Numerical study of the parameters α and β in the Navier–Stokes-$\alpha\beta$ equations for turbulence

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Abstract

We perform numerical studies of the Navier–Stokes-$\alpha\beta$ equations, which are based on a general framework for fluid-dynamical theories with gradient dependencies. Specifically, we examine the effect of the length scales α and β on the energy spectrum in three-dimensional statistically homogeneous and isotropic turbulent flows in a periodic cubic domain, including the limiting cases of the Navier–Stokes-α and Navier–Stokes equations. A significant increase in the accuracy arises for $\beta < \alpha$, but an optimal choice of these scales depends on the grid resolution.

1. Background

Subgrid models for turbulence seek to capture the influence of small-scale motions on physical phenomena at larger scales, allowing for accurate and efficient simulations of high Reynolds-number flows while only resolving large-scale structures. The Navier–Stokes-α (NS-α) equations, which combine Lagrangian-averaged dispersive nonlinearity with Newtonian viscosity, provide one such model. These equations involve a parameter $\alpha > 0$, with dimensions of length, and direct numerical simulations (DNS) demonstrate that they model faithfully the properties of flows involving eddy scales greater than α [6]. We examine a recent generalization, the Navier–Stokes-$\alpha\beta$ (NS-$\alpha\beta$) equations, that attempts to extend the applicability of the NS-α equations by accounting for the separation of scales between the inertial and dissipation ranges.

Chen et al. [1] obtained the NS-α equations by adding a viscous term to the Euler-α equations of Holm et al. [2]. In the context of Lagrangian averaging, α is the statistical correlation length of the excursion taken by a fluid particle away from its phase-averaged trajectory. Heuristically, α is the characteristic linear dimension of the smallest eddy that the model can resolve. Using a framework for fluid-dynamical theories involving gradient dependencies [3], Fried & Gurtin [4, 5] derived a slight generalization of the NS-α equations, called the NS-$\alpha\beta$ equations. For a fluid with density ρ and kinematic viscosity ν, these equations constitute a system

$$\left.\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\top}\mathbf{v} \\ = -\operatorname{grad}\frac{p}{\rho} + \nu(1-\beta^2\triangle)\triangle\mathbf{u}, \\ \mathbf{v} = (1-\alpha^2\triangle)\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0,\end{aligned}\right\} \quad (1)$$

for unfiltered and filtered velocities $\mathbf{v}$ and $\mathbf{u}$ and a filtered pressure p. Like α, $\beta > 0$ has dimensions of length. Setting β equal to α in (1) yields the NS-α equations. When $(1)_2$ is used to eliminate $\mathbf{v}$ from $(1)_1$, a regularized Navier–Stokes equation for $\mathbf{u}$ results; this equation contains a dispersive term, of energetic origin, with coefficient α, and a dissipative term with coefficient β. Phenomenologically, α represents eddie scales in the inertial range, where the dissipationless transfer of kinetic energy between intermediate scale eddies occurs, and β represents eddie scales in the dissipation range, where viscous damping converts the kinetic energy contained in the smallest eddies into heat. [4, 5]. Accordingly, one expects that $\beta < \alpha$. Since the specialization of the NS-$\alpha\beta$ equations to the NS-α equations requires equating β to α, the NS-α equations involve an implicit equating of disparate length scales. This is a vestige of the conventional derivation of the NS-α equations, a derivation which associates viscosity with the stretching tensor determined by the *unfiltered* velocity.

2. Pseudospectral formulation

This work investigates the influence of the length scales α and β on the energy spectrum predicted by (1) for three-dimensional homogeneous and isotropic turbulent flows in a periodic cubic domain.

The identity $\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\top}\mathbf{v} + \mathbf{u}\times\mathbf{q}$, with $\mathbf{q} = \operatorname{curl}\mathbf{v}$, allows us to rewrite $(1)_1$ as

$$\frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{u})\mathbf{v} - \mathbf{u}\times\mathbf{q} = -\operatorname{grad}\frac{p}{\rho} + \nu(1-\beta^2\triangle)\triangle\mathbf{u}. \quad (2)$$

By $(1)_{2,3}$, the unfiltered velocity $\mathbf{v}$ is solenoidal:

$$\operatorname{div}\mathbf{v} = 0. \quad (3)$$

Also, $(1)_2$ and (3) imply $(1)_3$. When supplemented by the identification $\mathbf{q} = \operatorname{curl}\mathbf{v}$, the system (1) is equivalent to $(1)_2$, (2), and (3).

Approximating the unfiltered velocity $\mathbf{v}$ by a finite collection $\mathcal{I}$ of Fourier modes $\mathbf{v}_{\mathbf{k}}$ via $\mathbf{v}(\mathbf{x},t) = \sum_{\mathcal{I}} \mathbf{v}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$ and substituting into $(1)_2$, (2), and (3), we obtain pseudospectral equations of the form

$$\left.\begin{aligned}\frac{\mathrm{d}\mathbf{v}_{\mathbf{k}}}{\mathrm{d}t} = \mathbf{P}_{\mathbf{k}}(\mathbf{u}\times\mathbf{q})_{\mathbf{k}} - \nu\frac{1+k^2\beta^2}{1+k^2\alpha^2}k^2\mathbf{v}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}}, \\ \mathbf{v}_{\mathbf{k}} = (1+\alpha^2k^2)\mathbf{u}_{\mathbf{k}}, \quad \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} = 0,\end{aligned}\right\} \quad (4)$$

where $\mathbf{P}_{\mathbf{k}} = \mathbf{1} - \mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2$ is the spectral projector, $k = |\mathbf{k}|$ is the wave-number, and the forcing $\mathbf{f}_{\mathbf{k}}$ is introduced to ensure that the total energy in each of the first two wave-number shells remains constant in time with ratio obeying Kolmogorov's $-5/3$ law.

The pseudo-spectral (4) equations were discretized using full dealiasing and a second-order Adams–Bashforth time-stepping scheme. The code is based on one developed by Albertson [7] and discussed by Albertson & Partlage [8]. A cubic flow domain with side-length $2\pi l$ and periodic boundary conditions, as is conventional for DNS of statistically homogeneous and isotropic turbulence, was used. To enforce periodicity, the set $\mathcal{I}$ was restricted to appropriate wave-numbers. The initial velocity was taken to be a Gaussian field having prescribed energy spectrum proportional to $k^4\exp(-2(k/k_0)^2)$, with $k_0 l = 5$.

3. Results

Flow statistics were calculated by averaging over several large eddy turnover times $2\pi l/u'$, where u' is the mean velocity fluctuation. All results reported are for times no shorter than that required for 5 large-eddy turnovers. More extensive results are provided by Kim et al. [9], who also consider the effect of α and β on the alignment of vorticity structures.

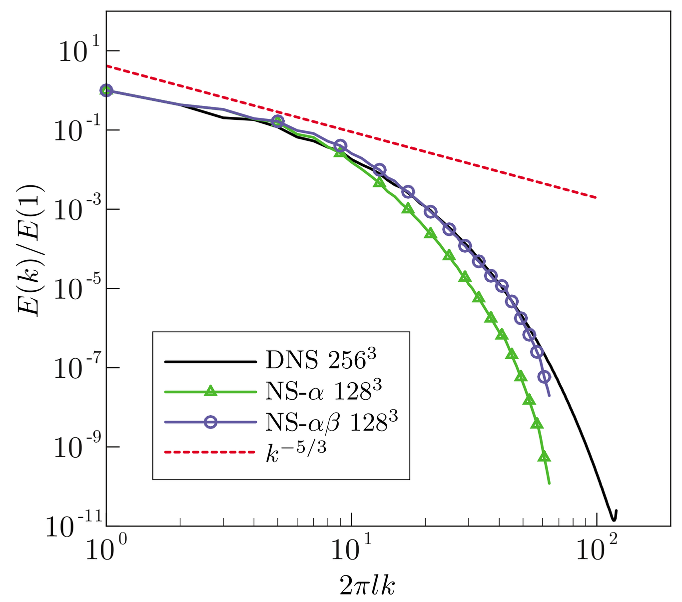

Figure 1: *Log-log plot of the energy spectrum for NS-α ($\alpha = \pi l/4$, $R_\lambda \approx 58$) and NS-$\alpha\beta$ ($\alpha = \pi l/4$, $\beta = \pi l/6$ at a grid resolution of 128^3, $R_\lambda \approx 51$) compared with the NS-equation simulation ($R_\lambda \approx 52$) at a grid resolution of 256^3.*

Figure 1 shows the normalized energy spectrum for the NS-$\alpha\beta$ equations obtained using using a grid resolution of 128^3 and a normalized kinematic viscosity $\nu\tau/l^2 = 0.01$. Plots are provided for three choices of α and β: $\alpha = \beta = 0$ (NS); $\alpha = \pi l/4$, $\beta = \alpha$ (NS-α); and $\alpha = \pi l/4$, $\beta = \pi l/6$ (NS-$\alpha\beta$). The corresponding computed values of the Taylor microscale Reynolds number R_λ are all similar: 52 (NS), 58 (NS-α), and 51 (NS-$\alpha\beta$). The result for NS-α is similar to that reported by Chen et al. [6] Although the NS-α result follows Kolomogorv 's $-5/3$ law in the inertial range, it does not agree with the NS results for $k\alpha > 1$. However, the NS-$\alpha\beta$ and NS results are indistinguishable in both the inertial and dissipation ranges. These results are consistent with the realization that setting β equal to α overdamps the response and that setting $\beta < \alpha$ allows for a greater portion of the inertial range to be captured. Although no effort was made to optimize the choice of β when α is taken to be $\pi l/4$, choosing $\beta = \pi l/6$ allows recovery of an additional decade of the energy spectrum. More optimal choices of β are likely to exist.

We now consider the sensitivity of the energy spectrum to variations in α and β and grid resolution. To study flows at slightly higher Reynolds numbers, we increased the magnitude of the forcing in the first two wave-number shells. Figures 2 and 3 shows results obtained with grid resolutions of 64^3 and 128^3, respectively. The lower resolution (64^3) results were obtained using a normalized kinematic viscosity $\nu\tau/l^2 = 0.0115$ and resulted in Taylor microscale Reynolds numbers R_λ between 65 and 100. For a given kinematic viscosity and forcing, decreasing α while holding β fixed increases R_λ. Conversely, decreasing β while holding α fixed decreases R_λ. The finer resolution (128^3) results were obtained using a slightly smaller kinematic viscosity $\nu\tau/l^2 = 0.0058$, resulting in still larger values of R_λ. Both studies show that, as β is decreased with respect to α, Kolmogorov's $-5/3$ law is better preserved by the DNS results and the decay in the dissipation range becomes steeper. On the other hand, agreement with the $-5/3$ law breaks down at higher wave-numbers and the decay of the energy spectrum in the dissipation range does not steepen with decreasing α.

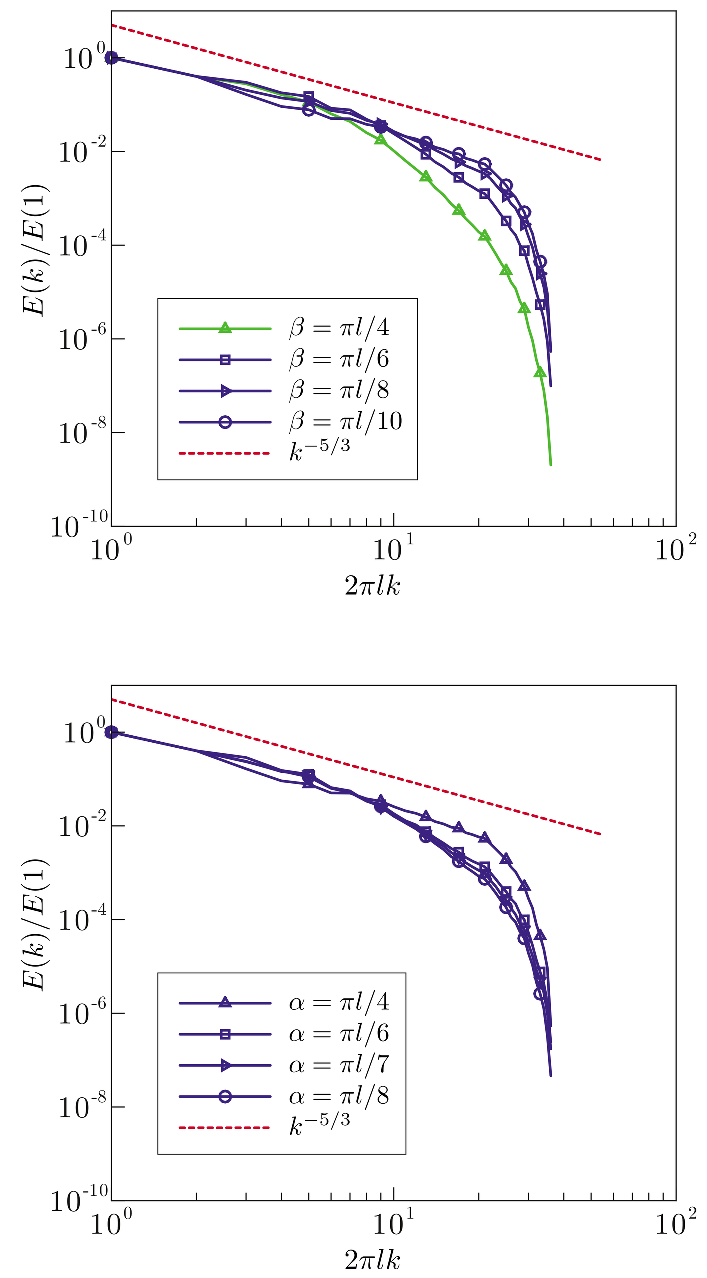

Figure 2: *Influence of the length scales (a) β and (b) α for the NS-$\alpha\beta$ model with the grid resolution 64^3. In (a) $\alpha = \pi l/4$, whereas in (b) $\beta = \pi l/10$.*

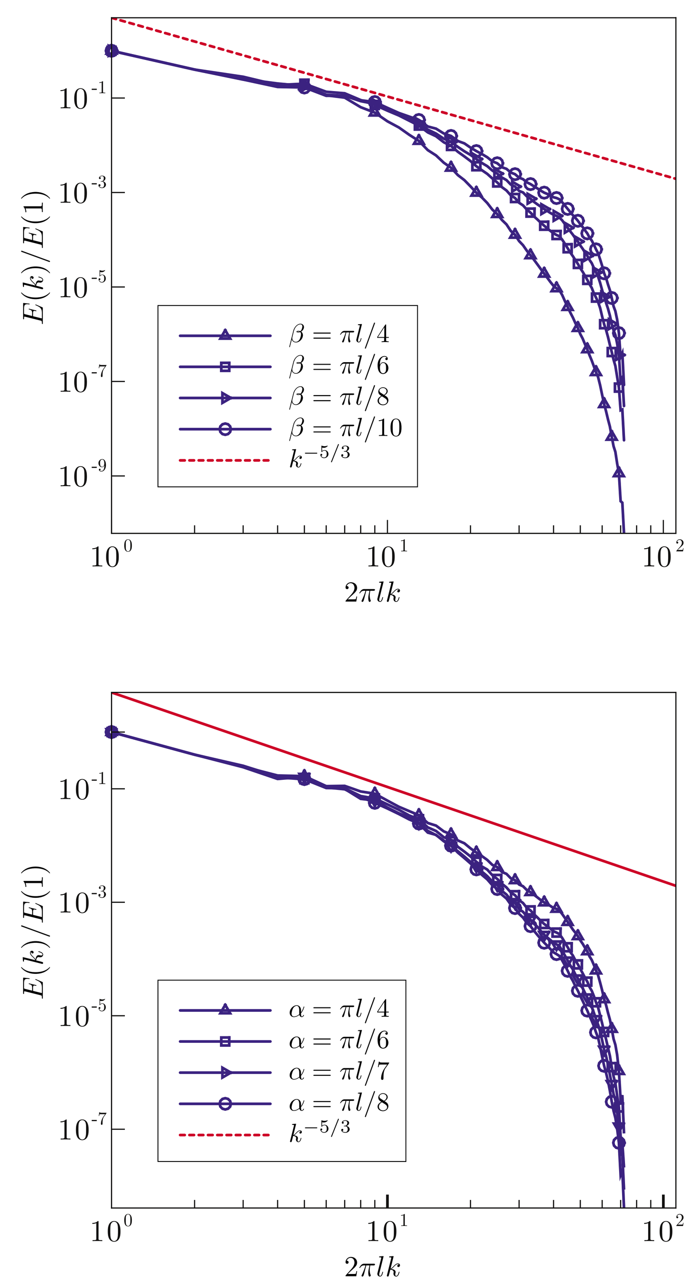

Figure 3: *Influence of the length scales (a) β and (b) α for the NS-$\alpha\beta$ model with the grid resolution 128^3. In (a) $\alpha = \pi l/4$, whereas in (b) $\beta = \pi l/10$.*

4. Discussion

Numerical evidence clearly indicates better agreement with the standard Navier–Stokes results for $\beta < \alpha$, consistent with the conjecture of Fried & Gurtin [4]. Appropriate choices of both α and β provide a means to capture both large and intermediate scale features of flows, even at coarser grid resolutions. The decay of the energy spectrum in the inertial range is mostly affected by α and the extent of the inertial range is mostly affected by β. While results are encouraging, questions remain concerning the optimal choice of β for given choices of α and the grid resolution. This is a subject for future work.

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