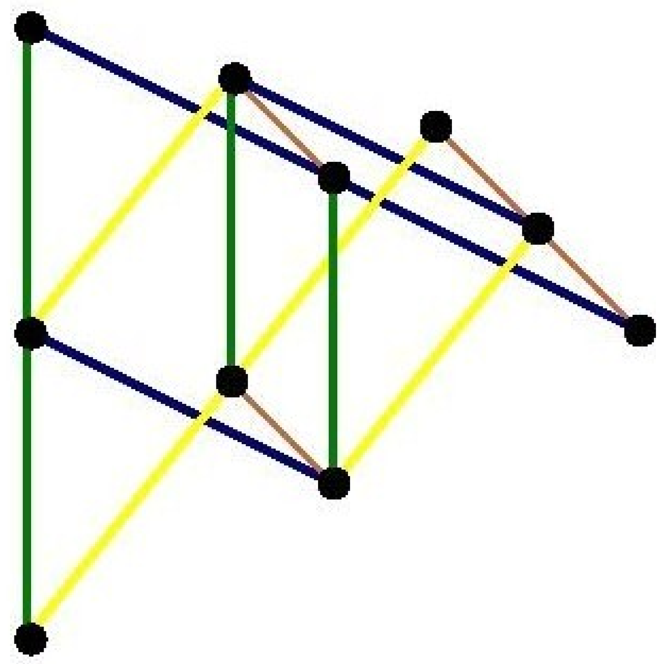

The poset perspective on alternating sign matrices

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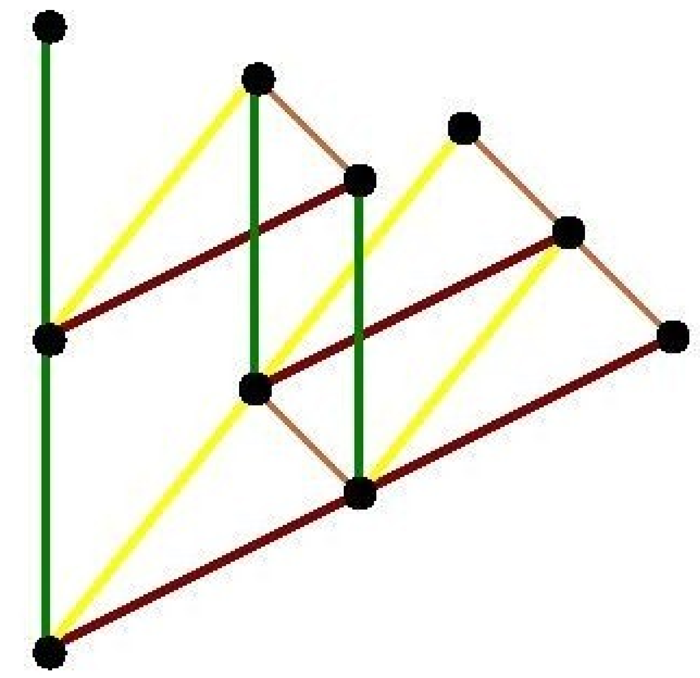


1 Introduction

Alternating sign matrices (ASMs) are defined as square matrices with entries 0, 1, or -1 whose rows and columns sum to 1 and alternate in sign. Totally symmetric self–complementary plane partitions (TSSCPPs) are plane partitions, each equal to its complement and invariant under all permutations of the coordinate axes. TSSCPPs inside a $2n \times 2n \times 2n$ box are equinumerous with $n \times n$ ASMs, but no explicit bijection between these two sets of objects has previously been found. We present a new perspective which sheds light on ASMs and TSSCPPs and puts them in a larger context.

2 The tetrahedral poset

Theorem 1. *$n \times n$ alternating sign matrices are in bijection with semistandard Young tableaux (SSYT) of staircase shape $\delta_n = n\,(n-1)\,(n-2)\ldots 3\;2\;1$ with entries $y_{i,j}$ at most n such that $y_{i,j} \le y_{i+1,j-1}$. Denote this set as SSA_n.*

ASM		Monotone triangle		Rotated by $\frac{\pi}{4}$
$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\Longleftrightarrow$	2 1 4 1 3 4 1 2 3 4	$\Longleftrightarrow$	1 1 1 2 2 3 4 3 4 4

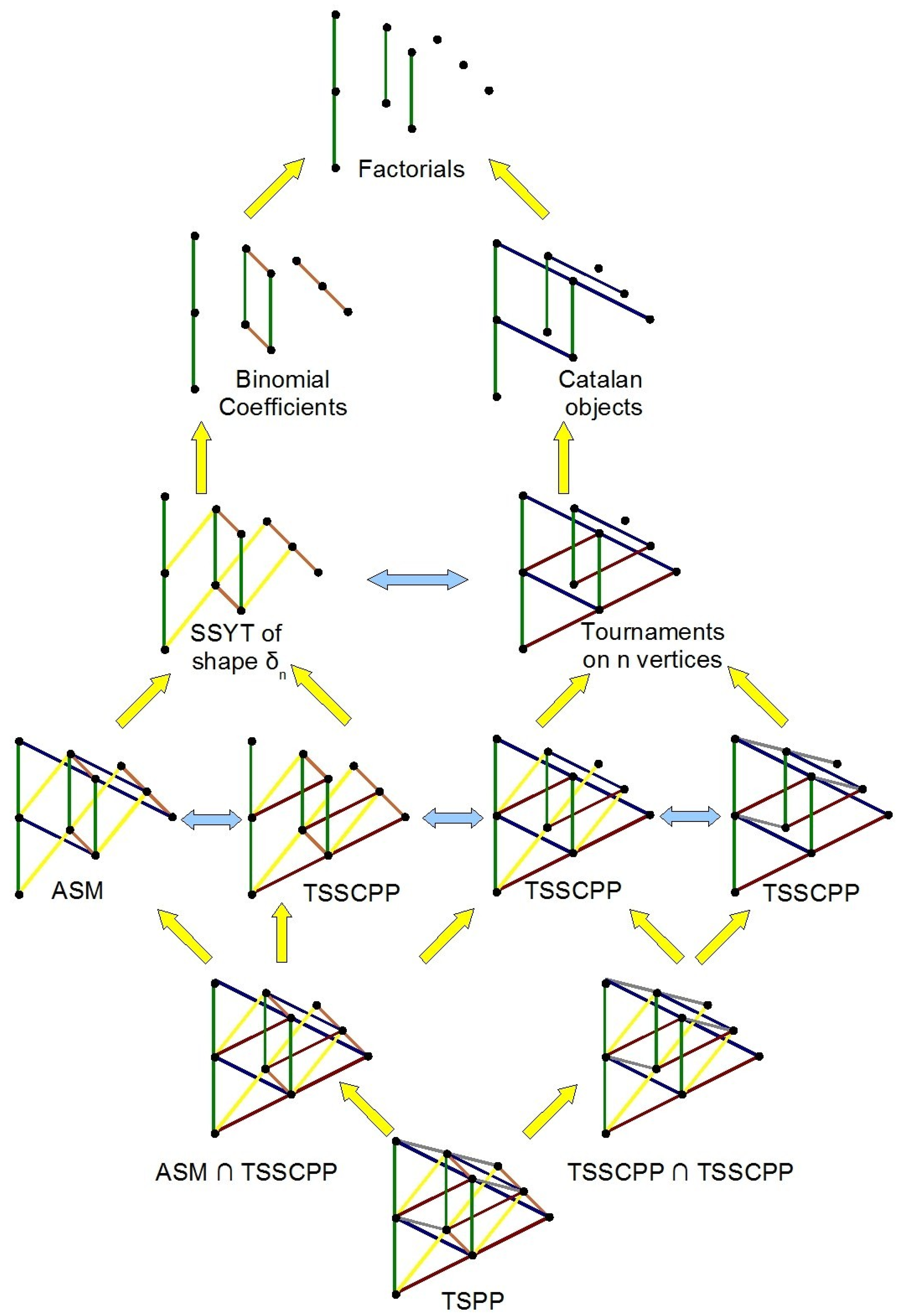
Ordered by componentwise comparison of the entries, SSA_n forms a distributive lattice $J(P)$ where the Hasse diagram of the poset of join-irreducibles P (for $n = 4$) is shown above in the upper left corner.

Theorem 2. *Totally symmetric self–complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with SSYT of staircase shape δ_n with entries $y_{i,j}$ at most n such that $y_{i,j} \le y_{i-1,j+1} + 1$. Denote this set as SST_n.*

TSSCPP		Fundamental domain		Horizontally reflected		Rotated by $\frac{\pi}{4}$		i added to row i
8 8 8 8 7 7 6 4 8 8 7 7 5 4 4 2 8 7 7 7 5 4 4 1 8 7 7 5 4 3 3 1 7 5 5 4 3 1 1 · 7 4 4 3 1 1 1 · 6 4 4 3 1 1 · · 4 2 1 1 · · · ·	$\Leftrightarrow$	3 1 1 1 1 · · · · ·	$\Leftrightarrow$	3 1 1 · 1 1 · · · ·	$\Leftrightarrow$	· · 1 3 · 1 1 · 1 ·	$\Leftrightarrow$	1 1 2 4 2 3 3 3 4 4

Ordered by componentwise comparison of the entries, SST_n forms a distributive lattice $J(Q)$ where the poset of join-irreducibles Q (for $n = 4$) is shown above in the upper right corner. Suppose we put the posets P and Q together and consider SSYT with both conditions on the diagonals. Our new poset looks like a tetrahedron with one direction of edges missing. If we insert those extra edges we obtain a tetrahedral poset whose order ideals we prove are in bijection with totally symmetric plane partitions (TSPPs) inside an $(n-1) \times (n-1) \times (n-1)$ box.

We assign a color to each of the six directions of edges in this tetrahedral poset. Surprisingly, for almost all the 2^6 posets made up of the different combinations of edge colors, we found a nice product formula for the number of order ideals and a bijection between these order ideals and an interesting set of combinatorial objects. The big picture of inclusions and bijections between the order ideals of these posets is shown below, and for each grouping of isomorphic posets, we show a representative poset for $n = 4$ and label which combinatorial objects correspond to the order ideals. The two sided arrows represent bijections between the order ideals. The one sided arrows represent inclusions of one set of order ideals into another.



We use this poset perspective to prove an expansion of the tournament generating function as a sum over TSSCPPs which is analogous to the following known formula involving ASMs.

Theorem 3 (Robbins–Rumsey). *Let A_n be the set of $n \times n$ ASMs, $I(A)$ the inversion number of A, and $N(A)$ the number of -1s in A, then*

$$\prod_{1\le i<j\le n}(x_i + \lambda x_j) = \sum_{A\in A_n} \lambda^{I(A)}\left(1+\lambda^{-1}\right)^{N(A)} \prod_{i,j=1}^{n} x_j^{(n-i)A_{ij}}.$$

We first translate this theorem into an expansion over SSA_n.

Theorem 4. *The generating function for tournaments on n vertices can be expanded as the following sum over the ASM tableaux SSA_n*

$$\prod_{1\le i<j\le n}(x_i + \lambda x_j) = \sum_{\alpha\in SSA_n} \lambda^{E(\alpha)}(1+\lambda)^{N(\alpha)} \prod_{k=1}^{n} x_k^{C_k(\alpha)-1}$$

where $E(\alpha)$ is the number of entries of α equaling the entry to the southwest, $C_k(\alpha)$ is the number of entries of α equal to k, and $N(\alpha)$ is the number of entries of α strictly greater than the entry to the west and strictly less than the entry to the southwest.

The following theorem gives the analogous expansion of the tournament generating function over SST'_n, which denotes the tableaux corresponding to a slightly different manifestation of TSSCPPs inside the tetrahedral poset.

Theorem 5. *Let $E_i(\alpha)$ be the number of entries in row i equaling the entry to the southwest and $E^i(\alpha)$ be the number of entries in antidiagonal i equal to the entry to the southwest. Then the generating function for tournaments on n vertices can be expanded as a sum over SST'_n as follows*

$$\prod_{1\le i<j\le n}(x_i + \lambda x_j) = \sum_{\alpha\in SST'_n} \lambda^{E(\alpha)} \prod_{i=1}^{n-1} x_i^{n-i-E_i(\alpha)} \sum_{\alpha'} \prod_{j=1}^{n-1} x_j^{E^j(\alpha')}$$

where the second sum is over all row shuffles α' of α.

We show below SST'_3 the corresponding contribution toward the RHS above.

1 1 1 2 2 3	1 1 2 2 2 3	1 1 2 2 3 3	1 1 3 2 3 3	1 2 2 2 2 3	1 2 2 2 3 3	1 2 3 2 3 3
$x_1^2x_2$	$\lambda x_1x_2(x_2+x_3)$	$\lambda x_1^2x_3$	$\lambda^2 x_1x_3^2$	$\lambda^2 x_2x_2x_3$	$\lambda^2 x_1x_2x_3$	$\lambda^3 x_2x_3^2$

Setting the x's to 1 we obtain the following expansions.

Expansion over ASMs: $$(1+\lambda)^{\binom{n}{2}} = \sum_{\alpha\in SSA_n} \lambda^{E(\alpha)}(1+\lambda)^{N(\alpha)}$$

Expansion over TSSCPPs: $$(1+\lambda)^{\binom{n}{2}} = \sum_{\alpha\in SST'_n} \lambda^{E(\alpha)} \prod_{1\le i\le k\le n-1} \binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)}$$

where $C_{i,k}(\alpha)$ is the number of entries in row i with value k and $E_{i,k}(\alpha)$ is the number of entries of value k in row i equaling the entry to the southwest.