Local Fields in Nonlinear Power Law Materials

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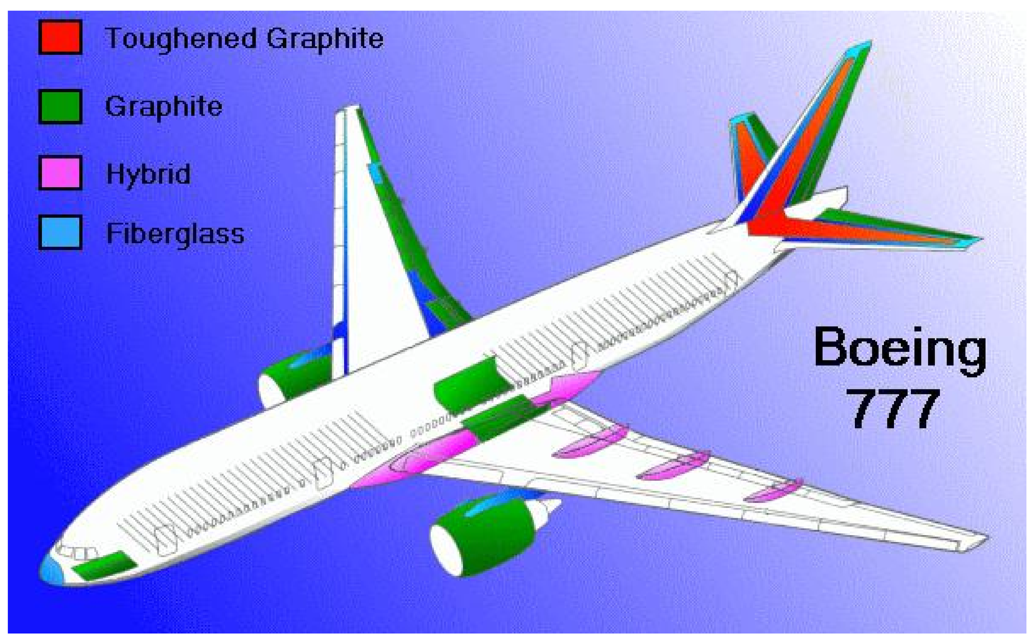
Abstract

Oscillations appear everywhere in nature and applied sciences. They naturally appear in many contexts including waves and transport phenomena in highly heterogeneous media. The mathematics of oscillations and associated transport phenomena including heat conduction, diffusion and porous media flow is now often referred to as *Homogenization Theory*.

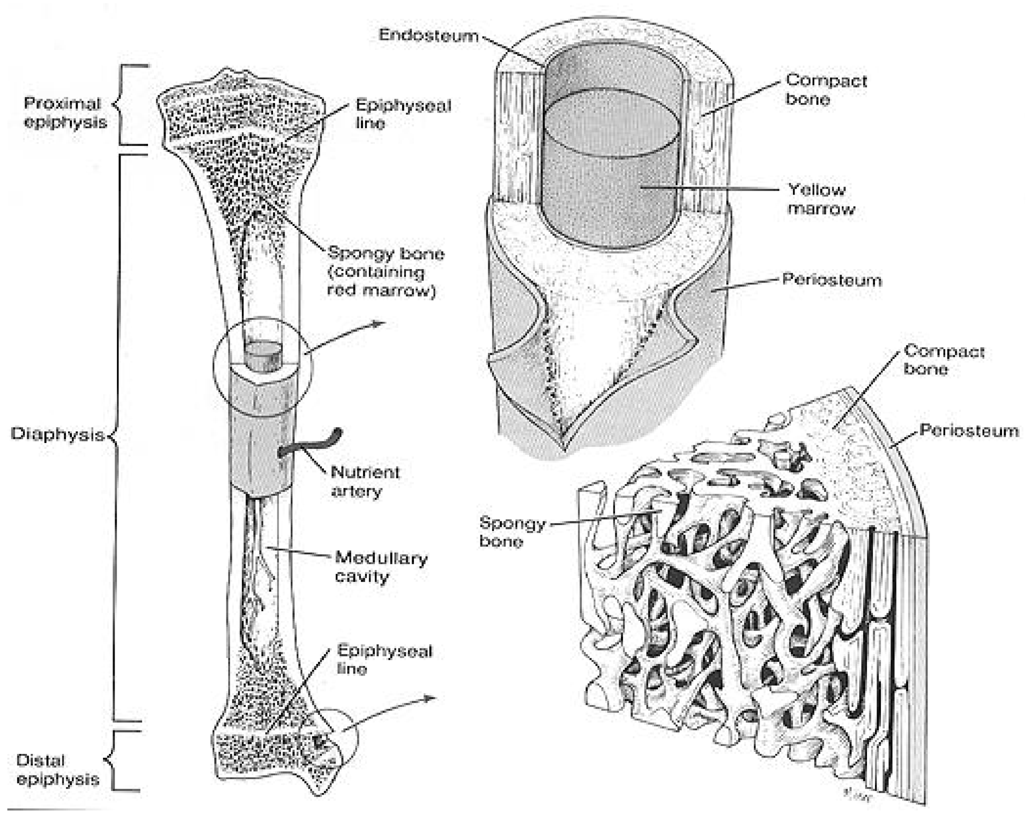
We provide an overview and background for the Homogenization Theory and outline new developments in tracking the behavior of gradients of solutions to nonlinear partial differential equations with highly oscillatory coefficients.

1. Introduction

Composite materials are materials made from two or more constituent materials with significantly different physical or chemical properties and which remain separate and distinct on a macroscopic level within the finished structure.



Many composite structures are hierarchical in nature and are made up of substructures distributed across several length scales. Examples include fiber reinforced laminates as well as naturally occurring structures like bone. In composite materials, failure initiation is a multiscale phenomena. A load applied at the structural scale is often amplified by the microstructure creating local zones of high field concentration, therefore it is of relevance to assess the load transfer between macroscopic and microscopic length scales.



2. Homogenization Theory

Consider the Dirichlet problem

$$\begin{cases} -div\left(A^{\epsilon_k}(x,\nabla u^{\epsilon_k})\right) = f \text{ on } \Omega, \\ u^{\epsilon_k} \in \mathbf{W}_0^{1,p}(\Omega);\ f \in \mathbf{W}^{-1,q}(\Omega). \end{cases} \quad (1)$$

- Ω is a bounded open subset of $\mathbf{R}^n$, which represents a sample of the material.
- Let $\mathbf{Y} = (0,1)^n$ denote the unit cube in $\mathbf{R}^n$.
- Here $p \geq 2$ and let q such that $\frac{1}{p} + \frac{1}{q} = 1$.
- We consider N-phase materials. The characteristic function for the i-th material $\chi_i(y)$ is Y-periodic and $\sum_{i=1}^{N} \chi_i(y) = 1$.
- The function $A : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is defined by

$$A(y,\lambda) = \sum_{i=1}^{N} \chi_i(y) a_i \left|\lambda\right|^{p-2} \lambda, \text{ with } a_i \geq 0.$$

Set $\epsilon_k = \frac{1}{k} > 0$, with $k \in \mathbb{N}$. We denote

$$A^{\epsilon_k}(x,\lambda) = A\left(x/\epsilon_k, \lambda\right), \text{ and } \chi_i^{\epsilon_k}(x) = \chi_i\left(x/\epsilon_k\right).$$

2.1 Homogenization Theorem

We have that the solutions u_{ϵ_k} of (1) satisfy $u_{\epsilon_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$ as $\epsilon_k \to 0$, where u is solution of

$$\begin{cases} -div\left(b\left(\nabla u\right)\right) = f \text{ on } \Omega, \\ u \in \mathbf{W}_0^{1,p}(\Omega). \end{cases} \quad (2)$$

The monotone map $b : \mathbf{R}^n \to \mathbf{R}^n$, which is independent of f and Ω, is defined for all $\xi \in \mathbf{R}^n$ by

$$b(\xi) = \int_Y A(y, p(y,\xi))dy, \quad (3)$$

where $p(y,\xi) = \xi + \nabla v_\xi(y)$, and v_ξ is the solution to the cell problem:

$$\begin{cases} \int_Y \left(A(y, \xi + \nabla v_\xi), \nabla w\right) dy = 0 \text{ for every } w \in \mathbf{W}_{per}^{1,p}(Y), \\ v_\xi \in \mathbf{W}_{per}^{1,p}(Y). \end{cases} \quad (4)$$

Here $W_{per}^{1,p}(Y)$ be the set of all functions $u \in \mathbf{W}^{1,p}(Y)$ with mean value zero which have the same trace on the opposite faces of Y.

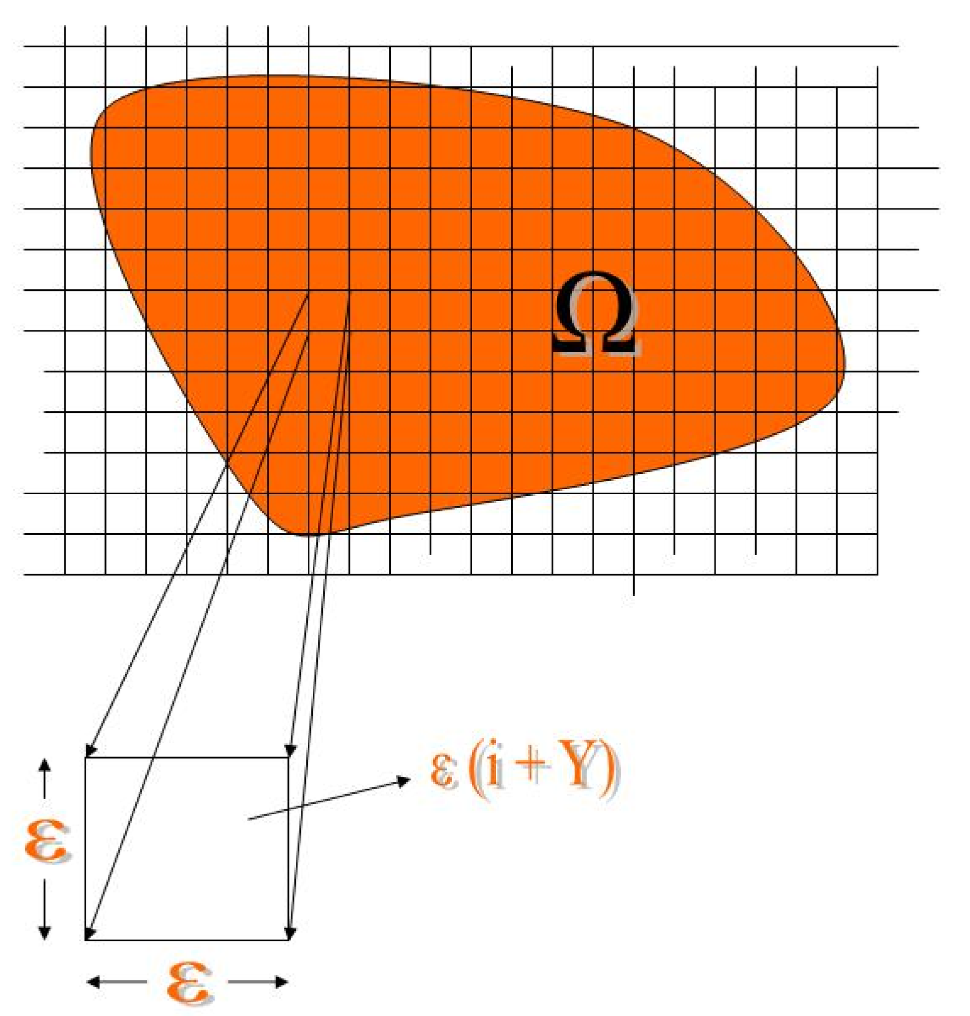
3. Corrector Theory

3.1 Notation

- Denote $Y_{\epsilon_k}^i = \epsilon_k(i + Y)$, where $i \in \mathbf{Z}^n$. The translated and rescaled image of Y.
- Let $I_{\epsilon_k} = \left\{i \in \mathbf{Z}^n : Y_{\epsilon_k}^i \subset \Omega\right\}$.
- Define M_{ϵ_k} at ∇u by

$$M_{\epsilon_k}(\nabla u)(x) = \sum_{i \in I_{\epsilon_k}} \chi_{Y_{\epsilon_k}^i}(x) \frac{1}{|Y_{\epsilon_k}^i|} \int_{Y_{\epsilon_k}^i} \nabla u(y)dy,$$

which takes the average of the field in every $Y_{\epsilon_k}^i$ inside Ω.



The Homogenization Theorem allows us to approximate ∇u_{ϵ_k}, where u_{ϵ_k} is the solution of (1), in terms of ∇u, where u is the solution of the (2), up to a remainder which converges to 0 weakly in $L^p(\Omega;\mathbb{R}^n)$. The Corrector Theorem gives a strong convergence result that captures the asymptotic behavior of the gradients ∇u_{ϵ_k}, as ϵ_k tends to 0.

3.2 Corrector Theorem

Let $f \in W^{-1,q}(\Omega)$, let u_{ϵ_k} be the solution to the problem (1), and let u be the solution to problem (2). Then,

$$\left\| p\left(\frac{x}{\epsilon_k}, M_{\epsilon_k}(\nabla u)(x)\right) - \nabla u_{\epsilon_k}(x) \right\|_{\mathbf{L}_n^p(\Omega)} \to 0, \quad (5)$$

4. Lower Bounds on Field Concentrations

We use the Corrector Theorem to study the behavior of gradients of solutions ∇u_{ϵ_k} of (1). The Corrector Theorem and the Theory of Young measures allow us to bound nonlinear quantities of the gradients from below in terms of the local solution p and the homogenized gradient.

$$\int_D \int_Y \Psi\left(x, p\left(y, \nabla u(x)\right)\right) dydx \leq \liminf_{k \to \infty} \int_D \Psi\left(x, \nabla u_{\epsilon_k}(x)\right) dx,$$

for $D \subset \Omega$ measurable, and for all Carathéodory functions $\Psi \geq 0$. Here functions of the form Ψ are often used as failure criteria. In particular,

$$\int_D \int_Y \left|p(y, \nabla u^H(x))\right|^p dydx \leq \liminf_{k \to \infty} \int_D |\nabla u_{\epsilon_k}(x)|^p dx, \quad (6)$$

for any $p > 1$.

We note that if the sequence $\Psi\left(x, \nabla u_{\epsilon_k}(x)\right)$ is weakly convergent in $L^1(\Omega)$, then the inequality becomes an equality.

5. Current and Future Research

Currently, we are studying composite materials obtained by mixing two different nonlinear power-law materials with different exponents. We want to develop corrector theory for the study of local fields inside mixtures of two power law materials and applied this corrector theory to find lower bounds for the field concentrations in these two-phase composites. We consider two nonlinear power law materials periodically distributed inside Ω. We describe the geometry of the mixture through the characteristic functions χ_1^ϵ and χ_2^ϵ corresponding to each of the materials. Here $\chi_1^\epsilon = 1$ in material one and zero outside, and $\chi_2^\epsilon = 1 - \chi_1^\epsilon$.

Since the mixture is periodic, we will use the unit period cell Y to define χ_1^ϵ and χ_2^ϵ. Here the indicator function of phase one in the unit cell Y is $\chi_1(y)$ and the indicator functions of phase two is $\chi_2(y) = 1 - \chi_1(y)$. Then the ϵ periodic mixture inside Ω is described by

$$\chi_1^\epsilon(x) = \chi_1(x/\epsilon) \text{and } \chi_2^\epsilon(x) = \chi_2(x/\epsilon).$$

The exponents for each of the materials are denoted α_1 and α_2 and satisfy $2 < \alpha_1 \leq \alpha_2$, and we denote their Hölder conjugates by β_2 and β_1 respectively. The piecewise power law material is defined by the constitutive law $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$A\left(x,\xi\right) = \alpha_1 \chi_1\left(x\right) |\xi|^{\alpha_1 - 2} \xi + \alpha_2 \chi_2\left(x\right) |\xi|^{\alpha_2 - 2} \xi,$$

and the constitutive law for the ϵ-periodic composite is given by

$$A_\epsilon(x,\xi) = A\left(\frac{x}{\epsilon}, \xi\right), \text{ for every } \epsilon > 0.$$

For a given source term $f \in W^{-1,\beta_2}(\Omega)$, we work with the following Dirichlet problem

$$-div\left(A_\epsilon\left(x, \nabla u_\epsilon\right)\right) = f \text{ on } \Omega, \quad (7)$$

with solution $u_\epsilon \in W_0^{1,\alpha_1}(\Omega)$.

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