

# Universal families and quantum control in infinite dimensions

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# Quantum control in infinite dimensions

## The infinite dimensional unitary group

Wu, Tarn and Li (2005) established *controllability criteria on the  $\infty$ -dim. manifolds generated by non-compact Lie algebras*. Left open was the question of **when these manifolds are dense on the Hilbert sphere**, a key requirement for full controllability in infinite dimensions.

An **alternative approach** starts from the study of the infinite dimensional unitary group,  $U(\infty)$ , transitive in the Hilbert sphere, and the study of ways to generate it from a finite number of generators.

*The proper mathematical setting for  $U(\infty)$* : A Gelfand triplet

$$S^* \supset L^2(\mathbb{R}^d) \supset S$$

$S$  a dense nuclear subspace, limit  $S = \bigcap_n S_n$  of a sequence of spaces with increasing Hilbertian norms,  $\|\xi\|_n = \|A^n \xi\|$

$$S^* \supset \cdots \supset S_{-n} \supset \cdots \supset L^2(\mathbb{R}^d) \supset \cdots \supset S_n \supset \cdots \supset S$$

For  $g \in U(\infty)$ ,  $\|g\xi\| = \|\xi\|$  and, by duality,  $U(\infty)$  is also defined on  $S^*$ .

# Quantum control in infinite dimensions

## The infinite dimensional unitary group

*Quantum scattering states are in  $S^*$  not in  $L^2(\mathbb{R}^d)$ .*

**For harmonic analysis one needs functionals on  $S^*$ .**

$U(\infty)$  being a complexification of  $O(\infty)$ , a standard result states that a  $O(\infty)$ -invariant measure must be a sum of a delta and a Gaussian.

Hence, we are led to the  $(L^2)$  space of functionals on  $S^*$  with a  $O(\infty)$ -invariant Gaussian measure  $(L^2) = L^2(S^*, B, \mu)$ ,  $B$  generated by cylinder sets in  $S^*$  and  $\mu$  the measure

$$C(f) = \int_{S^*} e^{i\langle x, f \rangle} d\mu(x) = e^{-\frac{1}{2}\|f\|^2}, \quad x \in S^*, f \in S$$

**In conclusion:** *The mathematical setting for  $U(\infty)$  is the complex white noise space  $(S_c^*, B_c, \mu_c)$*

**Consequences:**

1 - The regular representation of  $U(\infty)$ ,  $U_g \varphi(z) = \varphi(g^* z)$ , decomposes into irreducible representations corresponding to the Fock space (chaos expansion) decomposition of  $(L_c^2)$

# Quantum control in infinite dimensions

The infinite dimensional unitary group

$$(L^2) = \bigoplus_{n=0}^{\infty} \left( \bigoplus_{k=0}^n H_{n-k,k} \right)$$

$H_{n-k,k}$  = complex Fourier-Hermite polynomials

## 2 - Subgroups

A) Whiskers

B) Subgroups based on coordinate vectors

B1) All transformations approximated by finite-dimensional unitary transformations. Example:  $G_{\infty} = \text{proj.lim}_{n \rightarrow \infty} U_n$

B2) Subgroups with *essentially infinite dimensional* elements, meaning that **infinitely many coordinates**  $\langle x, \xi_n \rangle$  **change significantly**. Example:

The Lévy group

*Conclusion:* To generate  $U(\infty)$  some essentially infinite dimensional elements are needed.

■ **The next result shows that one such transformation is enough.**

# A control result in the infinite-dimensional Hilbert sphere

Separable Hilbert space is isomorphic to  $\ell^2(\mathbb{Z})$

$$a = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\} \in \ell^2(\mathbb{Z})$$

with basis  $e_k = \{\cdots, 0, 0, 1_k, 0, 0, \cdots\}$ ;  $a = \sum_{-\infty}^{\infty} a_k e_k$

**Define:**

**(i)** A linear operator  $T_+$  and its inverse

$$T_+ e_k = e_{k+1}, \quad k \in \mathbb{Z}$$

$$T_+^{-1} e_k = e_{k-1}, \quad k \in \mathbb{Z}$$

**(ii)** A linear operator  $\Pi$

$$\Pi e_0 = e_1$$

$$\Pi e_1 = e_0$$

$$\Pi e_k = e_k, \quad k \in \mathbb{Z} \setminus \{0, 1\}$$

Then  $\Pi_n = T_+^n \Pi T_+^{-n}$  exchanges  $a_n$  with  $a_{n+1}$  in  $a = \sum_{-\infty}^{\infty} a_k e_k$

**(iii)** A  $U(2)$  group operating in  $\{e_0, e_1\}$  and as the identity on  $\ell^2(\mathbb{Z}) \ominus \{e_0, e_1\}$ . In particular  $\Pi \in U(2)$ .

# A control result in the infinite-dimensional Hilbert sphere

## Lemma (1)

Given  $a \in \ell^2(\mathbb{Z})$ ,  $k \in \mathbb{Z}$ ,  $l \in \mathbb{Z}$ , the operator  $\Pi_{k,k+l}a = \Pi_k \Pi_{k+1} \cdots \Pi_{k+l-2} \Pi_{k+l-1} \cdots \Pi_{k+1} \Pi_k a$  exchanges the coefficients of  $e_k$  and  $e_{k+l}$ .

## Theorem (1)

Let  $G(T_+, \Pi)$  be the group generated by  $T_+$ ,  $T_+^{-1}$  and  $\Pi$ . For any  $0 \neq a \in \ell^2(\mathbb{Z})$  the linear span of  $G(T_+, \Pi)a$  is dense in  $\ell^2(\mathbb{Z})$ .

## Lemma (2)

Suppose  $0 \neq a \in \ell^2(\mathbb{Z})$  is a terminating normalized sequence. Then, there is  $g \in G(T_+, U(2))$  such that  $ge_0 = a$ .

## Theorem (2)

For any  $0 \neq a \in \ell^2(\mathbb{Z})$  the set  $G(T_+, U(2))a$  is dense in  $\ell^2(\mathbb{Z})$ .

■ **In conclusion: given any initial state  $0 \neq a \in \ell^2(\mathbb{Z})$  it is possible by the unitary action of an element in  $G(T_+, U(2))$  to approach as close as desired any other state  $b$  in  $\ell^2(\mathbb{Z})$ .**

- *Universal families* - Given a topological space  $X$  and a family of continuous mappings  $T_\alpha : X \rightarrow X$  with  $\alpha$  belonging to some index set  $I$ , an element  $x \in X$  is called *universal* if the set  $\{T_\alpha x : \alpha \in I\}$  is dense in  $X$ . The family  $\{T_\alpha : \alpha \in I\}$  will be called *universal* if there is at least one universal element  $x \in X$ .

- For *open systems* consider evolutions by completely positive trace-preserving maps  $\Phi$ ,  $\Phi(\rho) = \sum K_i \rho K_i^\dagger$ ,

- *The problem of quantum control = search for a universal family of operators acting in the operator algebra of bounded operators  $B(H)$*

- No countable subset of  $B(H)$  can be dense in the operator norm topology. The problem has no practical sense in this topology. Instead one should discuss density in the strong operator topology, in which  $B(H)$  is separable.

# Open systems in infinite-dimensions

Consider in  $\ell^2(\mathbb{Z})$ , the shift operator  $T_+$ , its inverse  $T_+^{-1}$ , as well as a  $U(2)$  group acting on the subspace  $\{e_0, e_1\}$ .

This set, generates all random-unitary transformations but not all trace-preserving completely positive operations. A new operator must be added. May be chosen to be the projection on a basis state, for example  $P_0 = |e_0\rangle\langle e_0|$ .

## Theorem (3)

*$P_0, T_+, T_+^{-1}$  and  $U(2)$  generate a (strong operator topology-) universal family in the set of all density operators in infinite dimensions, with a dense set of universal elements.*

### - Smaller sets ?

Weakly supercyclic unitary operators for  $L^2(\mu)$  in Kronecker sets

### References

W. Karwowski, RVM (2004); Phys. Lett A 322 , 282-285.

RVM (2009); arXiv:0902.0561.