

Statistical Behavior of the Self-Guided Dynamics Algorithms

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Abstract

The self-guided dynamics algorithms are a pair of techniques to increase the speed of sampling from the possible conformations for a macromolecule. Wu and Wang (1998), and later Wu and Brooks (2003), proposed a numerical scheme that introduced a biasing term based on a running average of past states of the system into the equations of motion to increase the rate of barrier-crossings. In the present work, an analytical form for the guiding force is derived from the numerical methods. That analytical form is used to study the sampling properties of the two algorithms. It is shown that, up to a separability assumption, the systems do not sample from the Boltzmann distribution, and the magnitudes of the error are estimated. Finally, a new thermostat for the self-guided algorithms that conserves the Boltzmann distribution is proposed.

Background

The self-guided algorithms add a guiding force $\mathbf{g}(\mathbf{r}, \mathbf{p}, \mathbf{t})$ to the equations of motion in order to increase the rate of barrier crossings. The general self-guided equations are:

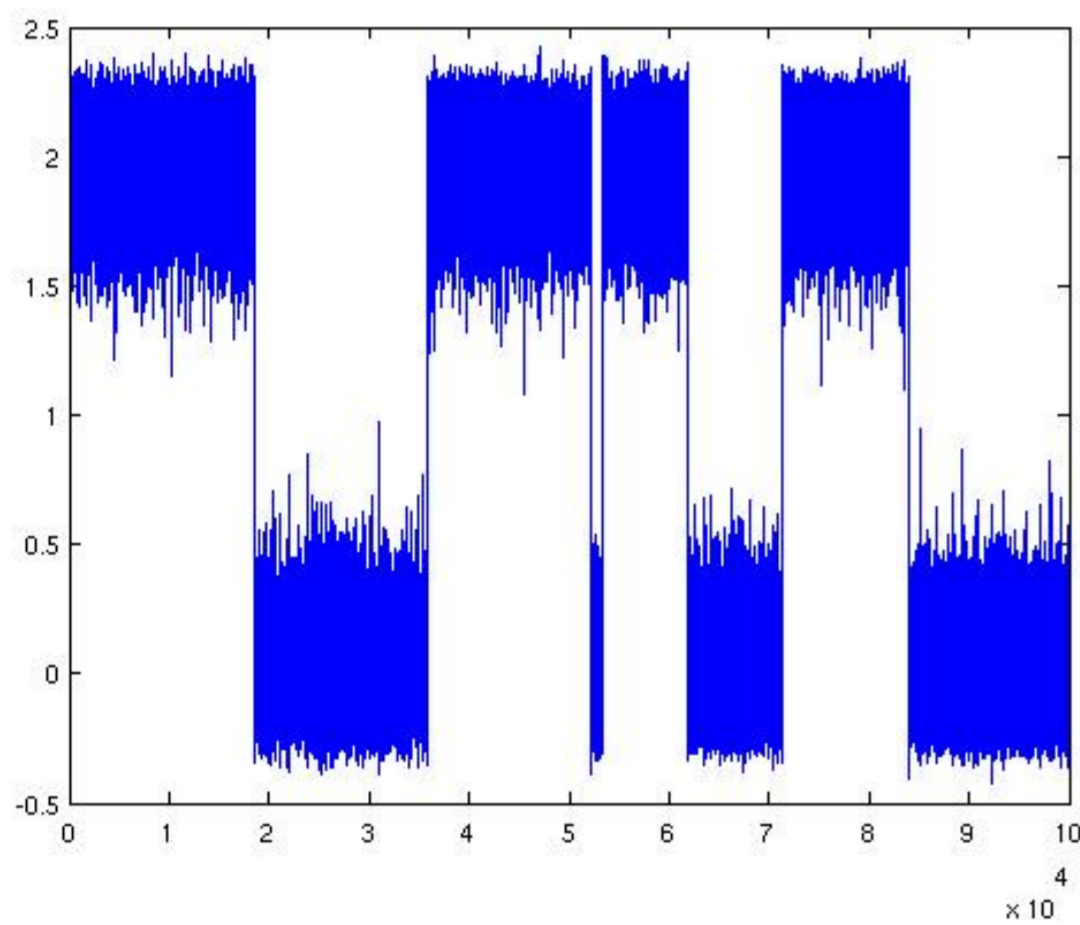
$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{m} \quad \dot{\mathbf{p}} = -\mathbf{V}_r + \lambda \mathbf{g}(\mathbf{r}, \mathbf{p}, \mathbf{t}) + \mathbf{T}(\mathbf{p}, \mathbf{t}) \quad (1)$$

An easy test case to probe the rate of barrier crossings is a particle in a double-well potential:

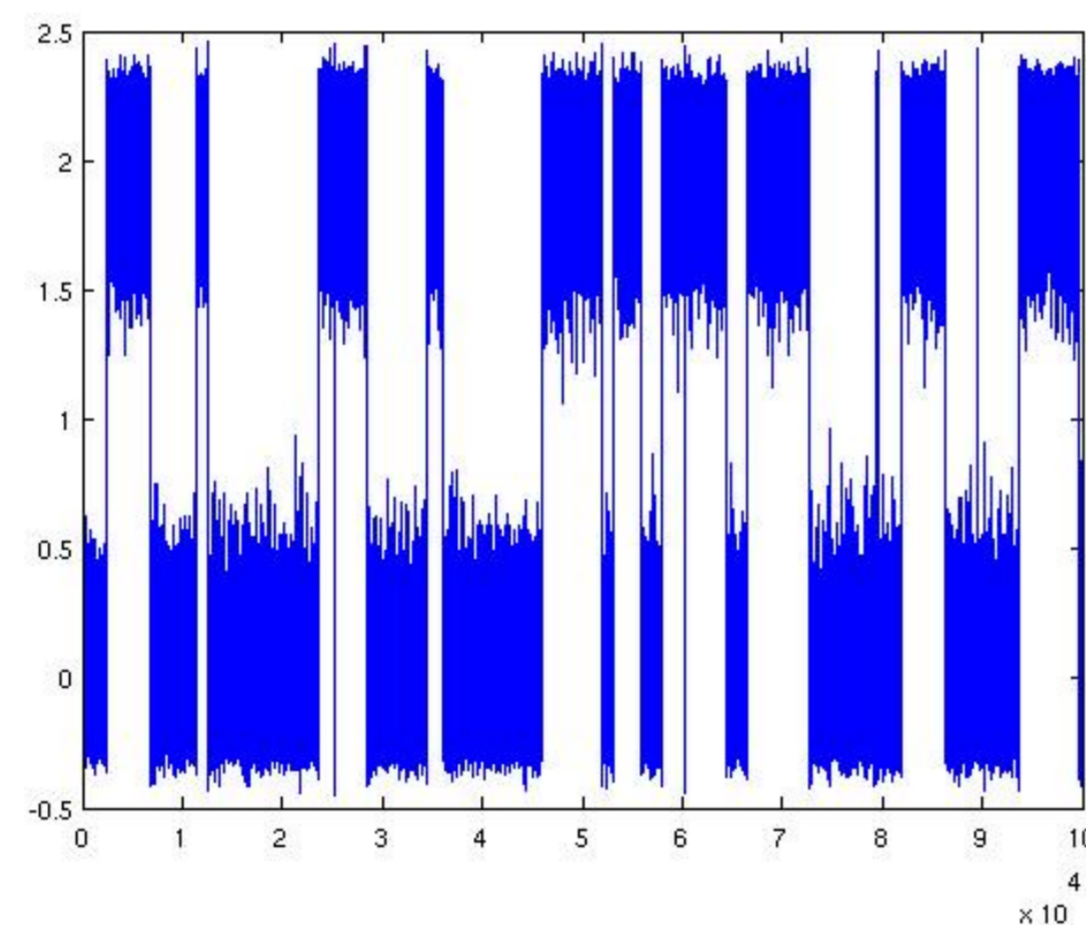
$$V(\mathbf{x}) = \mathbf{x}^2(\mathbf{x} - 2)^2$$

Running simulations with the Langevin thermostat and two different values of λ shows a large increase in the rate of crossings between the local minima at $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = 2$.

For $\lambda = 0$:



For $\lambda > 0$:



Derivation

The guiding force in the self-guided molecular dynamics algorithm is defined by the numerical formula:

$$\mathbf{g}(\mathbf{t} + \delta \mathbf{t}) = \left(1 - \frac{\delta \mathbf{t}}{t_\ell}\right) \mathbf{g}(\mathbf{t}) + \frac{\delta \mathbf{t}}{t_\ell} (-\mathbf{V}_r(\mathbf{t}) + \lambda \mathbf{g}(\mathbf{t})) \quad (2)$$

Rearranging:

$$\frac{\mathbf{g}(\mathbf{t} + \delta \mathbf{t}) - \mathbf{g}(\mathbf{t})}{\delta \mathbf{t}} = \frac{1}{t_\ell} ((\lambda - 1)\mathbf{g}(\mathbf{t}) - \mathbf{V}_r(\mathbf{t})) \quad (3)$$

Taking the limit as $\delta \mathbf{t} \rightarrow 0$

$$\dot{\mathbf{g}} = t_\ell^{-1} ((\lambda - 1)\mathbf{g} - \mathbf{V}_r) \quad (4)$$

The self-guided molecular dynamics algorithm with the Nosé-Hoover thermostat is now given by:

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m} & \dot{\mathbf{p}} &= -\mathbf{V}_r + \lambda \mathbf{g} - \xi \mathbf{p} \\ \dot{\mathbf{g}} &= t_\ell^{-1} (-\zeta \mathbf{g} - \mathbf{V}_r) & \dot{\xi} &= \frac{\mathbf{p}^2}{m} - NkT \end{aligned} \quad (5)$$

where $\zeta = 1 - \lambda > 0$.

Boltzmann

A necessary (but not sufficient) condition for the self-guided molecular dynamics system to sample from a given distribution is that such a distribution must be an equilibrium solution of Liouville's equation:

$$\frac{\partial}{\partial \mathbf{t}} \rho + \nabla \cdot ((\dot{\mathbf{r}}, \dot{\mathbf{p}}, \dot{\mathbf{g}}, \dot{\xi}) \rho) = 0 \quad (6)$$

Since we are primarily interested in the Boltzmann distribution, $\mathbf{B}(\mathbf{r}, \mathbf{p}) = \exp(-(\mathbf{kT})^{-1}(\mathbf{p}^2/2m + \mathbf{V}(\mathbf{r})))$, we will look for equations of the form:

$$\rho(\mathbf{r}, \mathbf{p}, \mathbf{g}, \xi) \propto \mathbf{B}(\mathbf{r}, \mathbf{p}) \varrho(\mathbf{g}, \xi) \quad (7)$$

Boltzmann cont'd

Putting ρ into Liouville's equation (6) with $\frac{\partial}{\partial \mathbf{t}} \rho = 0$:

$$\nabla \cdot ((\dot{\mathbf{r}}, \dot{\mathbf{p}}, \dot{\mathbf{g}}, \dot{\xi}) \mathbf{B} \varrho) = 0 \quad (8)$$

$$\begin{aligned} \mathbf{B}(\mathbf{kT})^{-1} \left(\varrho \frac{\lambda \mathbf{g} \mathbf{p}}{m} + \mathbf{kT} t_\ell^{-1} \frac{\partial}{\partial \mathbf{g}} \varrho ((\lambda - 1)\mathbf{g} - \mathbf{V}_r) + \varrho NkT(\lambda - 1) \right) \\ + \mathbf{B}(\mathbf{kT})^{-1} \left(\varrho \xi \left(\frac{\mathbf{p}^2}{m} - \mathbf{kT} \right) + \frac{\partial}{\partial \xi} \left(\frac{\mathbf{p}^2}{m} - \mathbf{kT} \right) \right) = 0 \end{aligned} \quad (9)$$

The dependence of $(\frac{\partial}{\partial \mathbf{p}} + \frac{\partial}{\partial \mathbf{g}} \varrho)$ on \mathbf{r} and \mathbf{p} is non-trivial, which is in contradiction of our assumption that ϱ depended solely upon \mathbf{g} and ξ .

Error Estimate

A perturbation expansion heuristic will be used to estimate the order of the errors in λ . We will look for a solution to equation (6) of the form:

$$\rho = \sum_{k=0}^{\infty} \lambda^k \rho_k \quad (10)$$

For ρ_0 , the solution should be of the form $\rho = \mathbf{B}(\mathbf{r}, \mathbf{p}) \exp(-\xi^2/2) \mathbf{f}(\mathbf{g})$. However, \mathbf{f} must then be a solution to the equation:

$$\frac{\partial}{\partial \mathbf{t}} \rho_0 + \frac{\partial}{\partial \mathbf{g}} (-\mathbf{g} - \mathbf{V}_r) \mathbf{f}(\mathbf{g}) = 0 \quad (11)$$

A change of variables is needed to get an appropriate solution for ρ_0 .

$$\mathbf{g}^* = \lambda \mathbf{g} \quad \dot{\mathbf{g}}^* = \lambda \dot{\mathbf{g}} = \lambda(-\zeta \mathbf{g} - \mathbf{V}_r) = -\zeta \mathbf{g}^* - \lambda \mathbf{V}_r \quad (12)$$

With the change of variables, ρ_0 has solution:

$$\rho_0 = \mathbf{B}(\mathbf{r}, \mathbf{p}) \exp\left(-\frac{\xi^2}{2}\right) \delta(\mathbf{g}) \quad (13)$$

Writing out the equation for ρ_1 :

$$\frac{\partial}{\partial \mathbf{t}} \rho_1 + \nabla \cdot ((\dot{\mathbf{r}}, \dot{\mathbf{p}}, \dot{\mathbf{g}}, \dot{\xi}) \rho) = -\mathbf{V}_r \mathbf{B}(\mathbf{r}, \mathbf{p}) \exp\left(\frac{\xi^2}{2}\right) \delta'(\mathbf{g}) \quad (14)$$

Therefore, the errors in \mathbf{r} are of order λ . Taking a cue from self-guided Langevin dynamics, averaging over the momentum gives:

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m} & \dot{\mathbf{p}} &= -\mathbf{V}_r + \mathbf{g} - \xi \mathbf{p} \\ \dot{\mathbf{g}} &= t_\ell^{-1} (-\zeta \mathbf{g} - \lambda \mathbf{p}) & \dot{\xi} &= \frac{\mathbf{p}^2}{m} - NkT \end{aligned} \quad (15)$$

The system defined by (15) gives the same solution for ρ_0^* and ρ_1^* fulfills the equation:

$$\frac{\partial}{\partial \mathbf{t}} \rho_1^* + \nabla \cdot ((\dot{\mathbf{r}}, \dot{\mathbf{p}}, \dot{\mathbf{g}}, \dot{\xi}) \rho_1^*) = -\mathbf{p} \mathbf{B}(\mathbf{r}, \mathbf{p}) \exp\left(\frac{\xi^2}{2}\right) \delta'(\mathbf{g}) \quad (16)$$

which implies that the errors in \mathbf{r} are of order λ^2 .

New Thermostat

It is possible to change the thermostat such that

$$\rho(\mathbf{r}, \mathbf{p}, \mathbf{g}, \xi) = \mathbf{B}(\mathbf{r}, \mathbf{p}) \exp\left(-\left(\frac{\mathbf{g}^2}{2\lambda} + Q \frac{\xi^2}{2}\right)\right) \quad (17)$$

is an equilibrium solution. The general equations are:

$$\begin{aligned} \dot{\mathbf{r}} &= \frac{\mathbf{p}}{m} & \dot{\mathbf{p}} &= -\mathbf{V}_r + \mathbf{g} - \xi \mathbf{p} \\ \dot{\mathbf{g}} &= t_\ell^{-1} (-\zeta \mathbf{g} - \lambda \mathbf{p}) & \dot{\xi} &= \frac{\mathbf{p}^2}{m} - NkT + \Gamma(\xi, \mathbf{p}, \mathbf{g}) \end{aligned} \quad (18)$$

Evaluating (6) gives the expression for Γ :

$$\frac{\partial}{\partial \xi} \Gamma - \xi \Gamma = -\mathbf{g} \frac{\mathbf{p}}{m} - \zeta \frac{\mathbf{g}^2}{\lambda} + \mathbf{g} \mathbf{p} + NkT \zeta \quad (19)$$

$$\Gamma(\xi, \mathbf{p}, \mathbf{g}) = \left(-\mathbf{g} \frac{\mathbf{p}}{m} - \zeta \frac{\mathbf{g}^2}{\lambda} + \mathbf{g} \mathbf{p} + NkT \zeta\right) \exp\left(\frac{\xi^2}{2}\right) \text{erf}(\xi) \quad (20)$$

References

- X. Wu, S. Wang, J. Chem. Phys. 110 (1999) 9401.
- X. Wu, B.R. Brooks, Chem. Phys. Lett. 381 (2003) 512.