{u}_n,q_n)$ of the solution. Here the right-hand side of the transport equation (7) (we denote it by $\mathbf{B}(\mathbf{f},\mathbf{u},q)$) is written in the divergence form as

$$\mathbf{B}(\mathbf{f},\mathbf{u},q)=\operatorname{div}\mathbf{C}(\mathbf{H},\mathbf{u},q). \tag{12}$$

Asymptotics

Next we have to prove that $\mathcal{M}$ maps sufficiently large balls in proper weighted spaces into themselves. We use the following space

$$\begin{aligned}V=\Big\{(\mathbf{u},q):\ &\mathbf{u}\in L^p(\Omega,\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R})),\\ &\nabla\mathbf{u},\nabla^2\mathbf{u}\in L^p(\Omega,\mu_{1-\frac{2}{p}}^{\frac32-\frac{3}{p},\omega}(\cdot,\mathcal{R})),\\ &q,\nabla q\in L^p(\Omega,\mu_{\frac12-\frac{2}{p}}^{\frac32-\frac{3}{p},\omega}(\cdot,\mathcal{R}))\Big\}\end{aligned} \tag{13}$$

for $p>6$ and $\omega>0$ but small. We start with

$$\|(\mathbf{w},s)\|_V\le C_0 \tag{14}$$

and $(\mathbf{w},s)$ satisfying (11).

Transport equation estimates

First we estimate

$$\|\mathbf{B}(\mathbf{f},\mathbf{w},s)\|_X,\|\mathbf{C}(\mathbf{H},\mathbf{w},s)\|_X\le C((\mathcal{R}+\mathcal{W})C_0^2+\mathcal{R}^{1-\omega}\mathcal{W}C_0), \tag{15}$$

where

$$X=L^p(\Omega,\mu_{1-\frac{2}{p}}^{\frac32-\frac{3}{p},2\omega}(\cdot,\mathcal{R})). \tag{16}$$

Therefore also the solution of the steady transport equation (7)

$$\|\mathbf{z}\|_X\le C((\mathcal{R}+\mathcal{W})C_0^2+\mathcal{R}^{1-\omega}\mathcal{W}C_0) \tag{17}$$

and since

$$\mathbf{z}=\operatorname{div}\left[\mathbf{C}(\mathbf{H},\mathbf{w},s)-\mathcal{W}((\mathbf{w}+\mathbf{e}_1)\cdot\nabla)\mathbf{z}\right]=\operatorname{div}\mathbf{Z} \tag{18}$$

also

$$\|\mathbf{Z}\|_X\le C((\mathcal{R}+\mathcal{W})C_0^2+\mathcal{R}^{1-\omega}\mathcal{W}C_0). \tag{19}$$

Oseen equation - key estimates

Now we have to use integral representation formulas for solutions of Oseen equation (6) with the right hand side $\mathbf{z}=\operatorname{div}\mathbf{Z}$. The key step is to estimate the volume terms. For estimates of the volume terms we use the results of boundedness of convolutions with gradients of the Oseen kernel, namely for the volume part of the velocity $\mathbf{u}^V$ we get

$$\left\|\mathbf{u}^V\right\|_{L^p(\mu_{1-\frac{2}{p}}^{1-\frac{3}{p},\omega}(\cdot,\mathcal{R}))}\le C\mathcal{R}^{-1+\omega}\|\mathbf{Z}\|_X. \tag{20}$$

For $\nabla\mathbf{u}$ and $\nabla^2\mathbf{u}$ we get using results of Koch [1] the following

$$\left\|(\nabla\mathbf{u})^V,(\nabla^2\mathbf{u})^V\right\|_{L^p(\mu_{1-\frac{2}{p}}^{\frac32-\frac{3}{p},\omega}(\cdot,\mathcal{R}))}\le C\mathcal{R}^\omega\|\mathbf{Z}\|_X. \tag{21}$$

Similar estimates we get also for the volume parts of the pressure and its gradient. Thus together with (19) we get that all volume parts can be made sufficiently small by taking $\mathcal{R}$ and $\mathcal{W}$ sufficiently small.

Oseen equation - boundary integrals

In order to estimate the boundary integrals in the representation formulas we decompose our domain into three parts: $\Omega=\Omega_1\cup\Omega_2\cup\Omega_3$, where

$$\begin{aligned}\Omega_1&=\{\mathbf{x}\in\Omega,|\mathbf{x}|\le1\}\\ \Omega_2&=\left\{\mathbf{x}\in\Omega,1\le|\mathbf{x}|\le\frac{1}{\mathcal{R}}\right\}\\ \Omega_3&=\left\{\mathbf{x}\in\Omega,|\mathbf{x}|\ge\frac{1}{\mathcal{R}}\right\}.\end{aligned} \tag{22}$$

In Ω_1 every weight behaves like a constant and therefore instead of using the integral representation we estimate L^p norms of solution and its gradients using estimates (11). The norms can be made small in comparison with C_0 by taking C_0 sufficently large.
In domain Ω_2 we use estimates of the type

$$\left|\nabla^k\boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R})\right|\le C\frac{\mathcal{R}^{\frac{k}{2}}}{|\mathbf{x}|^{1+\frac{k}{2}}} \tag{23}$$

for $k\ge0$. This together with (11) leads to integration of functions $|\mathbf{x}|^{-\alpha}$ for $\alpha>3$. Therefore all arising terms can be made small in comparison with C_0 again by taking C_0 sufficently large.
In domain Ω_3 we use estimates of the type

$$\left|\nabla^k\boldsymbol{\mathcal{O}}(\mathbf{x},\mathcal{R})\right|\le C\frac{\mathcal{R}^{\frac{k}{2}}}{|\mathbf{x}|^{1+\frac{k}{2}}(1+s(\mathcal{R}\mathbf{x}))^{1+\frac{k}{2}}} \tag{24}$$

for $k\ge0$. Consequently this leads to need of integration of functions of the type $\mathcal{R}^\gamma\eta_{-b}^{-a}(\mathbf{x},1)$ with $\gamma>0$, which are for arising a,b always integrable.
Puting all calculations together, chosing $\mathcal{R}$ and $\mathcal{W}$ sufficiently small and C_0 sufficiently large, we end up with

$$\|(\mathbf{u},q)\|_V\le C_0, \tag{25}$$

therefore $\mathcal{M}$ maps balls of radius C_0 into themselves and the proof is finished.

References

[1] Koch, H.: *Partial differential equations and singular integrals*, Dispersive nonlinear problems in mathematical physics, Quad. Mat., **15**, Dept. Math., Seconda Univ. Napoli, Caserta (2004) 59–122.

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