Speed of KPP fronts with a cut-off: rigorous results

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Abstract:

We study the reaction diffusion equation $u_t = u_{xx} + f(u)$ with a cut-off in the reaction term (Fig 1). We show that for reaction terms of the form $f(u) = u - \varphi(u)$ where $|\varphi(u)| < K u^p$ the asymptotic speed of the front satisfies

$$K/p \leq c^2 - 4\sin^2\phi_* \leq 0 \qquad \text{where} \qquad \phi_* \tan\phi_* = |\ln\epsilon|^{-1}/2$$

In the limit of small ϵ $\quad 2\sin\phi_* = 2 - \pi^2/(\ln\epsilon)^2 + \mathcal{O}(1/|\ln\epsilon|^3).$

In the limit $\epsilon \to 1$, $\quad 2\sin\phi_* = \sqrt{2(1-\epsilon)} + \ldots$

The asymptotic speed of the front, for arbitrary reaction terms f(u) such that f(0) = f(1) =0 can be derived from the variational principle

$$c^2 = \sup_{u(s)} 2\,\frac{F(1)/s_0 + \int_0^{s_0} F(u(s))/s^2\,ds}{\int_0^{s_0} (du/ds)^2\,ds}, \qquad \text{where} \qquad F(u) = \int^u f(q)\,dq,$$

s_0 is an arbitrary parameter and where the supremum is taken over positive increasing functions u(s) such that u(0) =0, $u(s_0) = 1$.

We apply the variational principle to a reaction term
$$f(u) = \tilde{f}(u)\Theta(u-\epsilon) = \begin{cases} 0 & \text{if } 0 \leq u \leq \epsilon \\ u - \varphi(u) & \text{if } \epsilon < u < 1. \end{cases}$$

The upper bound is constructed observing that the reaction function f(u), (hence the corresponding F(u)) is smaller than the reaction term shown with a solid line in Fig. 1. For the reaction term shown with the solid line there is a function *u(s)* for which the supremum is attained. This function can be calculated explicitly, and the upper bound follows immediately.

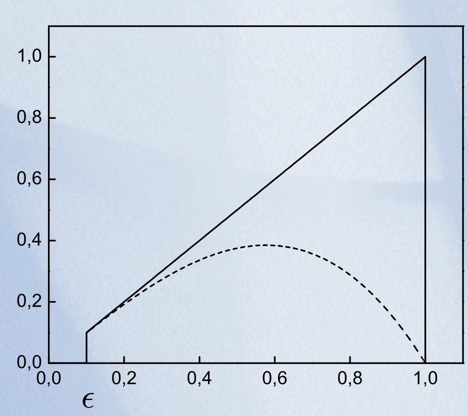
The lower bound is obtained using as a trial function the function *u(s)* described above which is given by

$$u(s) = \begin{cases} s & \text{if } 0 \leq s \leq \epsilon \\ A\sqrt{s}\cos(\phi(s)) & \text{if } \epsilon < s < s_0, \end{cases} \qquad \text{where} \qquad A = \sqrt{\epsilon}\sec(\phi_*), \quad s_0 = 1/\epsilon, \quad \phi(s) = \tfrac{1}{2}\cot(\phi_*)\ln(s/\epsilon) - \phi_*.$$

The bounds allow us to determine the range of validity of the Brunet-Derrida formula, which corresponds to the limiting case $\epsilon \to 0$. Different scaling is obtained in the opposite case $\epsilon \to 1$.

The effect of a cut-off on the reaction difffusion convection equation and on the hyperbolic reaction diffusion equation is work in progress.

Figure 1



References:

Benguria, R, Depassier M. C. and Loss, M, *Upper and lower bounds for the speed of pulled fronts with a cut-off,* European Phys. Journal **61** (2008) 331.

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