Variational models for incompressible Euler equations

Alessio Figalli

The University of Texas at Austin,
Austin TX 78712, USA

Incompressible Euler equations

The Euler equations describe the evolution in time of the velocity $\boldsymbol{u}$ of an incompressible fluid:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p & \text{in } [0,T]\times D, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } [0,T]\times D, \\ \boldsymbol{u}\cdot n = 0 & \text{on } [0,T]\times \partial D. \end{cases}$$

Flow of $\boldsymbol{u}$:

$$\begin{cases} \dot{g}(t,a) = \boldsymbol{u}(t, g(t,a)), \\ g(0,a) = a. \end{cases}$$

Incompressible Euler equations

The Euler equations describe the evolution in time of the velocity $\boldsymbol{u}$ of an incompressible fluid:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p & \text{in } [0, T] \times D, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } [0, T] \times D, \\ \boldsymbol{u} \cdot n = 0 & \text{on } [0, T] \times \partial D. \end{cases}$$

Flow of $\boldsymbol{u}$:

$$\begin{cases} \dot{g}(t, a) = \boldsymbol{u}(t, g(t, a)), \\ \\ g(0, a) = a. \end{cases}$$

Incompressible Euler equations

The Euler equations describe the evolution in time of the velocity $\boldsymbol{u}$ of an incompressible fluid:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p & \text{in } [0, T] \times D, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } [0, T] \times D, \\ \boldsymbol{u} \cdot n = 0 & \text{on } [0, T] \times \partial D. \end{cases}$$

Flow of $\boldsymbol{u}$:

$$\begin{cases} \dot{g}(t, a) = \boldsymbol{u}(t, g(t, a)), \\ g(0, a) = a. \end{cases}$$

If $\boldsymbol{u}$ is smooth, then $g(t,\cdot) : D \to D$ is a diffeomorphism and

$$\operatorname{div} \boldsymbol{u} = 0 \quad \Rightarrow \quad g(t,\cdot)_\# \operatorname{vol}_D = \operatorname{vol}_D,$$

i.e. $\operatorname{vol}_D(g(t,\cdot)^{-1}(E)) = \operatorname{vol}_D(E)$ for all E Borel.
(vol_D = volume measure in D, renormalized to be a probability)

$$\text{Euler} \quad \Longleftrightarrow \quad \begin{cases} \ddot{g}(t,a) = -\nabla p(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases}$$

$$\mathrm{SDiff}(D) = \{\text{measure preserving diffeomorphisms}\}.$$

If $\boldsymbol{u}$ is smooth, then $g(t,\cdot) : D \to D$ is a diffeomorphism and

$$\operatorname{div} \boldsymbol{u} = 0 \quad \Rightarrow \quad g(t,\cdot)_{\#} \operatorname{vol}_D = \operatorname{vol}_D,$$

i.e. $\operatorname{vol}_D(g(t,\cdot)^{-1}(E)) = \operatorname{vol}_D(E)$ for all E Borel.
($\operatorname{vol}_D =$ volume measure in D, renormalized to be a probability)

$$\text{Euler} \quad \Longleftrightarrow \quad \begin{cases} \ddot{g}(t,a) = -\nabla p(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases}$$

$$\mathrm{SDiff}(D) = \{\text{measure preserving diffeomorphisms}\}.$$

If $\boldsymbol{u}$ is smooth, then $g(t,\cdot) : D \to D$ is a diffeomorphism and

$$\operatorname{div} \boldsymbol{u} = 0 \quad \Rightarrow \quad g(t,\cdot)_\# \mathrm{vol}_D = \mathrm{vol}_D,$$

i.e. $\mathrm{vol}_D(g(t,\cdot)^{-1}(E)) = \mathrm{vol}_D(E)$ for all E Borel.
($\mathrm{vol}_D =$ volume measure in D, renormalized to be a probability)

$$\text{Euler} \iff \begin{cases} \ddot{g}(t,a) = -\nabla p\,(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases}$$

$$\mathrm{SDiff}(D) = \{\text{measure preserving diffeomorphisms}\}.$$

If $\boldsymbol{u}$ is smooth, then $g(t,\cdot) : D \to D$ is a diffeomorphism and

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \Rightarrow \quad g(t,\cdot)_{\#} \mathrm{vol}_D = \mathrm{vol}_D,$$

i.e. $\mathrm{vol}_D(g(t,\cdot)^{-1}(E)) = \mathrm{vol}_D(E)$ for all E Borel.
(vol_D = volume measure in D, renormalized to be a probability)

$$\text{Euler} \iff \begin{cases} \ddot{g}(t,a) = -\nabla p(t, g(t,a)) & (t,a) \in [0,T] \times D, \\ g(0,a) = a & a \in D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & t \in [0,T]. \end{cases}$$

$$\mathrm{SDiff}(D) = \{\text{measure preserving diffeomorphisms}\}.$$

Weak solutions

$d = 2$: well-posedness of distributional solutions with bounded vorticity (Yudovich, 1963).

$d \geq 3$: "very" weak solutions:
Young measure solutions (DiPerna-Majda, 1987);
Maximally dissipative solutions (P.L.Lions, 1995).

Weak solutions

$d = 2$: well-posedness of distributional solutions with bounded vorticity (Yudovich, 1963).

$d \geq 3$: "very" weak solutions:
Young measure solutions (DiPerna-Majda, 1987);
Maximally dissipative solutions (P.L.Lions, 1995).

Weak solutions

$d = 2$: well-posedness of distributional solutions with bounded vorticity (Yudovich, 1963).

$d \geq 3$: "very" weak solutions:
Young measure solutions (DiPerna-Majda, 1987);
Maximally dissipative solutions (P.L.Lions, 1995).

Arnold interpratation

Arnold (1966):

$$(SDiff, \|\cdot\|_{L^2}) \simeq \text{ infinite dimensional manifold}$$

$$\text{Tangent space } \simeq \text{ divergence free vector fields}$$

$$\text{Euler equations } \simeq \text{ geodesic equations}$$

One looks for solutions to Euler equations by minimizing

$$\int_0^1 \frac{1}{2} \int_D |\dot{g}(t,x)|^2 \, d\,\mathrm{vol}_D(x)\, dt$$

with $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$, $g(0,\cdot) = f$ and $g(1,\cdot) = h$ (by right invariance, one may assume $f = \boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

$$\delta(f,h) := \text{Arnold distance.}$$

Arnold interpratation

Arnold (1966):

$$(SDiff, \|\cdot\|_{L^2}) \simeq \text{ infinite dimensional manifold}$$

$$\text{Tangent space } \simeq \text{ divergence free vector fields}$$

$$\text{Euler equations } \simeq \text{ geodesic equations}$$

One looks for solutions to Euler equations by minimizing

$$\int_0^1 \frac{1}{2} \int_D |\dot{g}(t,x)|^2 \, d\,\mathrm{vol}_D(x)\, dt$$

with $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$, $g(0,\cdot) = f$ and $g(1,\cdot) = h$ (by right invariance, one may assume $f = \boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

$$\delta(f,h) := \text{Arnold distance.}$$

Arnold interpratation

Arnold (1966):

$$(SDiff, \|\cdot\|_{L^2}) \simeq \text{ infinite dimensional manifold}$$

$$\text{Tangent space } \simeq \text{ divergence free vector fields}$$

$$\text{Euler equations } \simeq \text{ geodesic equations}$$

One looks for solutions to Euler equations by minimizing

$$\int_0^1 \frac{1}{2} \int_D |\dot{g}(t,x)|^2 \, d\,\mathrm{vol}_D(x)\, dt$$

with $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$, $g(0,\cdot) = f$ and $g(1,\cdot) = h$ (by right invariance, one may assume $f = \boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

$$\delta(f,h) := \text{Arnold distance.}$$

Arnold interpratation

Arnold (1966):

$$(SDiff, \|\cdot\|_{L^2}) \simeq \text{ infinite dimensional manifold}$$

$$\text{Tangent space } \simeq \text{ divergence free vector fields}$$

$$\text{Euler equations } \simeq \text{ geodesic equations}$$

One looks for solutions to Euler equations by minimizing

$$\int_0^1 \frac{1}{2} \int_D |\dot{g}(t,x)|^2 \, d\,\mathrm{vol}_D(x)\, dt$$

with $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$, $g(0,\cdot) = f$ and $g(1,\cdot) = h$ (by right invariance, one may assume $f = \boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

$\delta(f,h) :=$ Arnold distance.

Arnold interpratation

Arnold (1966):

$$(SDiff, \|\cdot\|_{L^2}) \simeq \text{ infinite dimensional manifold}$$

$$\text{Tangent space } \simeq \text{ divergence free vector fields}$$

$$\text{Euler equations } \simeq \text{ geodesic equations}$$

One looks for solutions to Euler equations by minimizing

$$\int_0^1 \frac{1}{2} \int_D |\dot{g}(t,x)|^2 \, d\,\mathrm{vol}_D(x)\, dt$$

with $g(t,\cdot) : [0,1] \to \mathrm{SDiff}(D)$, $g(0,\cdot) = f$ and $g(1,\cdot) = h$ (by right invariance, one may assume $f = \boldsymbol{i}$).

Existence: Ebin-Marsden (1970), $g \circ f^{-1} \sim \boldsymbol{i}$.

$$\delta(f,h) := \text{Arnold distance.}$$

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.
(b) Necessary and sufficient optimality conditions.
(c) Regularity of the pressure field.
(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point.

Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.

(b) Necessary and sufficient optimality conditions.

(c) Regularity of the pressure field.

(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.
(b) Necessary and sufficient optimality conditions.
(c) Regularity of the pressure field.
(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.

(b) Necessary and sufficient optimality conditions.

(c) Regularity of the pressure field.

(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.

(b) Necessary and sufficient optimality conditions.

(c) Regularity of the pressure field.

(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.

(b) Necessary and sufficient optimality conditions.

(c) Regularity of the pressure field.

(d) Regularity of (relaxed) curves with minimal length.

This variational problem is different from solving Euler equations: one does not fix the initial velocity of the geodesic, but the final point. Nevertheless, the investigation of this problem leads to difficult and still not completely understood questions (typical of Calculus of Variations) namely:

(a) Non-attainment and Lavrentiev (gap) phenomena.

(b) Necessary and sufficient optimality conditions.

(c) Regularity of the pressure field.

(d) Regularity of (relaxed) curves with minimal length.

Non-existence results

Shnirelman:

$$\exists\, g \in \mathrm{SDiff}([0,1]^2) \quad \text{s.t.} \quad \delta(\boldsymbol{i}, g) = +\infty.$$

Moreover, if

$$h(x_1, x_2, x_3) := (g(x_1, x_2), x_3) \in \mathrm{SDiff}([0,1]^3), \qquad g \text{ as above},$$

then $\delta(\boldsymbol{i}, h)$ is not attained:
if a minimizer $t \mapsto g(t)$ exists and $\boldsymbol{u} := \dot{g} \circ g^{-1}$, then

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \boldsymbol{u}(x_1, x_2, \eta(x_3)), \qquad \eta(x_3) := \min\{2x_3, 2 - 2x_3\},$$

induces a path $\tilde{g}$ with strictly less action and still joining $\boldsymbol{i}$ to h.
Minimizing sequences exhibit oscillations on small scales $\rightsquigarrow$ need of introducing some relaxed solutions.

Non-existence results

Shnirelman:

$$\exists\, g \in \mathrm{SDiff}([0,1]^2) \quad \text{s.t.} \quad \delta(\boldsymbol{i}, g) = +\infty.$$

Moreover, if

$$h(x_1, x_2, x_3) := (g(x_1, x_2), x_3) \in \mathrm{SDiff}([0,1]^3), \qquad g \text{ as above},$$

then $\delta(\boldsymbol{i}, h)$ is not attained:
if a minimizer $t \mapsto g(t)$ exists and $\boldsymbol{u} := \dot{g} \circ g^{-1}$, then

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \boldsymbol{u}(x_1, x_2, \eta(x_3)), \qquad \eta(x_3) := \min\{2x_3, 2 - 2x_3\},$$

induces a path $\tilde{g}$ with strictly less action and still joining $\boldsymbol{i}$ to h.
Minimizing sequences exhibit oscillations on small scales $\rightsquigarrow$ need of introducing some relaxed solutions.

Non-existence results

Shnirelman:

$$\exists\, g \in \mathrm{SDiff}([0,1]^2) \quad \text{s.t.} \quad \delta(\boldsymbol{i}, g) = +\infty.$$

Moreover, if

$$h(x_1, x_2, x_3) := (g(x_1, x_2), x_3) \in \mathrm{SDiff}([0,1]^3), \qquad g \text{ as above},$$

then $\delta(\boldsymbol{i}, h)$ is not attained:
if a minimizer $t \mapsto g(t)$ exists and $\boldsymbol{u} := \dot{g} \circ g^{-1}$, then

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \boldsymbol{u}(x_1, x_2, \eta(x_3)), \qquad \eta(x_3) := \min\{2x_3, 2 - 2x_3\},$$

induces a path $\tilde{g}$ with strictly less action and still joining $\boldsymbol{i}$ to h.
Minimizing sequences exhibit oscillations on small scales $\rightsquigarrow$ need of introducing some relaxed solutions.

Non-existence results

Shnirelman:

$$\exists\, g \in \mathrm{SDiff}([0,1]^2) \quad \text{s.t.} \quad \delta(\boldsymbol{i}, g) = +\infty.$$

Moreover, if

$$h(x_1, x_2, x_3) := (g(x_1, x_2), x_3) \in \mathrm{SDiff}([0,1]^3), \qquad g \text{ as above},$$

then $\delta(\boldsymbol{i}, h)$ is not attained:
if a minimizer $t \mapsto g(t)$ exists and $\boldsymbol{u} := \dot{g} \circ g^{-1}$, then

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \boldsymbol{u}(x_1, x_2, \eta(x_3)), \qquad \eta(x_3) := \min\{2x_3, 2 - 2x_3\},$$

induces a path $\tilde{g}$ with strictly less action and still joining $\boldsymbol{i}$ to h.

Minimizing sequences exhibit oscillations on small scales $\rightsquigarrow$ need of introducing some relaxed solutions.

Non-existence results

Shnirelman:

$$\exists\, g \in \mathrm{SDiff}([0,1]^2) \quad \text{s.t.} \quad \delta(\boldsymbol{i}, g) = +\infty.$$

Moreover, if

$$h(x_1, x_2, x_3) := (g(x_1, x_2), x_3) \in \mathrm{SDiff}([0,1]^3), \qquad g \text{ as above},$$

then $\delta(\boldsymbol{i}, h)$ is not attained:
if a minimizer $t \mapsto g(t)$ exists and $\boldsymbol{u} := \dot{g} \circ g^{-1}$, then

$$\tilde{\boldsymbol{u}}(x_1, x_2, x_3) := \boldsymbol{u}(x_1, x_2, \eta(x_3)), \qquad \eta(x_3) := \min\{2x_3, 2 - 2x_3\},$$

induces a path $\tilde{g}$ with strictly less action and still joining $\boldsymbol{i}$ to h.

Minimizing sequences exhibit oscillations on small scales $\rightsquigarrow$ need of introducing some relaxed solutions.

Goal of the talk:
introduce the theory of relaxed solutions (Brenier, 1989-1999) and study the properties of minimizers.

References:
Brenier, 1989-1999;
Shnirelman, 1994;
Ambrosio-F., 2008;
Bernot-F.-Santambrogio, 2008.

Goal of the talk:
introduce the theory of relaxed solutions (Brenier, 1989-1999) and study the properties of minimizers.

References:
Brenier, 1989-1999;
Shnirelman, 1994;
Ambrosio-F., 2008;
Bernot-F.-Santambrogio, 2008.

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \ g_{\#}\,\mathrm{vol}_D = \mathrm{vol}_D\}\,.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ (\pi_1)_{\#}\eta = (\pi_2)_{\#}\eta = \mathrm{vol}_D\}\,,$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_{\#}\,\mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \ g_\# \mathrm{vol}_D = \mathrm{vol}_D\}.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ (\pi_1)_\# \eta = (\pi_2)_\# \eta = \mathrm{vol}_D\},$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \; g_\# \mathrm{vol}_D = \mathrm{vol}_D\}.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \; (\pi_1)_\# \eta = (\pi_2)_\# \eta = \mathrm{vol}_D\},$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t, \cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \; g_\# \, \mathrm{vol}_D = \mathrm{vol}_D\} .$$

2nd relaxation: introduce multivalued maps

$g(t, \cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \; (\pi_1)_\# \eta = (\pi_2)_\# \eta = \mathrm{vol}_D\} ,$$

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y, \qquad \forall \, (x, y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \, \mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \; g_{\#}\,\mathrm{vol}_D = \mathrm{vol}_D\}\,.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \; (\pi_1)_{\#}\eta = (\pi_2)_{\#}\eta = \mathrm{vol}_D\}\,,$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_{\#}\,\mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \ g_\# \mathrm{vol}_D = \mathrm{vol}_D\}.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ (\pi_1)_\#\eta = (\pi_2)_\#\eta = \mathrm{vol}_D\},$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall (x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \ g_\# \,\mathrm{vol}_D = \mathrm{vol}_D\}\,.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \ (\pi_1)_\#\eta = (\pi_2)_\#\eta = \mathrm{vol}_D\}\,,$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \,\mathrm{vol}_D \in \Gamma(D).$$

Relaxed solutions

1st relaxation: relax the smoothness constraint

$g(t,\cdot)$ is a *measure-preserving map* (not necessarily injective):

$$S(D) := \{g : D \to D : \; g_\# \,\mathrm{vol}_D = \mathrm{vol}_D\}\,.$$

2nd relaxation: introduce multivalued maps

$g(t,\cdot)$ is not necessarily a map, but a *measure preserving plan* (roughly speaking, a multivalued map which preserves the measure):

$$\Gamma(D) := \{\eta \in \mathscr{P}(D \times D) : \; (\pi_1)_\#\eta = (\pi_2)_\#\eta = \mathrm{vol}_D\}\,,$$

$$\pi_1(x,y) = x, \quad \pi_2(x,y) = y, \qquad \forall\,(x,y) \in D \times D.$$

$S(D) \subset \Gamma(D)$ with the identification

$$S(D) \ni g \simeq (\boldsymbol{i} \times g)_\# \,\mathrm{vol}_D \in \Gamma(D).$$

- Lagrangian approach.

- Relaxations:
  - No smoothness.
  - No injectivity.
  - Particles split (forward and backward).

- Lagrangian approach.
- Relaxations:
  - No smoothness.
  - No injectivity.
  - Particles split (forward and backward).

Brenier (1989):

$$\Omega(D) := C([0,1];D)\,, \qquad e_t(\omega) := \omega(t),\ t \in [0,1].$$

Minimize the action functional

$$\mathscr{A}(\eta) := \int_{\Omega(D)} \frac{1}{2}\int_0^1 |\dot\omega|^2\,dt\,d\eta(\omega), \qquad \eta \in \mathscr{P}(\Omega(D)),$$

$$(e_0, e_1)_\# \eta = (\boldsymbol{i} \times h)_\# \,\mathrm{vol}_D, \qquad (e_t)_\# \eta = \mathrm{vol}_D \ \ \forall\, t \in [0,1].$$

Curves in $\mathrm{SDiff}(D)$ induce *generalized incompressible flows*:

$$g(t,a) \quad \rightsquigarrow \quad \eta := (\Phi_g)_\# \,\mathrm{vol}_D,$$

where

$$\Phi_g : D \to \Omega(D), \qquad \Phi_g(a) := g(\cdot, a).$$

Brenier (1989):

$$\Omega(D) := C([0,1]; D), \qquad e_t(\omega) := \omega(t),\ t \in [0,1].$$

Minimize the action functional

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega), \qquad \boldsymbol{\eta} \in \mathscr{P}(\Omega(D)),$$

$$(e_0, e_1)_\# \boldsymbol{\eta} = (\boldsymbol{i} \times h)_\# \mathrm{vol}_D, \qquad (e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D \quad \forall\, t \in [0,1].$$

Curves in $\mathrm{SDiff}(D)$ induce *generalized incompressible flows*:

$$g(t,a) \quad \rightsquigarrow \quad \boldsymbol{\eta} := (\Phi_g)_\# \mathrm{vol}_D,$$

where

$$\Phi_g : D \to \Omega(D), \qquad \Phi_g(a) := g(\cdot, a).$$

Brenier (1989):

$$\Omega(D) := C([0,1];D), \qquad e_t(\omega) := \omega(t),\ t \in [0,1].$$

Minimize the action functional

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega(D)} \frac{1}{2}\int_0^1 |\dot\omega|^2\,dt\,d\boldsymbol{\eta}(\omega), \qquad \boldsymbol{\eta} \in \mathscr{P}(\Omega(D)),$$

$$(e_0, e_1)_\#\boldsymbol{\eta} = (\boldsymbol{i} \times h)_\#\,\mathrm{vol}_D, \qquad (e_t)_\#\boldsymbol{\eta} = \mathrm{vol}_D \quad \forall\, t \in [0,1].$$

Curves in $\mathrm{SDiff}(D)$ induce *generalized incompressible flows*:

$$g(t,a) \quad \rightsquigarrow \quad \boldsymbol{\eta} := (\Phi_g)_\#\,\mathrm{vol}_D,$$

where

$$\Phi_g : D \to \Omega(D), \qquad \Phi_g(a) := g(\cdot, a).$$

Brenier (1989):

$$\Omega(D) := C([0,1]; D), \qquad e_t(\omega) := \omega(t), \ t \in [0,1].$$

Minimize the action functional

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega), \qquad \boldsymbol{\eta} \in \mathscr{P}(\Omega(D)),$$

$$(e_0, e_1)_\# \boldsymbol{\eta} = (\boldsymbol{i} \times h)_\# \, \mathrm{vol}_D, \qquad (e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D \ \ \forall \, t \in [0,1].$$

Curves in $\mathrm{SDiff}(D)$ induce *generalized incompressible flows*:

$$g(t,a) \quad \rightsquigarrow \quad \boldsymbol{\eta} := (\Phi_g)_\# \, \mathrm{vol}_D,$$

where

$$\Phi_g : D \to \Omega(D), \qquad \Phi_g(a) := g(\cdot, a).$$

$\bar{\delta}(f, h) :=$ induced distance.

If

$$D = [0,1]^d \quad \text{or} \quad D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d,$$

then $\bar{\delta}(\boldsymbol{i}, h) \leq \sqrt{d}$.

Compactness $+$ semicontinuity $\Rightarrow$ existence of minimizers.

Problem: how to connect $f \in S(D)$ to $h \in S(D)$? (clear only if f is invertible).

Actually, we need a more general model which allows to connect $\eta = \eta_a \otimes \mathrm{vol}_D \in \Gamma(D)$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D \in \Gamma(D)$.

$$\bar{\delta}(f,h) := \text{induced distance.}$$

If

$$D = [0,1]^d \quad \text{or} \quad D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d,$$

then $\bar{\delta}(\boldsymbol{i}, h) \leq \sqrt{d}$.

Compactness + semicontinuity ⇒ existence of minimizers.

Problem: how to connect $f \in S(D)$ to $h \in S(D)$? (clear only if f is invertible).

Actually, we need a more general model which allows to connect $\eta = \eta_a \otimes \mathrm{vol}_D \in \Gamma(D)$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D \in \Gamma(D)$.

$$\bar{\delta}(f, h) := \text{induced distance.}$$

If

$$D = [0,1]^d \quad \text{or} \quad D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d,$$

then $\bar{\delta}(\boldsymbol{i}, h) \leq \sqrt{d}$.

Compactness + semicontinuity $\Rightarrow$ existence of minimizers.

Problem: how to connect $f \in S(D)$ to $h \in S(D)$? (clear only if f is invertible).

Actually, we need a more general model which allows to connect $\eta = \eta_a \otimes \text{vol}_D \in \Gamma(D)$ to $\gamma = \gamma_a \otimes \text{vol}_D \in \Gamma(D)$.

$$\bar{\delta}(f, h) := \text{induced distance.}$$

If

$$D = [0,1]^d \quad \text{or} \quad D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d,$$

then $\bar{\delta}(\boldsymbol{i}, h) \leq \sqrt{d}$.

Compactness $+$ semicontinuity $\Rightarrow$ existence of minimizers.

Problem: how to connect $f \in S(D)$ to $h \in S(D)$? (clear only if f is invertible).

Actually, we need a more general model which allows to connect $\eta = \eta_a \otimes \mathrm{vol}_D \in \Gamma(D)$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D \in \Gamma(D)$.

$$\bar{\delta}(f,h) := \text{induced distance.}$$

If

$$D = [0,1]^d \quad \text{or} \quad D = \mathbb{T}^d \sim \mathbb{R}^d/\mathbb{Z}^d,$$

then $\bar{\delta}(\boldsymbol{i}, h) \leq \sqrt{d}$.

Compactness $+$ semicontinuity $\Rightarrow$ existence of minimizers.

Problem: how to connect $f \in S(D)$ to $h \in S(D)$? (clear only if f is invertible).

Actually, we need a more general model which allows to connect $\eta = \eta_a \otimes \mathrm{vol}_D \in \Gamma(D)$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D \in \Gamma(D)$.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_{\#}\eta \in \Gamma(D) \setminus S(D)$, $t \in (0, 1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0, \pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\eta := \frac{1}{2\pi^2}(\omega_{x,\theta})_{\#}\left(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi)\right)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_\# \eta \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\eta := \frac{1}{2\pi^2}(\omega_{x,\theta})_\#\left(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi)\right)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_{\#}\boldsymbol{\eta} \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\boldsymbol{\eta} := \frac{1}{2\pi^2}(\omega_{x,\theta})_{\#}(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi))$. $\boldsymbol{\eta}$ is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_\# \boldsymbol{\eta} \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\boldsymbol{\eta} := \frac{1}{2\pi^2}(\omega_{x,\theta})_\#\left(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi)\right)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_\# \eta \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\eta := \frac{1}{2\pi^2}(\omega_{x,\theta})_\# \left(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi)\right)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_\# \eta \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\eta := \frac{1}{2\pi^2}(\omega_{x,\theta})_\# \left(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0,2\pi)\right)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Motivation for the extension to $\Gamma(D)$

There exists a minimizing η connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ which is non-deterministic in between: $(e_0, e_t)_\# \eta \in \Gamma(D) \setminus S(D)$, $t \in (0,1)$.

Example (Brenier (1989), Bernot-F.-Santambrogio (2008))

$D = B_1(0) \subset \mathbb{R}^2$ and we want to connect (x_1, x_2) to $-(x_1, x_2)$. Two classical solutions:

$$[0,\pi] \ni t \mapsto (x_1 \cos \pm t + x_2 \sin \pm t, x_1 \sin \pm t + x_2 \cos \pm t).$$

On the other hand, one can consider $\omega_{x,\theta}$ from x to $-x$

$$\omega_{x,\theta}(t) := x \cos t + \sqrt{1 - |x|^2}(\cos\theta, \sin\theta) \sin t, \qquad \theta \in (0, 2\pi),$$

and set $\eta := \frac{1}{2\pi^2}(\omega_{x,\theta})_\# \big(\mathscr{L}^2 \lfloor D \times \mathscr{L}^1 \lfloor (0, 2\pi)\big)$. η is optimal and non-deterministic in between. Moreover, there exists infinitely many other non-deterministic minimizers.

Extended model

Need to connect $\eta = \eta_a \otimes \mathrm{vol}_D$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D$ (here we are disintegrating both the initial and final plan with respect to the first variable).
Idea: "double" the state space, adding to the Eulerian state space D a Lagrangian state space A. Even though A could be thought as an identical copy of D, it is convenient to denote it by a different symbol.
Set

$$\Omega^*(D) := \Omega(D) \times D,$$

and consider probability measures $\eta = \eta_a \otimes \mathrm{vol}_D$ on $\Omega^*(D)$, with $\eta_a \in \mathscr{P}(\Omega(D))$.

Extended model

Need to connect $\eta = \eta_a \otimes \mathrm{vol}_D$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D$ (here we are disintegrating both the initial and final plan with respect to the first variable).

Idea: "double" the state space, adding to the Eulerian state space D a Lagrangian state space A. Even though A could be thought as an identical copy of D, it is convenient to denote it by a different symbol. Set

$$\Omega^*(D) := \Omega(D) \times D,$$

and consider probability measures $\eta = \eta_a \otimes \mathrm{vol}_D$ on $\Omega^*(D)$, with $\eta_a \in \mathscr{P}(\Omega(D))$.

Extended model

Need to connect $\eta = \eta_a \otimes \mathrm{vol}_D$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D$ (here we are disintegrating both the initial and final plan with respect to the first variable).
Idea: "double" the state space, adding to the Eulerian state space D a Lagrangian state space A. Even though A could be thought as an identical copy of D, it is convenient to denote it by a different symbol.
Set

$$\Omega^*(D) := \Omega(D) \times D,$$

and consider probability measures $\eta = \eta_a \otimes \mathrm{vol}_D$ on $\Omega^*(D)$, with $\eta_a \in \mathscr{P}(\Omega(D))$.

Extended model

Need to connect $\eta = \eta_a \otimes \mathrm{vol}_D$ to $\gamma = \gamma_a \otimes \mathrm{vol}_D$ (here we are disintegrating both the initial and final plan with respect to the first variable).
Idea: "double" the state space, adding to the Eulerian state space D a Lagrangian state space A. Even though A could be thought as an identical copy of D, it is convenient to denote it by a different symbol.
Set

$$\Omega^*(D) := \Omega(D) \times D,$$

and consider probability measures $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mathrm{vol}_D$ on $\Omega^*(D)$, with $\boldsymbol{\eta}_a \in \mathscr{P}(\Omega(D))$.

We minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

under the incompressibility constraints $(e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D$ and

$$(e_0)_\# \boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_\# \boldsymbol{\eta}_a = \eta_a \qquad \text{for } \mathrm{vol}_D\text{-a.e. } a \in D.$$

$$\overline{\delta}(\eta, \gamma)^2 := \text{ minimal action.}$$

$(\Gamma(D), \overline{\delta})$ is a metric space. Moreover, it is *complete* and a *length* space, whose convergence is stronger than narrow convergence.

We minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

under the incompressibility constraints $(e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D$ and

$$(e_0)_\# \boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_\# \boldsymbol{\eta}_a = \eta_a \qquad \text{for } \mathrm{vol}_D\text{-a.e. } a \in D.$$

$$\overline{\delta}(\eta, \gamma)^2 := \text{ minimal action.}$$

$(\Gamma(D), \overline{\delta})$ is a metric space. Moreover, it is *complete* and a *length* space, whose convergence is stronger than narrow convergence.

We minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

under the incompressibility constraints $(e_t)_{\#}\boldsymbol{\eta} = \mathrm{vol}_D$ and

$$(e_0)_{\#}\boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_{\#}\boldsymbol{\eta}_a = \eta_a \qquad \text{for } \mathrm{vol}_D\text{-a.e. } a \in D.$$

$$\overline{\delta}(\eta, \gamma)^2 := \text{ minimal action.}$$

$(\Gamma(D), \overline{\delta})$ is a metric space. Moreover, it is *complete* and a *length* space, whose convergence is stronger than narrow convergence.

We minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

under the incompressibility constraints $(e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D$ and

$$(e_0)_\# \boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_\# \boldsymbol{\eta}_a = \eta_a \qquad \text{for } \mathrm{vol}_D\text{-a.e. } a \in D.$$

$$\overline{\delta}(\eta, \gamma)^2 := \text{ minimal action.}$$

$(\Gamma(D), \overline{\delta})$ is a metric space. Moreover, it is *complete* and a *length* space, whose convergence is stronger than narrow convergence.

We minimize the action

$$\mathscr{A}(\boldsymbol{\eta}) := \int_{\Omega^*(D)} \frac{1}{2} \int_0^1 |\dot{\omega}|^2 \, dt \, d\boldsymbol{\eta}(\omega, a)$$

under the incompressibility constraints $(e_t)_\# \boldsymbol{\eta} = \mathrm{vol}_D$ and

$$(e_0)_\# \boldsymbol{\eta}_a = \gamma_a, \quad (e_1)_\# \boldsymbol{\eta}_a = \eta_a \qquad \text{for vol}_D\text{-a.e. } a \in D.$$

$$\overline{\delta}(\eta, \gamma)^2 := \text{ minimal action.}$$

$(\Gamma(D), \overline{\delta})$ is a metric space. Moreover, it is *complete* and a *length* space, whose convergence is stronger than narrow convergence.

Density results and relaxation

$$S(D) = \{(\boldsymbol{i} \times g)_\# \mathrm{vol}_D\} \subset \Gamma(D) \quad \text{dense.}$$

Moreover, if $D = [0,1]^d$, $d \geq 2$,

$$S(D) = \overline{\mathrm{SDiff}(D)}^{L^2(\mathrm{vol}_D)} \quad \text{(Brenier-Gangbo, 2003).}$$

Natural question:
relation between the distance $\bar{\delta}(\boldsymbol{i}, \cdot)$ and the relaxation δ_* of the Arnold distance, i.e. for any $h \in S(D)$

$$\delta_*(h) := \inf\left\{ \liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \; h_n \in \mathrm{SDiff}(D), \; \int_D |h_n - h|^2 \, d\,\mathrm{vol}_D \to 0 \right\},$$

Density results and relaxation

$$S(D) = \{(\boldsymbol{i} \times g)_\# \mathrm{vol}_D\} \subset \Gamma(D) \quad \text{dense.}$$

Moreover, if $D = [0,1]^d$, $d \geq 2$,

$$S(D) = \overline{\mathrm{SDiff}(D)}^{L^2(\mathrm{vol}_D)} \quad \text{(Brenier-Gangbo, 2003).}$$

Natural question:
relation between the distance $\bar{\delta}(\boldsymbol{i}, \cdot)$ and the relaxation δ_* of the Arnold distance, i.e. for any $h \in S(D)$

$$\delta_*(h) := \inf\left\{ \liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \ h_n \in \mathrm{SDiff}(D), \ \int_D |h_n - h|^2 \, d\,\mathrm{vol}_D \to 0 \right\},$$

Density results and relaxation

$$S(D) = \{(\boldsymbol{i} \times g)_\# \mathrm{vol}_D\} \subset \Gamma(D) \quad \text{dense.}$$

Moreover, if $D = [0,1]^d$, $d \geq 2$,

$$S(D) = \overline{\mathrm{SDiff}(D)}^{L^2(\mathrm{vol}_D)} \quad \text{(Brenier-Gangbo, 2003).}$$

Natural question:
relation between the distance $\bar{\delta}(\boldsymbol{i}, \cdot)$ and the relaxation δ_* of the Arnold distance, i.e. for any $h \in S(D)$

$$\delta_*(h) := \inf\left\{ \liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \ h_n \in \mathrm{SDiff}(D), \ \int_D |h_n - h|^2 \, d\,\mathrm{vol}_D \to 0 \right\},$$

Density results and relaxation

$$S(D) = \{(\boldsymbol{i} \times g)_\# \mathrm{vol}_D\} \subset \Gamma(D) \quad \text{dense.}$$

Moreover, if $D = [0,1]^d$, $d \geq 2$,

$$S(D) = \overline{\mathrm{SDiff}(D)}^{L^2(\mathrm{vol}_D)} \quad \text{(Brenier-Gangbo, 2003).}$$

Natural question:
relation between the distance $\overline{\delta}(\boldsymbol{i}, \cdot)$ and the relaxation δ_* of the Arnold distance, i.e. for any $h \in S(D)$

$$\delta_*(h) := \inf\left\{\liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \; h_n \in \mathrm{SDiff}(D), \; \int_D |h_n - h|^2 \, d\,\mathrm{vol}_D \to 0\right\},$$

Theorem (Density of smooth flows (Shnirelman, 1994))

$D = [0,1]^d$, $d > 2$. *For any generalized incompressible flow* $\boldsymbol{\eta}$ *between* $\boldsymbol{i}$ *and* $h \in \mathrm{SDiff}(D)$ *there exist smooth flows* g_k *connecting* $\boldsymbol{i}$ *to* h *satisfying:*

(a) $\mathscr{A}(g_k) \to \mathscr{A}(\eta)$;

(b) *the probability measures* $\boldsymbol{\eta}_k$ *induced by* g_k *narrowly converge to* $\boldsymbol{\eta}$.

Moreover, $\delta \sim \|\cdot\|_{L^2}$:

$$\frac{1}{\sqrt{2}}\|f-g\|_{L^2(\mathrm{vol}_D)} \le \delta(f,g) \le C\|f-g\|^{\alpha}_{L^2(\mathrm{vol}_D)}.$$

Hence, no gap phenomenon occurs when $d > 2$:

$d > 2 \quad \Rightarrow \quad \delta_*(h) = \overline{\delta}(\boldsymbol{i}, h) \quad \forall\, h \in S(D)$.

Theorem (Density of smooth flows (Shnirelman, 1994))

$D = [0,1]^d$, $d > 2$. For any generalized incompressible flow $\boldsymbol{\eta}$ between $\boldsymbol{i}$ and $h \in \mathrm{SDiff}(D)$ there exist smooth flows g_k connecting $\boldsymbol{i}$ to h satisfying:

(a) $\mathscr{A}(g_k) \to \mathscr{A}(\eta)$;

(b) *the probability measures $\boldsymbol{\eta}_k$ induced by g_k narrowly converge to $\boldsymbol{\eta}$.*

Moreover, $\delta \sim \|\cdot\|_{L^2}$:

$$\frac{1}{\sqrt{2}}\|f-g\|_{L^2(\mathrm{vol}_D)} \le \delta(f,g) \le C\|f-g\|^{\alpha}_{L^2(\mathrm{vol}_D)}.$$

Hence, no gap phenomenon occurs when $d > 2$:

$d > 2 \quad \Rightarrow \quad \delta_*(h) = \overline{\delta}(\boldsymbol{i}, h) \quad \forall\, h \in S(D)$.

Theorem (Density of smooth flows (Shnirelman, 1994))

$D = [0,1]^d$, $d > 2$. For any generalized incompressible flow $\boldsymbol{\eta}$ between $\boldsymbol{i}$ and $h \in \mathrm{SDiff}(D)$ there exist smooth flows g_k connecting $\boldsymbol{i}$ to h satisfying:

(a) $\mathscr{A}(g_k) \to \mathscr{A}(\eta)$;

(b) *the probability measures $\boldsymbol{\eta}_k$ induced by g_k narrowly converge to $\boldsymbol{\eta}$.*

Moreover, $\delta \sim \|\cdot\|_{L^2}$:

$$\frac{1}{\sqrt{2}}\|f - g\|_{L^2(\mathrm{vol}_D)} \leq \delta(f,g) \leq C\|f - g\|^{\alpha}_{L^2(\mathrm{vol}_D)}.$$

Hence, no gap phenomenon occurs when $d > 2$:
$d > 2 \quad \Rightarrow \quad \delta_*(h) = \overline{\delta}(\boldsymbol{i}, h) \quad \forall\, h \in S(D)$.

Theorem (Density of smooth flows (Shnirelman, 1994))

$D = [0,1]^d$, $d > 2$. For any generalized incompressible flow $\boldsymbol{\eta}$ between $\boldsymbol{i}$ and $h \in \mathrm{SDiff}(D)$ there exist smooth flows g_k connecting $\boldsymbol{i}$ to h satisfying:

(a) $\mathscr{A}(g_k) \to \mathscr{A}(\eta)$;

(b) *the probability measures $\boldsymbol{\eta}_k$ induced by g_k narrowly converge to $\boldsymbol{\eta}$.*

Moreover, $\delta \sim \|\cdot\|_{L^2}$:

$$\frac{1}{\sqrt{2}}\|f-g\|_{L^2(\mathrm{vol}_D)} \le \delta(f,g) \le C\|f-g\|^{\alpha}_{L^2(\mathrm{vol}_D)}.$$

Hence, no gap phenomenon occurs when $d > 2$:

$d > 2 \quad \Rightarrow \quad \delta_*(h) = \overline{\delta}(\boldsymbol{i}, h) \quad \forall\, h \in S(D).$

For numerical purposes, it is important to ask the same question with non-deterministic final data γ.

Set

$$\delta_*(\gamma) := \inf \left\{ \liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \ (\boldsymbol{i} \times h_n)_\# \mathrm{vol}_D \to \gamma \text{ weakly} \right\}.$$

Theorem (Ambrosio-F., 2008)

No gap phenomenon occurs with non-deterministic final data γ:

$$d > 2 \quad \Rightarrow \quad \delta_*(\gamma) = \overline{\delta}(\boldsymbol{i}, \gamma) \quad \forall \gamma \in \Gamma(D).$$

For numerical purposes, it is important to ask the same question with non-deterministic final data γ.
Set

$$\delta_*(\gamma) := \inf\left\{\liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \ (\boldsymbol{i}\times h_n)_\# \mathrm{vol}_D \to \gamma \text{ weakly } \right\}.$$

Theorem (Ambrosio-F., 2008)

No gap phenomenon occurs with non-deterministic final data γ:

$$d > 2 \quad \Rightarrow \quad \delta_*(\gamma) = \overline{\delta}(\boldsymbol{i}, \gamma) \quad \forall\, \gamma \in \Gamma(D).$$

For numerical purposes, it is important to ask the same question with non-deterministic final data γ.
Set

$$\delta_*(\gamma) := \inf\left\{\liminf_{n\to\infty} \delta(\boldsymbol{i}, h_n) : \ (\boldsymbol{i} \times h_n)_\# \operatorname{vol}_D \to \gamma \text{ weakly } \right\}.$$

Theorem (Ambrosio-F., 2008)

No gap phenomenon occurs with non-deterministic final data γ:

$$d > 2 \quad \Rightarrow \quad \delta_*(\gamma) = \overline{\delta}(\boldsymbol{i}, \gamma) \quad \forall\, \gamma \in \Gamma(D).$$

The pressure field

Brenier (1993):

$\exists!$ pressure field (up to an additive time-dependent constant).

The pressure field appear when we relax the incompressibility constraint.

Consider *almost incompressibles flows* $\boldsymbol{\nu}$: if $\rho^{\boldsymbol{\nu}}$ denotes its density, i.e.

$$(e_t)_\# \boldsymbol{\nu} = \rho^{\boldsymbol{\nu}} \, \mathrm{vol}_D,$$

we say that $\boldsymbol{\nu}$ is almost incompressibles if $\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1} \leq 1/2$.

Theorem (pressure=Lagrange multiplier (Brenier, 1993; Ambrosio-F., 2008))

Let $\boldsymbol{\eta}$ be optimal between η and γ. Then there exists $p \in (C^1)^*$ s.t.

$$\mathscr{A}(\boldsymbol{\nu}) + \langle p, \rho^{\boldsymbol{\nu}} - 1 \rangle \geq \mathscr{A}(\boldsymbol{\eta})$$

for all almost incompressible flows $\boldsymbol{\nu}$ connecting η to γ.

The pressure field

Brenier (1993):

$\exists!$ pressure field (up to an additive time-dependent constant).

The pressure field appear when we relax the incompressibility constraint.
Consider *almost incompressibles flows* ν: if ρ^ν denotes its density, i.e.

$$(e_t)_\# \nu = \rho^\nu \, \mathrm{vol}_D,$$

we say that ν is almost incompressibles if $\|\rho^\nu - 1\|_{C^1} \leq 1/2$.

Theorem (pressure=Lagrange multiplier (Brenier, 1993; Ambrosio-F., 2008))

Let η be optimal between η and γ. Then there exists $p \in (C^1)^*$ s.t.

$$\mathscr{A}(\nu) + \langle p, \rho^\nu - 1 \rangle \geq \mathscr{A}(\eta)$$

for all almost incompressible flows ν connecting η to γ.

The pressure field

Brenier (1993):

> $\exists!$ pressure field (up to an additive time-dependent constant).

The pressure field appear when we relax the incompressibility constraint.

Consider *almost incompressibles flows* ν: if ρ^ν denotes its density, i.e.

$$(e_t)_\# \nu = \rho^\nu \,\mathrm{vol}_D,$$

we say that ν is almost incompressibles if $\|\rho^\nu - 1\|_{C^1} \leq 1/2$.

Theorem (pressure=Lagrange multiplier (Brenier, 1993; Ambrosio-F., 2008))

Let η be optimal between η and γ. Then there exists $p \in (C^1)^*$ s.t.

$$\mathscr{A}(\nu) + \langle p, \rho^\nu - 1 \rangle \geq \mathscr{A}(\eta)$$

for all almost incompressible flows ν connecting η to γ.

The pressure field

Brenier (1993):

$\exists!$ pressure field (up to an additive time-dependent constant).

The pressure field appear when we relax the incompressibility constraint.
Consider *almost incompressibles flows* $\boldsymbol{\nu}$: if $\rho^{\boldsymbol{\nu}}$ denotes its density, i.e.

$$(e_t)_{\#}\boldsymbol{\nu} = \rho^{\boldsymbol{\nu}}\,\mathrm{vol}_D,$$

we say that $\boldsymbol{\nu}$ is almost incompressibles if $\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1} \leq 1/2$.

Theorem (pressure=Lagrange multiplier (Brenier, 1993; Ambrosio-F., 2008))

Let $\boldsymbol{\eta}$ be optimal between η and γ. Then there exists $p \in (C^1)^*$ s.t.

$$\mathscr{A}(\boldsymbol{\nu}) + \langle p, \rho^{\boldsymbol{\nu}} - 1\rangle \geq \mathscr{A}(\boldsymbol{\eta})$$

for all almost incompressible flows $\boldsymbol{\nu}$ connecting η to γ.

The pressure field

Brenier (1993):

$\exists!$ pressure field (up to an additive time-dependent constant).

The pressure field appear when we relax the incompressibility constraint.

Consider *almost incompressibles flows* $\boldsymbol{\nu}$: if $\rho^{\boldsymbol{\nu}}$ denotes its density, i.e.

$$(e_t)_\# \boldsymbol{\nu} = \rho^{\boldsymbol{\nu}} \, \mathrm{vol}_D,$$

we say that $\boldsymbol{\nu}$ is almost incompressibles if $\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1} \leq 1/2$.

Theorem (pressure=Lagrange multiplier (Brenier, 1993; Ambrosio-F., 2008))

Let $\boldsymbol{\eta}$ be optimal between η and γ. Then there exists $p \in (C^1)^*$ s.t.

$$\mathscr{A}(\boldsymbol{\nu}) + \langle p, \rho^{\boldsymbol{\nu}} - 1 \rangle \geq \mathscr{A}(\boldsymbol{\eta})$$

for all almost incompressible flows $\boldsymbol{\nu}$ connecting η to γ.

1st variation: let $\boldsymbol{w}(t,x)$ be a smooth vector field vanishing for $t \sim 0, 1$, and consider the vatiation

$$\omega(t) \mapsto e^{\varepsilon \boldsymbol{w}(t)}(\omega(t)) = \omega(t) + \varepsilon \boldsymbol{w}(t, \omega(t)) + o(\varepsilon).$$

This induces a perturbation η_ε of η which is almost incompressible. Then the 1st variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot{\omega}(t) \cdot \frac{d}{dt} \boldsymbol{w}(t, \omega(t))\, dt\, d\eta(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w} \rangle = 0.$$

This equations uniquely determines p (up to an additive time-dependent constant), independently of the chosen minimizer η!

1st variation: let $\boldsymbol{w}(t,x)$ be a smooth vector field vanishing for $t \sim 0, 1$, and consider the vatiation

$$\omega(t) \mapsto e^{\varepsilon \boldsymbol{w}(t)}\big(\omega(t)\big) = \omega(t) + \varepsilon \boldsymbol{w}(t, \omega(t)) + o(\varepsilon).$$

This induces a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$ which is almost incompressible. Then the 1st variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot{\omega}(t) \cdot \frac{d}{dt} \boldsymbol{w}(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w} \rangle = 0.$$

This equations uniquely determines p (up to an additive time-dependent constant), independently of the chosen minimizer $\boldsymbol{\eta}$!

1st variation: let $\boldsymbol{w}(t, x)$ be a smooth vector field vanishing for $t \sim 0, 1$, and consider the vatiation

$$\omega(t) \mapsto e^{\varepsilon \boldsymbol{w}(t)}\big(\omega(t)\big) = \omega(t) + \varepsilon \boldsymbol{w}(t, \omega(t)) + o(\varepsilon).$$

This induces a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$ which is almost incompressible.

Then the 1st variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot{\omega}(t) \cdot \frac{d}{dt} \boldsymbol{w}(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w} \rangle = 0.$$

This equations uniquely determines p (up to an additive time-dependent constant), independently of the chosen minimizer $\boldsymbol{\eta}$!

1st variation: let $\boldsymbol{w}(t,x)$ be a smooth vector field vanishing for $t \sim 0, 1$, and consider the vatiation

$$\omega(t) \mapsto e^{\varepsilon \boldsymbol{w}(t)}\big(\omega(t)\big) = \omega(t) + \varepsilon \boldsymbol{w}(t, \omega(t)) + o(\varepsilon).$$

This induces a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$ which is almost incompressible. Then the 1st variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot\omega(t) \cdot \frac{d}{dt}\boldsymbol{w}(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w} \rangle = 0.$$

This equations uniquely determines p (up to an additive time-dependent constant), independently of the chosen minimizer η!

1st variation: let $\boldsymbol{w}(t,x)$ be a smooth vector field vanishing for $t \sim 0, 1$, and consider the vatiation

$$\omega(t) \mapsto e^{\varepsilon \boldsymbol{w}(t)}\big(\omega(t)\big) = \omega(t) + \varepsilon \boldsymbol{w}(t, \omega(t)) + o(\varepsilon).$$

This induces a perturbation $\boldsymbol{\eta}_\varepsilon$ of $\boldsymbol{\eta}$ which is almost incompressible. Then the 1st variation gives

$$\int_{\Omega^*(D)} \int_0^1 \dot\omega(t) \cdot \frac{d}{dt} \boldsymbol{w}(t, \omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) + \langle p, \operatorname{div} \boldsymbol{w} \rangle = 0.$$

This equations uniquely determines p (up to an additive time-dependent constant), independently of the chosen minimizer $\boldsymbol{\eta}$!

Since $\boldsymbol{w}$ is arbitrary, we get a “weak” formulation of the Euler equations:

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \operatorname{div}\left(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t(x)\right) + \nabla_x p(t,x) = 0,$$

where

$$\overline{\boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot{\omega}(t)\eta), \qquad \overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot{\omega}(t) \otimes \dot{\omega}(t)\eta).$$

As in general $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ (due to branching and multiple velocities), these solutions are a priori not distributional. However, in some special situations it may happen that $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t - \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ is a gradient (Bernot-F.-Santambrogio, 2008), so that $\overline{\boldsymbol{v}}_t$ is a distributional solution for a new “macroscopic” pressure.

Since $\boldsymbol{w}$ is arbitrary, we get a "weak" formulation of the Euler equations:

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \mathrm{div}\,(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t(x)) + \nabla_x p(t,x) = 0,$$

where

$$\overline{\boldsymbol{v}}_t\,\mathrm{vol}_D = (e_t)_\#(\dot\omega(t)\boldsymbol{\eta}), \qquad \overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t\,\mathrm{vol}_D = (e_t)_\#(\dot\omega(t)\otimes\dot\omega(t)\boldsymbol{\eta}).$$

As in general $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t\otimes\overline{\boldsymbol{v}}_t$ (due to branching and multiple velocities), these solutions are a priori not distributional.However, in some special situations it may happen that $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t - \overline{\boldsymbol{v}}_t\otimes\overline{\boldsymbol{v}}_t$ is a gradient (Bernot-F.-Santambrogio, 2008), so that $\overline{\boldsymbol{v}}_t$ is a distributional solution for a new "macroscopic" pressure.

Since $\boldsymbol{w}$ is arbitrary, we get a "weak" formulation of the Euler equations:

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \operatorname{div}\left(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t(x)\right) + \nabla_x p(t,x) = 0,$$

where

$$\overline{\boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot\omega(t)\boldsymbol{\eta}), \qquad \overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot\omega(t)\otimes\dot\omega(t)\boldsymbol{\eta}).$$

As in general $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ (due to branching and multiple velocities), these solutions are a priori not distributional. However, in some special situations it may happen that $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t - \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ is a gradient (Bernot-F.-Santambrogio, 2008), so that $\overline{\boldsymbol{v}}_t$ is a distributional solution for a new "macroscopic" pressure.

Since $\boldsymbol{w}$ is arbitrary, we get a "weak" formulation of the Euler equations:

$$\partial_t \overline{\boldsymbol{v}}_t(x) + \mathrm{div}\,(\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t(x)) + \nabla_x p(t,x) = 0,$$

where

$$\overline{\boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot\omega(t)\boldsymbol{\eta}), \qquad \overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \, \mathrm{vol}_D = (e_t)_\#(\dot\omega(t) \otimes \dot\omega(t)\boldsymbol{\eta}).$$

As in general $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ (due to branching and multiple velocities), these solutions are a priori not distributional.However, in some special situations it may happen that $\overline{\boldsymbol{v} \otimes \boldsymbol{v}}_t - \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ is a gradient (Bernot-F.-Santambrogio, 2008), so that $\overline{\boldsymbol{v}}_t$ is a distributional solution for a new "macroscopic" pressure.

Necessary and sufficient optimality conditions

Remark: if $q \in L^1([0,1] \times D)$ and $\int_D q(t,\cdot)\, d\,\mathrm{vol}_D = 0$, then

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\eta(\omega,a) = \int_0^1 \int_D q(t,x)\, d\,\mathrm{vol}_D(x)\, dt = 0$$

for all incompressible η. Set

$$c_q^{0,1}(x,y) := \inf\left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt : \ \omega(0) = x,\ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(t,x,v) := \frac{1}{2}|v|^2 - q(t,x)$. Then

$$\mathscr{A}(\eta) = \int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\eta(\omega,a) \geq \int_D c_q^{0,1}(a,h(a))\, d\,\mathrm{vol}_D(a)$$

for all incompressible flows η connecting ***i*** to h. Moreover equality holds iff η-a.e. (ω,a) is a $c_q^{0,1}$-minimizing curve.

Necessary and sufficient optimality conditions

Remark: if $q \in L^1([0,1] \times D)$ and $\int_D q(t,\cdot)\, d\,\mathrm{vol}_D = 0$, then

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega, a) = \int_0^1 \int_D q(t,x)\, d\,\mathrm{vol}_D(x)\, dt = 0$$

for all incompressible $\boldsymbol{\eta}$. Set

$$c_q^{0,1}(x,y) := \inf \left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt : \ \omega(0) = x, \ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(t,x,v) := \frac{1}{2}|v|^2 - q(t,x)$.
Then

$$\mathscr{A}(\eta) = \int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\eta(\omega, a) \geq \int_D c_q^{0,1}(a, h(a))\, d\,\mathrm{vol}_D(a)$$

for all incompressible flows η connecting $\boldsymbol{i}$ to h. Moreover equality holds iff η-a.e. (ω, a) is a $c_q^{0,1}$-minimizing curve.

Necessary and sufficient optimality conditions

Remark: if $q \in L^1([0,1] \times D)$ and $\int_D q(t,\cdot)\, d\,\mathrm{vol}_D = 0$, then

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega,a) = \int_0^1 \int_D q(t,x)\, d\,\mathrm{vol}_D(x)\, dt = 0$$

for all incompressible $\boldsymbol{\eta}$. Set

$$c_q^{0,1}(x,y) := \inf\left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt : \ \omega(0) = x,\ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(t,x,v) := \frac{1}{2}|v|^2 - q(t,x)$.

Then

$$\mathscr{A}(\boldsymbol{\eta}) = \int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\boldsymbol{\eta}(\omega,a) \geq \int_D c_q^{0,1}(a,h(a))\, d\,\mathrm{vol}_D(a)$$

for all incompressible flows $\boldsymbol{\eta}$ connecting $\boldsymbol{i}$ to h. Moreover equality holds iff $\boldsymbol{\eta}$-a.e. (ω,a) is a $c_q^{0,1}$-minimizing curve.

Necessary and sufficient optimality conditions

Remark: if $q \in L^1([0,1] \times D)$ and $\int_D q(t,\cdot)\, d\,\mathrm{vol}_D = 0$, then

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega,a) = \int_0^1 \int_D q(t,x)\, d\,\mathrm{vol}_D(x)\, dt = 0$$

for all incompressible $\boldsymbol{\eta}$. Set

$$c_q^{0,1}(x,y) := \inf\left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt : \ \omega(0) = x,\ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(t,x,v) := \frac{1}{2}|v|^2 - q(t,x)$. Then

$$\mathscr{A}(\boldsymbol{\eta}) = \int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\boldsymbol{\eta}(\omega,a) \geq \int_D c_q^{0,1}(a,h(a))\, d\,\mathrm{vol}_D(a)$$

for all incompressible flows $\boldsymbol{\eta}$ connecting $\boldsymbol{i}$ to h. Moreover equality holds iff $\boldsymbol{\eta}$-a.e. (ω,a) is a $c_q^{0,1}$-minimizing curve.

Necessary and sufficient optimality conditions

Remark: if $q \in L^1([0,1] \times D)$ and $\int_D q(t,\cdot)\, d\,\mathrm{vol}_D = 0$, then

$$\int_{\Omega^*(D)} \int_0^1 q(t,\omega(t))\, dt\, d\boldsymbol{\eta}(\omega,a) = \int_0^1 \int_D q(t,x)\, d\,\mathrm{vol}_D(x)\, dt = 0$$

for all incompressible $\boldsymbol{\eta}$. Set

$$c_q^{0,1}(x,y) := \inf\left\{ \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt :\ \omega(0) = x,\ \omega(1) = y \right\}$$

the value function for the Lagrangian $\mathcal{L}_q(t,x,v) := \frac{1}{2}|v|^2 - q(t,x)$. Then

$$\mathscr{A}(\boldsymbol{\eta}) = \int_{\Omega^*(D)} \int_0^1 \frac{1}{2}|\dot\omega|^2 - q(t,\omega)\, dt\, d\boldsymbol{\eta}(\omega,a) \geq \int_D c_q^{0,1}(a,h(a))\, d\,\mathrm{vol}_D(a)$$

for all incompressible flows $\boldsymbol{\eta}$ connecting $\boldsymbol{i}$ to h. Moreover equality holds iff $\boldsymbol{\eta}$-a.e. (ω,a) is a $c_q^{0,1}$-minimizing curve.

Theorem (Brenier, 1989)

Let $(\boldsymbol{u}, p)$ be a smooth solution of Euler such that p satisfies

$$(*) \qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| \le \pi^2.$$

Then the measure η induced by the flow of $\boldsymbol{u}$ is optimal on $[0, T]$ (and is the unique one if the inequality is strict).

Indeed $t \mapsto g(t,a)$ satisfies $\ddot{\omega}(t) = -\nabla p(t,\omega)$, and $(*)$ implies that stationary paths for the action are also minimal for $c_p^{0,T}$.

Question: How far are these conditions from being *necessary*?

For simplicity we consider $D = \mathbb{T}^d$, and set $\mathrm{vol}_{\mathbb{T}} = \mathrm{vol}_D$.

Theorem (Brenier, 1999; Ambrosio-F., 2008)

The pressure field belongs to
$L^2_{\mathrm{loc}}((0,1), BV(\mathbb{T}^d)) \subset L^2_{\mathrm{loc}}((0,1), L^{d/(d-1)}(\mathbb{T}^d))$.

Theorem (Brenier, 1989)

Let $(\boldsymbol{u}, p)$ be a smooth solution of Euler such that p satisfies

$$(*) \qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| \leq \pi^2.$$

Then the measure η induced by the flow of $\boldsymbol{u}$ is optimal on $[0,T]$ (and is the unique one if the inequality is strict).

Indeed $t \mapsto g(t,a)$ satisfies $\ddot{\omega}(t) = -\nabla p(t,\omega)$, and $(*)$ implies that stationary paths for the action are also minimal for $c_p^{0,T}$.

Question: How far are these conditions from being *necessary*?

For simplicity we consider $D = \mathbb{T}^d$, and set $\mathrm{vol}_{\mathbb{T}} = \mathrm{vol}_D$.

Theorem (Brenier, 1999; Ambrosio-F., 2008)

The pressure field belongs to

$L^2_{\mathrm{loc}}\big((0,1), BV(\mathbb{T}^d)\big) \subset L^2_{\mathrm{loc}}\big((0,1), L^{d/(d-1)}(\mathbb{T}^d)\big)$.

Theorem (Brenier, 1989)

Let $(\boldsymbol{u}, p)$ be a smooth solution of Euler such that p satisfies

$$(*) \qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| \le \pi^2.$$

Then the measure $\boldsymbol{\eta}$ induced by the flow of $\boldsymbol{u}$ is optimal on $[0,T]$ (and is the unique one if the inequality is strict).

Indeed $t \mapsto g(t,a)$ satisfies $\ddot{\omega}(t) = -\nabla p(t,\omega)$, and $(*)$ implies that stationary paths for the action are also minimal for $c_p^{0,T}$.
Question: How far are these conditions from being *necessary*?
For simplicity we consider $D = \mathbb{T}^d$, and set $\mathrm{vol}_{\mathbb{T}} = \mathrm{vol}_D$.

Theorem (Brenier, 1999; Ambrosio-F., 2008)

The pressure field belongs to
$L^2_{\mathrm{loc}}((0,1), BV(\mathbb{T}^d)) \subset L^2_{\mathrm{loc}}((0,1), L^{d/(d-1)}(\mathbb{T}^d))$.

Theorem (Brenier, 1989)

Let $(\boldsymbol{u}, p)$ be a smooth solution of Euler such that p satisfies

$$(*) \qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| \leq \pi^2.$$

Then the measure $\boldsymbol{\eta}$ induced by the flow of $\boldsymbol{u}$ is optimal on $[0,T]$ (and is the unique one if the inequality is strict).

Indeed $t \mapsto g(t,a)$ satisfies $\ddot{\omega}(t) = -\nabla p(t,\omega)$, and $(*)$ implies that stationary paths for the action are also minimal for $c_p^{0,T}$.

Question: How far are these conditions from being *necessary*?

For simplicity we consider $D = \mathbb{T}^d$, and set $\mathrm{vol}_{\mathbb{T}} = \mathrm{vol}_D$.

Theorem (Brenier, 1999; Ambrosio-F., 2008)

The pressure field belongs to
$L^2_{\mathrm{loc}}\big((0,1), BV(\mathbb{T}^d)\big) \subset L^2_{\mathrm{loc}}\big((0,1), L^{d/(d-1)}(\mathbb{T}^d)\big)$.

Theorem (Brenier, 1989)

Let $(\boldsymbol{u}, p)$ be a smooth solution of Euler such that p satisfies

$$(*) \qquad T^2 \sup_{t\in[0,T]} \sup_{x\in D} |\nabla_x^2 p(t,x)| \leq \pi^2.$$

Then the measure $\boldsymbol{\eta}$ induced by the flow of $\boldsymbol{u}$ is optimal on $[0,T]$ (and is the unique one if the inequality is strict).

Indeed $t \mapsto g(t,a)$ satisfies $\ddot{\omega}(t) = -\nabla p(t,\omega)$, and $(*)$ implies that stationary paths for the action are also minimal for $c_p^{0,T}$.

Question: How far are these conditions from being *necessary*?

For simplicity we consider $D = \mathbb{T}^d$, and set $\mathrm{vol}_{\mathbb{T}} = \mathrm{vol}_D$.

Theorem (Brenier, 1999; Ambrosio-F., 2008)

The pressure field belongs to
$L^2_{\mathrm{loc}}\left((0,1), BV(\mathbb{T}^d)\right) \subset L^2_{\mathrm{loc}}\left((0,1), L^{d/(d-1)}(\mathbb{T}^d)\right)$.

p not a priori continuous $\Rightarrow$ the value of the Lagrangian $\mathcal{L}_p$ along a curve ω is *not* invariant by changing p on a set of measure zero.

• We define a *precise* representative $\bar{p}$:

$$\bar{p}(t,x) := \liminf_{\varepsilon \downarrow 0} p(t,\cdot) * \phi_\varepsilon(x)$$

is a "good" definition.

• In the minization problem defining the value function we consider only curves ω which satisfy

$$Mp(t,\omega(t)) \in L^1_{\mathrm{loc}}(0,1),$$

where $Mp(t,\cdot) =$ spherical maximal function of $p(t,\cdot)$.

We can now talk about *locally minimizing curves* for the Lagrangian $\mathcal{L}_{\bar{p}}$ and we can define the corresponding value functions:

$$c_{\bar{p}}^{s,t} : \mathbb{T}^d \times \mathbb{T}^d \to [-\infty,+\infty], \qquad [s,t] \subset (0,1).$$

p not a priori continuous $\Rightarrow$ the value of the Lagrangian $\mathcal{L}_p$ along a curve ω is *not* invariant by changing p on a set of measure zero.

- We define a *precise* representative $\bar{p}$:

$$\bar{p}(t,x) := \liminf_{\varepsilon \downarrow 0} p(t,\cdot) * \phi_\varepsilon(x)$$

is a "good" definition.

- In the minization problem defining the value function we consider only curves ω which satisfy

$$Mp(t,\omega(t)) \in L^1_{\mathrm{loc}}(0,1),$$

where $Mp(t,\cdot) =$ spherical maximal function of $p(t,\cdot)$.

We can now talk about *locally minimizing curves* for the Lagrangian $\mathcal{L}_{\bar{p}}$ and we can define the corresponding value functions:

$$c^{s,t}_{\bar{p}} : \mathbb{T}^d \times \mathbb{T}^d \to [-\infty,+\infty], \qquad [s,t] \subset (0,1).$$

p not a priori continuous $\Rightarrow$ the value of the Lagrangian $\mathcal{L}_p$ along a curve ω is *not* invariant by changing p on a set of measure zero.

• We define a *precise* representative $\bar{p}$:

$$\bar{p}(t,x) := \liminf_{\varepsilon \downarrow 0} p(t,\cdot) * \phi_\varepsilon(x)$$

is a “good” definition.

• In the minization problem defining the value function we consider only curves ω which satisfy

$$Mp(t,\omega(t)) \in L^1_{\mathrm{loc}}(0,1),$$

where $Mp(t,\cdot) =$ spherical maximal function of $p(t,\cdot)$.

We can now talk about *locally minimizing curves* for the Lagrangian $\mathcal{L}_{\bar{p}}$ and we can define the corresponding value functions:

$$c^{s,t}_{\bar{p}} : \mathbb{T}^d \times \mathbb{T}^d \to [-\infty,+\infty], \qquad [s,t] \subset (0,1).$$

p not a priori continuous $\Rightarrow$ the value of the Lagrangian $\mathcal{L}_p$ along a curve ω is *not* invariant by changing p on a set of measure zero.

• We define a *precise* representative $\bar{p}$:

$$\bar{p}(t,x) := \liminf_{\varepsilon \downarrow 0} p(t,\cdot) * \phi_\varepsilon(x)$$

is a "good" definition.

• In the minization problem defining the value function we consider only curves ω which satisfy

$$Mp(t,\omega(t)) \in L^1_{\mathrm{loc}}(0,1),$$

where $Mp(t,\cdot) =$ spherical maximal function of $p(t,\cdot)$.

We can now talk about *locally minimizing curves* for the Lagrangian $\mathcal{L}_{\bar{p}}$ and we can define the corresponding value functions:

$$c^{s,t}_{\bar{p}} : \mathbb{T}^d \times \mathbb{T}^d \to [-\infty,+\infty], \qquad [s,t] \subset (0,1).$$

Theorem (Ambrosio-F., 2008)

Let $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mathrm{vol}_{\mathbb{T}}$ be an optimal incompressible flow between $\eta = \eta_a \otimes \mathrm{vol}_{\mathbb{T}}$ and $\gamma = \gamma_a \otimes \mathrm{vol}_{\mathbb{T}}$. Then

(i) *$\boldsymbol{\eta}$ is concentrated on locally minimizing paths for $\mathcal{L}_{\bar{p}}$;*

(ii) *for all intervals $[s, t]$, for $\mu_{\mathbb{T}}$-a.e. a, the plan $(e_s, e_t)_{\#}\boldsymbol{\eta}_a$ is $c_{\bar{p}}^{s,t}$-optimal, i.e.*

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x, y)\, d(e_s, e_t)_{\#}\boldsymbol{\eta}_a \leq \int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x, y)\, d\lambda$$

for any $\lambda \in \mathscr{P}(\mathbb{T}^d \times \mathbb{T}^d)$ having the same marginals of $(e_s, e_t)_{\#}\boldsymbol{\eta}_a$.

Conversely, if (i), (ii) hold with $\bar{p}$ replaced by some function q with $Mq \in L^1([0,1] \times \mathbb{T}^d)$, then η is optimal and q is the pressure field.

Condition (ii) becomes meaningful only when $(e_s)_{\#}\eta_a$ are *not* Dirac masses.

Theorem (Ambrosio-F., 2008)

Let $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mathrm{vol}_{\mathbb{T}}$ *be an optimal incompressible flow between* $\eta = \eta_a \otimes \mathrm{vol}_{\mathbb{T}}$ *and* $\gamma = \gamma_a \otimes \mathrm{vol}_{\mathbb{T}}$. *Then*

(i) $\boldsymbol{\eta}$ *is concentrated on locally minimizing paths for* $\mathcal{L}_{\bar{p}}$;

(ii) *for all intervals* $[s, t]$, *for* $\mu_{\mathbb{T}}$*-a.e. a, the plan* $(e_s, e_t)_{\#}\boldsymbol{\eta}_a$ *is* $c_{\bar{p}}^{s,t}$*-optimal, i.e.*

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x, y)\, d(e_s, e_t)_{\#}\boldsymbol{\eta}_a \le \int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x, y)\, d\lambda$$

for any $\lambda \in \mathscr{P}(\mathbb{T}^d \times \mathbb{T}^d)$ *having the same marginals of* $(e_s, e_t)_{\#}\boldsymbol{\eta}_a$.

Conversely, if (i), (ii) hold with $\bar{p}$ *replaced by some function q with* $Mq \in L^1\left([0, 1] \times \mathbb{T}^d\right)$, *then* $\boldsymbol{\eta}$ *is optimal and q is the pressure field.*

Condition (ii) becomes meaningful only when $(e_s)_{\#}\eta_a$ are *not* Dirac masses.

Theorem (Ambrosio-F., 2008)

Let $\boldsymbol{\eta} = \boldsymbol{\eta}_a \otimes \mathrm{vol}_{\mathbb{T}}$ be an optimal incompressible flow between $\eta = \eta_a \otimes \mathrm{vol}_{\mathbb{T}}$ and $\gamma = \gamma_a \otimes \mathrm{vol}_{\mathbb{T}}$. Then

(i) *$\boldsymbol{\eta}$ is concentrated on locally minimizing paths for $\mathcal{L}_{\bar{p}}$;*

(ii) *for all intervals $[s,t]$, for $\mu_{\mathbb{T}}$-a.e. a, the plan $(e_s, e_t)_\# \boldsymbol{\eta}_a$ is $c_{\bar{p}}^{s,t}$-optimal, i.e.*

$$\int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x,y)\, d(e_s, e_t)_\# \boldsymbol{\eta}_a \leq \int_{\mathbb{T}^d \times \mathbb{T}^d} c_{\bar{p}}^{s,t}(x,y)\, d\lambda$$

for any $\lambda \in \mathscr{P}(\mathbb{T}^d \times \mathbb{T}^d)$ having the same marginals of $(e_s, e_t)_\# \boldsymbol{\eta}_a$.

Conversely, if (i), (ii) hold with $\bar{p}$ replaced by some function q with $Mq \in L^1\left([0,1] \times \mathbb{T}^d\right)$, then $\boldsymbol{\eta}$ is optimal and q is the pressure field.

Condition (ii) becomes meaningful only when $(e_s)_\# \boldsymbol{\eta}_a$ are *not* Dirac masses.

Conclusions

• The above results show a connection with the theory of action-minimizing measures, though in our case the Lagrangian $\frac{1}{2}|v|^2 - \bar{p}(t,x)$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

• Nice variation on a classical theme of Calculus of Variations: we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

• This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

• Euler-Legrange equations (F.-Mandorino, 2009):

$$p \in L^p_t(W^{1,p_x}) \Rightarrow \ddot{\omega} = -\nabla p(t,\omega) \in L^p([0,1]) \ \eta\text{-a.e.}$$

In the BV case we may hope to have an analogous result, and deduce for instance $\dot{\omega} \in BV([0,1])$ η-a.e.

Conclusions

• The above results show a connection with the theory of action-minimizing measures, though in our case the Lagrangian $\frac{1}{2}|v|^2 - \bar{p}(t,x)$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

• Nice variation on a classical theme of Calculus of Variations: we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

• This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

• Euler-Legrange equations (F.-Mandorino, 2009):

$$p \in L^p_t(W^{1,p_x}) \Rightarrow \ddot{\omega} = -\nabla p(t,\omega) \in L^p([0,1]) \ \eta\text{-a.e.}$$

In the BV case we may hope to have an analogous result, and deduce for instance $\dot{\omega} \in BV([0,1])$ η-a.e.

Conclusions

• The above results show a connection with the theory of action-minimizing measures, though in our case the Lagrangian $\frac{1}{2}|v|^2 - \bar{p}(t,x)$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

• Nice variation on a classical theme of Calculus of Variations: we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

• This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

• Euler-Legrange equations (F.-Mandorino, 2009):

$$p \in L^p_t(W^{1,p_x}) \Rightarrow \ddot{\omega} = -\nabla p(t,\omega) \in L^p([0,1]) \ \eta\text{-a.e.}$$

In the BV case we may hope to have an analogous result, and deduce for instance $\dot{\omega} \in BV([0,1])$ η-a.e.

Conclusions

• The above results show a connection with the theory of action-minimizing measures, though in our case the Lagrangian $\frac{1}{2}|v|^2 - \bar{p}(t,x)$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

• Nice variation on a classical theme of Calculus of Variations: we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

• This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

• Euler-Legrange equations (F.-Mandorino, 2009):

$$p \in L^p_t(W^{1,p_x}) \Rightarrow \ddot{\omega} = -\nabla p(t,\omega) \in L^p([0,1]) \ \eta\text{-a.e.}$$

In the BV case we may hope to have an analogous result, and deduce for instance $\dot{\omega} \in BV([0,1])$ η-a.e.

Conclusions

• The above results show a connection with the theory of action-minimizing measures, though in our case the Lagrangian $\frac{1}{2}|v|^2 - \bar{p}(t, x)$ is possibly non-smooth and not given a priori, but *generated* by the variational problem itself.

• Nice variation on a classical theme of Calculus of Variations: we have a field of (possibly nonsmooth, or intersecting) *minimizers* which has to produce an incompressible flow *in the state space*.

• This structure seems to be rigid, and it might lead to new regularity results for the pressure field.

• Euler-Legrange equations (F.-Mandorino, 2009):

$$p \in L^p_t(W^{1,p_x}) \Rightarrow \ddot{\omega} = -\nabla p(t, \omega) \in L^p([0, 1]) \ \boldsymbol{\eta}\text{-a.e.}$$

In the BV case we may hope to have an analogous result, and deduce for instance $\dot{\omega} \in BV([0, 1])$ $\boldsymbol{\eta}$-a.e.

Thanks for your attention!