Stochastic Lagrangian Particle Systems for the Navier-Stokes and Burgers' equations

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The Euler and Navier-Stokes equations

Evolution of the velocity field of a fluid

$$
\text{(NS)} \qquad \begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \triangle u + \nabla p = f \\ \nabla \cdot u = 0 \qquad \text{(Incompressibility)} \end{cases}
$$

Where

u = the velocity field of the fluid.

f = External forces. [Usually 0 for this talk.]

$$
\nu = \text{Viscosity} \begin{cases} \nu > 0 & \text{Viscous dissipation [Navier Stokes]} \\ \nu = 0 & \text{Conservative (inviscid fluids) [Euler]} \end{cases}
$$

p = ‘Pressure’ [not physical for incompressible fluids]

- Navier-Stokes = Euler + Viscous dissipation. Expect a ‘natural’ stochastic representation as an average of Euler + noise.

A few exact stochastic representations of Navier-Stokes

1. Chorin (1973) introduced the *Random vortex method* in 2D: If $\omega = \nabla \times u$, then $u = -\triangle^{-1}\nabla \times \omega$ and

$$\partial_t \omega + (u \cdot \nabla)\, \omega - \nu \triangle \omega = 0.$$

2. LeJan and Sznitman (1997) Stochastic cascade by backward in time branching in Fourier space.

3. Yuri Glihcich (1997) Stochastic differential geometry.

4. Busnello, Flandoli and Romito (2005): Noisy flow paths + Girsanov.

5. 2006/7 Cipriano, Cruzerio / Eyink: Stochastic variational approaches.

A stochastic-Lagrangian formulation of Navier-Stokes

First consider a Lagrangian formulation of Euler. Then add noise to trajectories and average.

- Let X be the flow map of an inviscid fluid.

$$\dot{X} = u(X)$$
$$X_0(a) = a$$

- The Euler equations are equivalent to the assumptions

$$\nabla \cdot u = 0 \qquad \text{and} \qquad \ddot{X} = \nabla p$$

- The assumption on $\ddot{X}$ is not suitable when we add noise. Thus we eliminate time derivatives: Put $A_t = X_t^{-1}$, and integrate along trajectories. Get

$$u_t = \mathbf{P}\left[(\nabla^* A_t)(u_0 \circ A_t)\right]$$

where $\mathbf{P}$ is the Leray-Hodge projection.

A stochastic-Lagrangian formulation of Navier-Stokes

Euler		Navier-Stokes
$\dot{X} = u(X)$ $A_t = X_t^{-1}$ $u_t = \mathbf{P}\left[(\nabla^* A_t)(u_0 \circ A_t)\right]$	$\rightarrow$	$dX = u(X)\,dt + \sqrt{2\nu}dW_t$ $A_t = X_t^{-1}$ $u_t = E\mathbf{P}\left[(\nabla^* A_t)(u_0 \circ A_t)\right]$

Theorem 1 (Constantin, Iyer '06)**.** *Let X be the flow defined by*

$$dX_t = u_t\,dt + \sqrt{2\nu}\,dW_t$$

with $X_0(a) = a$ and $A = X^{-1}$. Then u is a solution of the incompressible Navier-Stokes equations with initial data u_0 if and only if

$$u_t = E\mathbf{P}[(\nabla^* A_t)(u_0 \circ A_t)] \qquad \text{(W)}$$

Idea behind the proof.

- Say

$$dX_t(a) = u_t(X_t(a))\,dt + \sqrt{2\nu}dW_t, \quad \text{with} \quad X_0(a) = a$$

- Claim: If θ is constant along trajectories of X, then θ satisfies

(*) $$d\theta_t + (u_t \cdot \nabla)\theta_t\,dt - \nu\triangle\theta_t\,dt + \sqrt{2\nu}\nabla\theta\,dW_t = 0$$

- In particular $A_t = X_t^{-1}$ and $\theta_t = \theta_0 \circ A_t$ both satisfy (*).
- Thus $\bar{\theta}_t = E\theta_0 \circ A_t$ satisfies the heat equation with initial data θ_0.

(H) $$\partial_t\bar{\theta} + (u \cdot \nabla)\,\bar{\theta} - \nu\triangle\bar{\theta} = 0$$

- When $\nu = 0$, this reduces to the method of characteristics.

A few consequences

- Representation for the vorticity. $\omega = \nabla \times u$ is given by

(3D) $$\omega_t = E\big([(\nabla X_t)\omega_0] \circ A_t\big)$$

(2D) $$\omega_t = E[\omega_0 \circ A_t]$$

- Conservation of circulation. For closed curves, for closed curves,

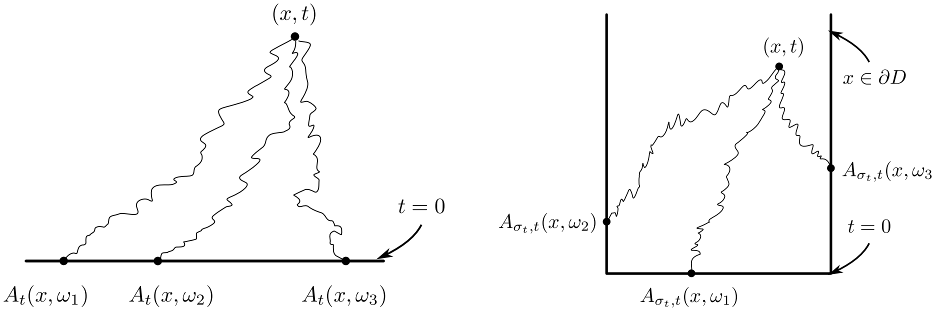
$$\oint_\Gamma u_t \cdot dl = E \oint_{A_t(\Gamma)} u_0 \cdot dl.$$

- Self-contained proofs of local existence in $C^{1,\alpha}$. Global existence for 2D, or $C^{1,\alpha}$ small initial data (Iyer '05, '07).

- Formulation for MHD and related equations & applications to Dynamos. (Eyink, '09).

- Time reversed formulation without spatial inverses, and existence theorems in Sobolev spaces (Zhang, '09).

Bounded domains (heat equation)



(a) No boundaries (b) Boundaries

Without boundaries: $\bar{\theta}(x,t) = E\theta_0(A_t(x))$. Average the initial data over all trajectories of X starting at time 0, which reach x at time t.

With boundaries: $\bar{\theta}(x,t) = E\theta_{\sigma_t(x)}(A_{\sigma_t(x),t}(x))$. Must additionally average the boundary conditions over trajectories of X starting on the sides of the cylinder $D \times [0,T]$.

Bounded domains (heat equation)

- Let $D \subset \mathbb{R}^3$ be Lipshitz. Assume u has a Lipshitz extension to $\mathbb{R}^3$.
- $X_{s,t}(a) = a + \int_s^t u_r(X_{s,r}(a))\,dr + \sqrt{2\nu}(W_t - W_s)$.
- $A_{s,t} = (X_{s,t})^{-1}$.
- $\sigma_t(x) = \inf\{s \mid \forall r > s,\ A_{r,t}(x) \in D\}$ ('Backward' exit time)
- Then $\bar{\theta}_t(x) = E\left[\chi_{\{\sigma_t(x)=0\}} f(A_{0,t}(x)) + \chi_{\{\sigma_t(x)>0\}} g_{\sigma_t(x)}(A_{\sigma_t(x),t}(x))\right]$ solves the heat equation with I.D. f, and B.C. g.
- Thus, expect $u_t = \mathbf{P}E\left[\nabla^* A_{\sigma_t,t}\left(\chi_{\{\sigma_t=0\}} u_0 \circ A_{0,t} + \chi_{\{\sigma_t>0\}} 0\right)\right]$ gives the desired stochastic representation for NS with *no-slip* boundary conditions.
- This is false! For Navier-Stokes, this is essentially missing the "vorticity produced at the boundary".

Navier-Stokes with no-slip B.C.

Theorem 2 (In preparation)**.** *For any* $\tilde{w} : \partial D \times [0, T]$*, if*

$$u_t = \mathbf{P}E\left[\nabla^* A_{\sigma_t,t}\left(\chi_{\{\sigma_t=0\}} u_0 \circ A_{0,t} + \chi_{\{\sigma_t>0\}} \tilde{w}_{\sigma_t} \circ A_{\sigma_t,t}\right)\right],$$

then u *be a solution of the Navier-Stokes equations. Further,* $\tilde{w}$ *can be chosen such that* u *satisfies the no-slip boundary conditions.*

- Given the vorticity ω on the boundary, $\tilde{w}$ must be the boundary values of the solution to

$$\partial_t \bar{w} + (u \cdot \nabla)\bar{w} - \nu \triangle \bar{w} + (\nabla^* u)\bar{w} = 0$$

$$\bar{w}\big|_{t=0} = u_0$$

$$\nabla \times \bar{w} = \omega \qquad \text{on } \partial D.$$

- Closely related to the Guage system.

- Proof is very different in the bounded domain case.

Particle systems[1]

Exact	Approximation
$dX_t = u_t(X_t)\,dt + \sqrt{2\nu}dW_t$	$dX_t^{i,N} = u_t^N(X_t^{i,N})\,dt + \sqrt{2\nu}\,dW_t^i$
$u = E\mathbf{P}\left[(\nabla^* A)\,u_0 \circ A\right]$	$u_t^N = \frac{1}{N}\sum_{i=1}^N \mathbf{P}\left[(\nabla^* A_t^{i,N})\,(u_0 \circ A_t^{i,N})\right]$

- Global existence in 2D, local in 3D.
- Convergence to Navier-Stokes:

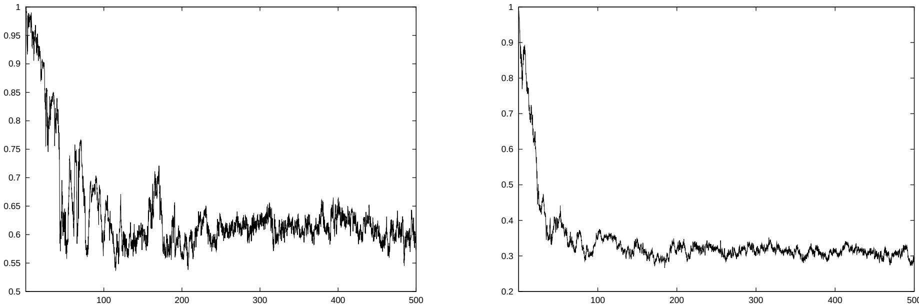
$$\sup_{t\in[0,T]} E\left\|u_t^N - u_t\right\|_{L^2}^2 \leqslant \frac{C}{N}$$

for any T in 2D (or for small T in 3D).

- Propagation of Chaos type estimates are harder to obtain.

[1] Joint work with Jonathan Mattingly

Anomalous long time energy dissipation.



(c) $N = 2$

(d) $N = 8$

Figure 1: Graph of $\|\omega_t^N\|_{L^2}^2$ vs time

Theorem 3. $\limsup_{t\to\infty} E\|\nabla u_t^N\|_{L^2}^2 \geqslant \frac{1}{L^2 N}\|u_0\|_{L^2}^2$

The Burgers' equation

(B) $$\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0$$

- When $\nu = 0$, classical solutions exist only for finite time.
- Classical solutions can be continued as weak solutions.
- The continued weak solutions are classical on regions separated by a shock.
- They satisfy a compatibility condition [Rankin-Hugonit] along the shock.
- One can choose a unique such solution satisfying Lax's entropy condition.
- For $\nu > 0$ global existence of classical solutions is known.

A stochastic Lagrangian approach

Inviscid	Viscous ($\nu = \frac{1}{2}$)	Particle system
$\dot{X}_t = u_t(X_t)$	$dX_t = u_t(X_t)\,dt + dW_t$	$dX_t^{i,N} = u_t^N\,dt + dW_t^i$
$u_t = u_0 \circ A_t$	$u_t = E u_0 \circ A_t$	$u_t^N = \frac{1}{N}\sum_{i=1}^{N} u_0 \circ A_t^{i,N}$
Shocks	Global smooth solutions	Shocks almost surely.

- Particle system shocks almost surely in finite time, even for large N.
- The shock time is bounded below almost surely, and before the shock time, u^N from the particle system converges to the smooth solution of the viscous Burgers' equation.
- Relevant equations here are second order SPDE's. A Rankin-Hugonit doesn't exist, and can't continue solutions past shocks.

Avoiding shocks via resetting.

Shocks can surprisingly be avoided by resetting:

- Fix N. Solve the particle system for short time δ_t.
- Replace the initial data u_0 with the solution at time δ_t, and restart.

Explicitly, let τ be a stopping time, $t_0 \geqslant 0$, u_{t_0} be given. If u satisfies

$$X^{i,N}_{t_0,t}(a) = a + \int_{t_0}^{\tau\wedge t} u_s(X^{i,N}_{t_0,s})\,ds + \int_{t_0}^{\tau\wedge t} dW^i_s$$

$$A^{i,N}_{t_0,t} = \left(X^{i,N}_{t_0,t}\right)^{-1}$$

$$u_t = \frac{1}{N}\sum_{i=1}^{N} u_{t_0} \circ A^{i,N}_{t_0,t},$$

then define $S^{N,\tau}_{t_0,t} u_{t_0} = u_t$.

Define iteratively the process

$$\begin{aligned} u_t^{\delta_t} &= S_{0,t}^{N,\tau} u_0 && \text{when } t \in (0, \delta_t] \\ u_t^{\delta_t} &= S_{\delta_t,t}^{N,\tau} u_{\delta_t} && \text{when } t \in (\delta_t, 2\delta_t] \\ &\text{etc.} \end{aligned}$$

Then for any T large, ε small and regular initial data, the solution to (PMB) is regular up to time T with probability at least $1 - \varepsilon$.

Theorem 4 (Iyer, Novikov '09)**.** *Let $s > 6 + \frac{1}{2}$, and suppose $u_0 \in H^s(\mathbb{T})$. Given any $T > 0$, $\varepsilon > 0$, there exists $\delta_T = \delta_T(T, \varepsilon, s, \|u_0\|_{H^s}) > 0$, such that for any $N > 1$, $\delta_t < \delta_T$, there exists a stopping time τ such that $P(\tau > T) > 1 - \varepsilon$ and the process u^{δ_t} defined by*

$$\text{(PMB)} \qquad \begin{cases} u_t^{\delta_t} = u_0 & \textit{when } t = 0, \\ u_t^{\delta_t} = S_{k\delta_t,t}^{N,\tau} u_{k\delta_t}^{\delta_t} & \textit{whenever } t \in (k\delta_t, (k+1)\delta_t] \textit{ for some } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

is in the space $\mathcal{C}^6([0,\tau]; \mathbb{T})$.

‘Debate’ on the validity of the theorem.

- **Argument against:**
  - When $\frac{1}{N}\sum_1^N$ is replaced by E, the system is Markovian – Restarting won't change anything!
  - Restarting should have no regularising effect. Consider a time-split version of $S = \tilde{S}^1 \tilde{S}^2$, where

$$\tilde{S}^1 = \text{Solution operator of the Burgers equation}$$

$$\tilde{S}^2 f(x) = \sum_{i=1}^{N} f(x + W_t^i)$$

    Then $\|\tilde{S}^2 f\| = \|f\|$ in all Hölder and Sobolev spaces.

- **Argument for: (‘Main message of this part’)**
  - Applying $\tilde{S}^2$ *repeatedly*, is smoothing with large probability (even if we intermix $\tilde{S}^2$ with $\tilde{S}^1$).

Key idea behind the proof.

First find a SPDE for u^{δ_t} on $[k\delta_t, (k+1)\delta_t]$. By Itô we get

$$du_t^{\delta_t} + \left(u_t^{\delta_t}\partial_x u_t^{\delta_t} - \tfrac{1}{2}\partial_x^2 u_t^{\delta_t}\right)\, dt + \frac{1}{N}\sum_{i=1}^{N} \partial_x (u_{k\delta_t}^{\delta_t} \circ A_{k\delta_t,t}^{i,N})\, dW_s^i = 0.$$

Expect $u_{k\delta_t}^{\delta_t} \circ A_{k\delta_t,t}^{i,N}$ for $i \in \{1, \dots, N\}$ to all be *approximately* equal when $t - k\delta_t$ is small. Thus expect u^{δ_t} to be close to the solution of

$$\text{(V)} \qquad dv_t + v_t\partial_x v\, dt - \tfrac{1}{2}\partial_x^2 v\, dt + \partial_x v_t \frac{1}{N}\sum_{1}^{N} dW_t^i = 0.$$

Lemma (Key). *Let v solve (V), with $v_0 = u_0$. If $\|u_t^{\delta_t}\|_{C^6}$ and $\|v_t\|_{C^6}$ are bounded a.s. up to time T_0, then $C = C(\|u^{\delta_t}\|_{C^6}, \|v\|_{C^6})$ such that*

$$\sup_{t \leqslant T_0} E\left\|u_t^{\delta_t} - v_t\right\|_{H^2} \leqslant C\sqrt{\delta_t}$$

"Applying $\tilde{S}_2$ often enough is smoothing" translates to the equation

$$\text{(V)} \qquad dv_t + v_t \partial_x v \, dt - \tfrac{1}{2} \partial_x^2 v \, dt + \partial_x v_t \frac{1}{N} \sum_1^N dW_t^i = 0.$$

being dissipative! Indeed, for $N > 1$, Itô's formula shows

$$d\|v_t\|_{L^2}^2 = -\left(1 - \frac{1}{N}\right) \|\partial_x v_t\|_{L^2}^2 \, dt$$

Lemma 5. $\sup_{t \geqslant 0} \|v_t\|_{H^s} \leqslant C(s, \|v_0\|_{H^s})$ *almost surely.*

Finally to finish, need local existence for u^{δ_t}:

Lemma 6. *Local existence for u^{δ_t} holds, and the existence time only depends on the C^1 norm. (On the existence interval, any additional smoothness of initial data is preserved).*

Sketch of the proof

- Let $V_1 = \sup_{t \geqslant 0} \|v_t\|_{C^1} < \infty$.
- Let $T_0 =$ local existence time for C^1 initial data smaller than $2V_1$.
- Let Ω_1 be the event $\{\|u^{\delta_t}_{T_0}\|_{C^1} \leqslant 2V_1\}$.

$$\begin{aligned}
P(\Omega_1) &\geqslant P\left(\left\|u^{\delta_t}_{T_0} - v_{T_0}\right\|_{C^1} \leqslant V_1\right) \\
&\geqslant P\left(\left\|u^{\delta_t}_{T_0} - v_{T_0}\right\|_{H^2} \leqslant \tfrac{V_1}{c_1}\right) && \text{[Sobolev embedding]} \\
&\geqslant 1 - \frac{c_1^2}{V_1^2} E\left(\left\|u^{\delta_t}_{T_0} - v_{T_0}\right\|^2_{H^2}\right) && \text{[Chebyshev's inequality]} \\
&\geqslant 1 - \frac{C\delta_t^{1/2}}{V_1^2} && \text{[Key lemma],}
\end{aligned}$$

- Now set $\tau_1 = \begin{cases} 2T_0 & \text{in } \Omega_1 \\ T_0 & \text{outside } \Omega_1, \end{cases}$ and iterate.

Super condensed idea behind the proof.

(To be used when I've run out of time)

- Show that as $\delta_t \to 0$, $u_t^{\delta_t} \to v$ and v satisfies

$$\text{(V)} \qquad dv_t + v_t \partial_x v \, dt - \tfrac{1}{2}\partial_x^2 v \, dt + \partial_x v_t \frac{1}{N}\sum_1^N dW_t^i = 0$$

- Eqn. (V) is dissipative for $N > 1$! Prove a strong norm of v is uniformly bounded in time.
- Show $\sup_{t \leqslant T} E\|u_t^{\delta_t} - v_t\|_{H^s}^2 \leqslant C\sqrt{\delta_t}$ almost surely.
- Gives a uniform in time bound on $\|u_t^{\delta_t}\|_{C^1}$ with large probability.
- Local existence depends only on $\|u_t^{\delta_t}\|_{C^1}$. $\implies$ Global existence.