IMAJAnnual Program Year Tutorial
An Introduction to Funny (Complex) Fluids: Rheology, Modeling and Theorems
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Understanding silly putty, snail slime and other funny fluids

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IPRIME
(Industrial Partnership for Research in Interfacial and Materials Engineering)
What is rheology?

ρειν (Greek) = to flow

τα παντα ρει = every thing flows

rheology = study of flow?, i.e. fluid mechanics?

rate of deformation

stress = f/area

viscosity = stress/rate

honey and mayonnaise

honey and mayo

rate of deformation
What is rheology?

ρειν (Greek) = to flow

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honey and mayo

rubber band and silly putty

viscosity

modulus = f/area

rate of deformation

time of deformation
4 key rheological phenomena
rheology = study of deformation of complex materials

fluid mechanician: simple fluids complex flows
materials chemist: complex fluids complex flows
rheologist: complex fluids simple flows

rheologist fits data to constitutive equations which
- can be solved by fluid mechanician for complex flows
- have a microstructural basis
DEDICATION

A.M.D.G. ad majorem Dei gloriam

This book has been written in the spirit that energized far greater scientists. Some of them express that spirit in the following quotations.

"This most beautiful system of the sun, planets and comets could only proceed from the counsel and dominion of an intelligent and powerful Being."

Isaac Newton

"Think what God has determined to do to all those who submit themselves to His righteousness and are willing to receive His gift."

James C. Maxwell

June 23, 1864

"In the distance tower still higher peaks, which will yield to those who ascend them still wider prospects, and deepen the feeling whose truth is emphasized by every advance in science, that 'Great are the works of the Lord' ."

J.J. Thomson,

Goal: Understand Principles of Rheology: (stress, strain, constitutive equations)

\[ \text{stress} = f \left( \text{deformation, time} \right) \]

Simplest constitutive relations:

Newton’s Law: \[ \tau = \eta \frac{d\gamma}{dt} = \eta \dot{\gamma} \]

Hooke’s Law: \[ \tau = G\gamma \]

Key Rheological Phenomena

- shear thinning (thickening) \[ \eta(\dot{\gamma}) \]
- time dependent modulus \[ G(t) \]
- normal stresses in shear \[ N_1 \]
- extensional > shear stress \[ \eta_u > \eta \]
The power of any spring is in the same proportion with the tension thereof.

Robert Hooke (1678)

\[ f \propto \Delta L \]
\[ f = k\Delta L \]
Uniaxial Extension

\[ T_{11} = \frac{f}{a} \]

\[ T_{11} = G \alpha^2 \frac{1}{\alpha} \]

\[ \alpha = \frac{L}{L'} \]

or strain \( \varepsilon = \frac{L - L}{L} = \alpha - 1 \)

natural rubber \( G = 400 \text{ kPa} \)
Shear gives different stress response

\[ \gamma = 0 \]

\[ \gamma = -0.4 \]

\[ \gamma = 0.4 \]

\[ T_{21} = G \gamma \]

\[ T_{11} - T_{22} = G \gamma^2 \]

Goal: explain different results in extension and shear obtain from Hooke’s Law in 3D
If use stress and deformation tensors

Silicone rubber \( G = 160 \text{ kPa} \)
Stress Tensor - Notation

\[ T = \hat{x}\hat{x}T_{xx} + \hat{x}\hat{y}T_{xy} + \hat{x}\hat{z}T_{xz} + \hat{y}\hat{x}T_{yx} + \hat{y}\hat{y}T_{yy} + \hat{y}\hat{z}T_{yz} + \hat{z}\hat{x}T_{zx} + \hat{z}\hat{y}T_{zy} + \hat{z}\hat{z}T_{zz} \]

\[ T = \hat{x}t_x + \hat{y}t_y + \hat{z}t_z \]

\[ \hat{x}\hat{y}T_{xy} \]

dyad

direction of stress on plane
plane stress acts on

\[ \hat{x}, \hat{y}, \hat{z} = \hat{x}_1, \hat{x}_2, \hat{x}_3 \]

\[ T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \]

\[ T = \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{x}_i \hat{x}_j T_{ij} = T_{ij} \]

Other notation besides $T_{ij}$: $\sigma_{ij}$ or $\Pi_{ij}$
Rheologists use very simple $\mathbf{T}$

1. Uniaxial Extension

\[ \mathbf{T} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ T_{22} = T_{33} = 0 \]

or

\[ \mathbf{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -T_{22} & 0 \\ 0 & 0 & -T_{33} \end{bmatrix} \]

\[ T_{11} = 0 \]

\[ T_{22} = T_{33} \]

$T_{11} - T_{22}$ causes deformation
Consider only normal stress components

\[
\mathbf{T}_{ij} = \begin{bmatrix}
T_{11} & 0 & 0 \\
0 & T_{22} & 0 \\
0 & 0 & T_{33}
\end{bmatrix}
\]

Hydrostatic Pressure

\[T_{11} = T_{22} = T_{33} = -p\]

\[
\begin{bmatrix}
-p & 0 & 0 & 1 & 0 & 0 \\
0 & -p & 0 & -p & 0 & 1 \\
0 & 0 & -p & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

\[\mathbf{T} = -p\mathbf{I} \quad \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\]

\[\mathbf{T} \mathbf{I} = \mathbf{T}\]

If a liquid is incompressible

\[G \neq f(p) \quad \eta \neq f(p)\]

Then only \(\mathbf{\tau}\) the extra or viscous stresses cause deformation

\[\mathbf{T} = -p\mathbf{I} + \mathbf{\tau}\]

and only the normal stress differences cause deformation

\[T_{11} - T_{22} = \tau_{11} - \tau_{22} \equiv N_1 \text{ (shear)}\]
Rheologists use very simple $T$

2. Simple Shear

But to balance angular momentum $T_{21} = T_{12}$

in general $T = T^T$

$t_n = \hat{n} \mathbf{g} \Gamma = T^T \mathbf{g} \hat{n}$

Stress tensor for simple shear

$T_{ij} = \begin{bmatrix} 0 & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$

Only 3 components:

$T_{12}$

$T_{11} - T_{22} = \tau_{11} - \tau_{22} \equiv N_1$

$T_{22} - T_{33} = \tau_{22} - \tau_{33} \equiv N_2$
Stress Tensor Summary

1. \( t_n = \hat{n} \) Stress at point on any plane

2. in general \( \mathbf{T} = f(\text{time or rate, strain}) \)

3. simple \( \mathbf{T} \) for rheologically complex materials:
   - extension and shear

4. \( \mathbf{T} = \text{pressure} + \text{extra stress} = -pI + \tau. \)

5. \( \tau \) causes deformation

6. normal stress differences cause deformation, \( \tau_{11} - \tau_{22} = T_{11} - T_{22} \)

7. symmetric \( \mathbf{T} = \mathbf{T}^T \) i.e. \( T_{12} = T_{21} \)

Hooke → Young → Cauchy → Gibbs Einstein

(1678) → (~1801) → (1830’s) → (1880,~1905)
Deformation Gradient Tensor

\[ F = \frac{\mathbf{x}}{x} \quad \text{or} \quad F_{ij} = \frac{x_i}{x_j} \]

\[ s = w - y = F \cdot s \]

\[ \mathbf{x} = \text{displacement function} \]
\[ \text{describes how material points move} \]
\[ \mathbf{w} = \mathbf{x}(\mathbf{w}) = \mathbf{x}(\mathbf{y} + \mathbf{s}) \]
\[ = \mathbf{x}(\mathbf{y}) + \frac{\mathbf{x}}{\mathbf{y}} \mathbf{s} + \mathbf{O}s^2 \]
\[ = \mathbf{y} + \mathbf{F} \cdot \mathbf{s} \]

\[ s' = w' - y' \]
\[ w' = y' + s' \]

\[ s = w - y \]
\[ w = y + s \]

\[ \mathbf{S}' \] is a vector connecting two very close points in the material, P and Q.
Apply $F$ to Uniaxial Extension

Displacement functions describe how coordinates of $P$ in undeformed state, $x_i'$, have been displaced to coordinates of $P$ in deformed state, $x_i$.

\[
x_i = \frac{\Delta x_i}{\Delta x_i'} x_i = \alpha_i x_i'
\]

\[
F_{ij} = \begin{bmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{bmatrix}
\]

1-17
Can we write Hooke’s Law as $\tau = G(\mathbf{F} - \mathbf{I})$?
Can we write Hooke’s Law as \( \tau = G( \mathbf{F} - \mathbf{I}) \) ?

Solid Body Rotation – expect no stresses

\[
\begin{align*}
x_1 &= x_\theta \cos \varphi - \varphi \sin \\
x_2 &= x_\theta \sin \varphi + \varphi \cos \\
x_3 &= x_3 \\
F_{ij} &= x_i / x_j = \\
&= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

For solid body rotation, expect \( \mathbf{F} = \mathbf{I} \)

\( \tau = 0 \)

But \( \mathbf{F} \neq \mathbf{I} \)

\( \mathbf{F} \neq \mathbf{F}^T \)

Need to get rid of rotation create a new tensor!
Finger Tensor

\[ F = VgR \]
\[ V = \text{stretch} \]
\[ R = \text{rotation} \]

\[ F \cdot F^T = (V \cdot R) \cdot (V \cdot R)^T = V \cdot R \cdot R^T \cdot V^T = V \cdot I \cdot V^T = V^2 \]

\[ B = FgF^T \quad \text{or} \quad B_{ij} = F_{ik}F_{jk} = \frac{x_i}{x_k} \frac{x_j}{x_k} \]

Solid Body Rotation

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[ B_{ij} = \begin{bmatrix}
\sin \theta & \cos \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \end{bmatrix} \]

\[ B_{ij} \text{ gives relative local change in area within the sample.} \]
Neo-Hookean Solid \( \tau = G(B - I) \) or \( T = -pI + GB \)

1. Uniaxial Extension

\[
\begin{align*}
\alpha_1 & \quad 0 & \quad 0 \\
F_y = & \quad 0 & \quad \alpha_1^{-1/2} & \quad 0 \\
& \quad 0 & \quad 0 & \quad \alpha_1^{-1/2} \\
\end{align*}
\]

\[
\begin{align*}
B_{ij} = F_{ik}F_{jk} = & \quad 0 & \quad \alpha_1^{-1/2} & \quad 0 & \quad 0 & \quad \alpha_1^{-1/2} & \quad 0 & \quad 0 & \quad \alpha_1^{-1} & \quad 0 \\
& \quad 0 & \quad 0 & \quad \alpha_1^{-1/2} & \quad 0 & \quad 0 & \quad \alpha_1^{-1/2} & \quad 0 & \quad 0 & \quad \alpha_1^{-1} \\
\end{align*}
\]

\[
\begin{align*}
\tau_{11} & = G(\alpha_1^2 - 1) \\
\tau_{22} & = G(\alpha_1^{-1} - 1) \\
\tau_{11} - \tau_{22} & = G(\alpha_1^2 - \frac{1}{\alpha_1}) = T_{11} - T_{22} = T_{11} = \frac{f_1}{a_1} \\
\tau_{22} - \tau_{33} & = 0 \quad \text{since} \quad T_{22} = 0 \\
\end{align*}
\]
2. Simple Shear

\[ x_1 = x_1 + \frac{s}{\Delta x'_2} x_2 = x_1 + \gamma x_2 \]

\[ x_2 = x_2 \]

\[ x_3 = x_3 \]

\[ F_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ B_{ij} = \begin{bmatrix} 0 & 1 & 0 & \gamma & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \tau = G(B - I) \]

\[ \tau_{21} = G\gamma \]

\[ \tau_{11} - \tau_{22} = G\gamma^2 \]

Silicone rubber \( G = 160 \text{ kPa} \)

\[ \gamma = \frac{s}{x_2} \]
Finger Deformation Tensor Summary

1. \( B = F \mathcal{G} F^T \) area change around a point on any plane

2. symmetric

3. eliminates rotation

4. gives Hooke’s Law in 3D fits rubber data fairly well predicts \( N_1 \), shear normal stresses

\[
\begin{align*}
&\text{for uniaxial tension} \quad B_{11} = \alpha_1 \quad T_{11} - T_{22} = T_{11} = \frac{f_1}{a_1} = G(\alpha_1^2 - \frac{1}{\alpha_1}) \\
&\quad \text{for} \quad \alpha_1 = 1 + \varepsilon \quad T_{11} = G \left( \frac{(1+\varepsilon)^3 - 1}{1+\varepsilon} \right) \quad \text{for} \quad \varepsilon << 1 \quad T_{11} = 3G\varepsilon \\
&\quad \frac{T_{11}}{\varepsilon} = \text{tensile modulus} = 3G
\end{align*}
\]
Course Goal: Understand Principles of Rheology: (constitutive equations)

stress = f (deformation, time)

Simplest constitutive relations:
Newton’s Law: \( \tau = \eta \frac{d\gamma}{dt} = \eta \dot{\gamma} \)
Hooke’s Law: \( \tau = G\gamma \)
\( \tau = G(B - I) \)

Key Rheological Phenomena

• shear thinning (thickening) \( \eta(\dot{\gamma}) \)
• time dependent modulus \( G(t) \)
• normal stresses in shear \( N_1 \)
• extensional > shear stress \( \eta_u > \eta \)
The resistance which arises from the lack of slipperiness originating in a fluid, other things being equal, is proportional to the velocity by which the parts of the fluids are being separated from each other.

Isaac S. Newton (1687)
\[ \tau = \eta \frac{dv}{dy} \quad \text{Bernoulli} \]

\[ \tau_{yx} = \eta \frac{dv_x}{dy} \quad \text{Stokes-Navier, 1845} \]

measured \( \eta \) in shear
1856 capillary (Poiseuille)
1880’s concentric cylinders
(Perry, Mallock, Couette, Schwedoff)

<table>
<thead>
<tr>
<th>Material</th>
<th>Approximate Viscosity (Pa - s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass</td>
<td>( 10^{40} )</td>
</tr>
<tr>
<td>Molten Glass (500°C)</td>
<td>( 10^{12} )</td>
</tr>
<tr>
<td>Asphalt</td>
<td>( 10^{8} )</td>
</tr>
<tr>
<td>Molten polymers</td>
<td>( 10^{3} )</td>
</tr>
<tr>
<td>Heavy syrup</td>
<td>( 10^{2} )</td>
</tr>
<tr>
<td>Honey</td>
<td>( 10^{1} )</td>
</tr>
<tr>
<td>Glycerin</td>
<td>( 10^{0} )</td>
</tr>
<tr>
<td>Olive oil</td>
<td>( 10^{-1} )</td>
</tr>
<tr>
<td>Light oil</td>
<td>( 10^{-2} )</td>
</tr>
<tr>
<td>Water</td>
<td>( 10^{-3} )</td>
</tr>
<tr>
<td>Air</td>
<td>( 10^{-5} )</td>
</tr>
</tbody>
</table>

Familiar materials have a wide range in viscosity

Adapted from Barnes et al. (1989).
\[ \tau = \eta \frac{dv}{dy} \quad \text{Bernoulli, 1687} \]

\[ \tau_{yx} = \eta \frac{dv_x}{dy} \quad \text{Stokes-Navier, 1845} \]

measured \( \eta \) in shear
1856  capillary (Poiseuille)
1880's  concentric cylinders
(Perry, Mallock, Couette, Schwedoff)

measured in extension
7  Trouton
\( \eta_u = 3\eta \)

“A variety of pitch which gave by the traction method \( \lambda = 4.3 \times 10^{10} \) (poise) was found by the torsion method to have a viscosity \( \mu = 1.4 \times 10^{10} \) (poise).”
F.T. Trouton (1906)

To hold his viscous pitch samples, Trouton forced a thickened end into a small metal box. A hook was attached to the box from which weights were hung.
Goal
2. Put Newton’s Law in 3 dimensions
   • rate of strain tensor 2D
   • show $\eta_u = 3\eta$
recall Deformation Gradient Tensor, $F$.

Separation and displacement of point $Q$ from $P$

$w' = x(y', t) + F \ s' \quad s' = w' - y'$

$s = w - y = F \ s$
Viscosity is “proportional to the velocity by which the parts of the fluids are being separated from each other.” —Newton

Velocity Gradient Tensor

\[ s = F \cdot s \]

rate of separation

\[ \frac{s}{t} = \frac{F}{t} \cdot s + F \cdot \frac{s}{t} \]

\[ \frac{s}{t} \cdot dv = F \cdot s = F \cdot dx \]

\[ dv = F \cdot dx = L \cdot dx \]

\[ \lim_{\frac{x}{x} \cdot \frac{x}{x}} F = I \quad \therefore \lim_{\frac{x}{x} \cdot \frac{x}{x}} F = L \]

Alternate notation:

\[ v = L^T = \hat{x}_i \hat{x}_j \frac{v_j}{x_i} \]
Can we write Newton’s Law for viscosity as \( \tau = \eta L \)?

\[
\begin{align*}
\tau &= \eta L \\
F &= V R \\
\dot{F} &= \dot{V}g \dot{R} + V \ddot{g} \dot{R} \\
\lim_{x \to x} V(t) &= \lim_{x \to x} R(t) = I \\
\lim_{x \to x} F &= L = V + R \\
L^T &= (V + R)^T = V - R \\
\dot{R} &\text{ is anti-symmetric}
\end{align*}
\]

Rate of Deformation Tensor \( D \)
\[
2 \dot{V} = 2D = L + L^T = (v)^T + v
\]

Vorticity Tensor \( W \)
\[
2 \dot{R} = 2W = L - L^T \\
L = D + W = (v)^T
\]

Other notation:
\[
2D = \gamma
\]
Example 2.2.4 Rate of Deformation Tensor is a Time Derivative of $\mathbf{B}$.

Show \[ \lim_{t \to t} \frac{d\mathbf{B}}{dt} = 2\mathbf{D} \]

Thus \[ \lim_{t \to t} \mathbf{B} = \mathbf{F} + \mathbf{F}^T \]

recall that \[ \lim_{x \to x} \mathbf{F} = \mathbf{L} \]

\[ \lim_{t \to t} \mathbf{B} = \mathbf{L} + \mathbf{L}^T = 2\mathbf{D} \]

Show that $2\mathbf{D} = 0$ for solid body rotation

\[
2\mathbf{D} = \mathbf{L} + \mathbf{L}^T = \begin{bmatrix} 0 & -\Omega & 0 & 0 & 0 & 0 \\ -\Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
2\mathbf{W} = \mathbf{L} - \mathbf{L}^T = \begin{bmatrix} 0 & \Omega & 0 & 0 & 0 & 0 \\ -\Omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]
Newtonian Liquid

Steady simple shear

Here planes of fluid slide over each other like cards in a deck.

\[ \tau = \eta 2 \mathbf{D} \quad \text{or} \quad \mathbf{T} = -p \mathbf{I} + \eta 2 \mathbf{D} \]

\[ x_1 = x_1 \gamma t \quad x_2 = x_2 \quad x_3 = x_3 \]

Time derivatives of the displacement functions for simple, shear

\[ \lim_{x_2} \frac{dx_1}{dt} = \frac{dy}{dt} x_2 = v_1 \quad \frac{dx_2}{dt} = 0 = v_2 \]

\[ v_1 = \gamma x_2 = \frac{dv_1}{dx_2} x_2 \quad \text{and} \quad v_2 = v_3 = 0 \quad (2.2.10) \]

\[
\begin{align*}
L_{ij} &= 0 \quad 0 \quad 0 \\
L_{ji} &= 0 \quad \gamma \quad 0 \\
0 &= 0 \quad 0 \quad 0 \\
0 &= 0 \quad 0 \quad 0 \\
2D_{ij} &= \gamma \quad 0 \quad 0 \\
T_{ij} &= -p \quad 0 \quad 1 \quad 0 \quad +\eta \quad \gamma \\
0 &= 0 \quad 1 \quad 0 \quad 0 \quad 0 \\
T_{12} &= \tau_{12} = \tau_{21} = \eta \gamma
\end{align*}
\]
Newtonian Liquid

Steady Uniaxial Extension

time derivatives of the displacement functions at $x_1$

\[ x \alpha = x \quad \frac{dx_1}{dt} = \frac{d\alpha_1}{dt} \quad \text{or} \quad \frac{d\alpha_1}{dt} = x \dot{\alpha} \]

Similarly

\[ v_2 = \dot{\alpha}_2 x_2 \quad v_3 = \dot{\alpha}_3 x_3 \]

\[ \dot{\alpha}_1 \quad 0 \quad 0 \]

\[ L_{ij} = 0 \quad \dot{\alpha}_2 \quad 0 \]

\[ 0 \quad 0 \quad \dot{\alpha}_3 \]

incompressible fluid (1.7.9)

\[ \nu = 0 \]

or $\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3 = 0$

symmetric, $v_2 = v_3$ and thus

\[ \dot{\alpha}_2 = \dot{\alpha}_3 \quad \dot{\alpha}_2 = \dot{\alpha}_3 = -\frac{\dot{\alpha}_1}{2} \]

\[ \dot{\alpha}_1 \quad 0 \quad 0 \quad \dot{\epsilon} \quad 0 \quad 0 \]

\[ L_{ij} = 0 \quad -\dot{\alpha}_1/2 \quad 0 = 0 \quad -\dot{\epsilon}/2 \quad 0 \]

\[ 0 \quad 0 \quad -\dot{\alpha}_1/2 \quad 0 \quad 0 \quad -\dot{\epsilon}/2 \]

\[ 2D_{ij} = (L_{ij} + L_{ji}) = 0 \quad -\dot{\epsilon} \quad 0 \]

\[ 0 \quad 0 \quad -\dot{\epsilon} \]
Newtonian Liquid

Apply to Uniaxial Extension

\[ \tau = \eta 2D \]

\[
\begin{pmatrix}
2\dot{\varepsilon} & 0 & 0 \\
0 & -\dot{\varepsilon} & 0 \\
0 & 0 & -\dot{\varepsilon}
\end{pmatrix}
\]

\[ 2D_{ij} = 
\begin{pmatrix}
0 & 0 \\
0 & -\dot{\varepsilon} \\
-\dot{\varepsilon} & 0
\end{pmatrix}
\]

\[ \tau_{11} = 2\eta \dot{\varepsilon} \]

\[ \tau_{22} = \tau_{33} = -\eta \dot{\varepsilon} \]

From definition of extensional viscosity

\[ \eta_u = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} = 3\eta \]

Newton’s Law in 3 Dimensions

- predicts \( \eta_0 \) low shear rate
- predicts \( \eta_u_0 = 3\eta_0 \)

but many materials show large deviation
Summary of Fundamentals

1. $t_n = \hat{n} T$ stress at point on plane
   simple $T$ - extension and shear
   $T = \text{pressure} + \text{extra stress} = -pI + \tau$.
   symmetric $T = T^T$ i.e. $T_{12}=T_{21}$

2. $B = FgF^T$ area change around a point on plane
   symmetric, eliminates rotation
   gives Hooke’s Law in 3D, $E=3G$

3. $2D = L + L^T = (v)^T + v$ rate of separation of particles
   symmetric, eliminates rotation
   gives Newton’s Law in 3D, $\eta_u = 3\eta$
Course Goal: Understand Principles of Rheology:

stress = f (deformation, time)

\[ \tau = G \left( B - I \right) \quad \tau = \eta 2D \]

Key Rheological Phenomena

- shear thinning (thickening) \( \eta(\dot{\gamma}) \)
- time dependent modulus \( G(t) \)
- normal stresses in shear \( N_1 \)
- extensional > shear stress \( \eta_u > \eta \)