Asymptotic analysis in thermodynamics of viscous fluids

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Contents

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1 Introduction

Continuum fluid mechanics describes fluids by means of observable *macroscopic* quantities: the mass density, the velocity field, the absolute temperature, among others. These quantities are represented by integrable functions or rather distributions defined in the underlying physical space. The *instantaneous* values of a quantity d at a time t, expressed in the *Eulerian* reference system, reads

$$d[B](t) = \int_B d(t,x)\ \mathrm{d}x,$$

where B is a fixed volume element of the physical space.

A *fluid* is a material that can flow, meaning a fluid cannot sustain any stress in the equilibrium state. In other words, any time a force is applied to a fluid, the latter starts and keep moving even if the original driving force is no longer active. Although *continuum* fluid mechanics is primarily concerned with macroscopic (phenomenological) quantities, the underlying conceptual idea views a fluid as a large sample of particles (atoms and molecules) subjected to basic principles of *classical* physics. Accordingly, the materials obey a constitutive theory amenable to these principles.

2 Mathematical theory of fluid dynamics

We develop a mathematical theory of simple but still complex fluid systems, to which all basic thermodynamic principles may be applied. We focus on *energetically closed* systems, where both the total mass of the fluid as well as its total energy are either constants of motion or their fluxes through the physical boundary are well controlled. To fix ideas, we take the mass density ϱ

and the absolute temperature ϑ as fundamental *state variables*, characterizing completely the fluid in equilibrium, while the velocity field $\mathbf{u}$ describes the mass transfer for fluids out of equilibrium states.

2.1 Thermal systems in equilibrium

A simple thermal system in equilibrium is characterized by the value of state variables ϱ, ϑ and the associated *thermodynamic functions*: the internal energy $e = e(\varrho, \vartheta)$, the pressure $p = p(\varrho, \vartheta)$, and the entropy $s = s(\varrho, \vartheta)$. The (specific) entropy s is a remarkable quantity, being a function of the state variables with the following attributes (see Callen [2], Rajagopal and Srinivasa [28]):

- the entropy s is an increasing function of the total energy e,

$$\frac{\partial s}{\partial e} = \frac{1}{\vartheta} > 0;$$

- maximization of the total entropy

$$S = \int \varrho s \, \mathrm{d}x$$

  over the set of all allowable states of the system yields the equilibrium state provided the system is mechanically and thermally insulated;

- (**Third law of thermodynamics**) the entropy tends to zero when the absolute temperature tends to zero;

- the entropy remains constant in those processes, where the material respond *elastically*;

The introduction of specific entropy as a function of other state variables is usually referred to as the entropy form of the equation of state.

2.2 Gibbs' equation and thermodynamic stability

In accordance with the general principles of statistical mechanics (cf. Gallavotti [14]), the thermodynamic functions e, s, and p are interrelated through

GIBBS' EQUATION:

$$\vartheta D s(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{2.1}$$

In addition to (2.1), we shall assume that the thermodynamic functions satisfy

Hypothesis of thermodynamic stability:

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \tag{2.2}$$

$$\frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \tag{2.3}$$

for any ϱ, $\vartheta > 0$.

Condition (2.2) means that *compressibility* of the fluid is always positive, while $\partial_\vartheta e$ is the *specific heat at constant volume*. Hypotheses (2.2), (2.3) play a crucial role in the asymptotic analysis of the underlying fluid systems.

On the other hand, given e and p, the entropy s can be reconstructed from (2.1) that can be reformulated in the form of *Maxwell's equation*

$$\frac{\partial e(\varrho,\vartheta)}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta \frac{\partial p(\varrho,\vartheta)}{\partial \vartheta} \right). \tag{2.4}$$

The explicit relation $p = p(\varrho,\vartheta)$ is termed *thermal equation of state*, while $e = e(\varrho,\vartheta)$ is referred to as *caloric equation of state*.

2.3 Balance laws

The time evolution of thermal fluid systems *out of equilibrium* is governed by *balance laws*. To each observable property of the physical system we assign its density d, the flux vector $\mathbf{F}$, and a source term s. If the fluid occupies a spatial domain $\Omega \subset R^3$ during a time interval $I \subset R$, we define these quantities as numerical functions (distributions) of the spatial position $x \in \Omega$ and the time $t \in I$. This is the *Eulerian description* of the motion.

Assuming continuity of the fields d, $\mathbf{F}$, and s, the corresponding balance law may be written as an integral identity

$$\int_V \Big(d(t_2,x) - d(t_1,x) \Big) \ \mathrm{d}x + \int_{t_1}^{t_2} \int_{\partial V} \mathbf{F}(t,x) \cdot \mathbf{n} \ \mathrm{d}S_x \ \mathrm{d}t =$$

$$\int_{t_1}^{t_2} \int_V s(t,x) \ \mathrm{d}x \ \mathrm{d}t$$

for any $t_1 < t_2$, $t_1, t_2 \in I$, and any volume element $V \subset \Omega$, where $\mathbf{n}$ stands for the outer normal vector to ∂V.

The expression on the left-hand side of the above identity may be viewed as the *normal trace* of the 4-dimensional vector field $[d, \mathbf{F}]$ on the boundary of the time-space domain $[t_1, t_2] \times V$, in other words,

$$\text{normal trace}_{\partial([t_1,t_2]\times V)}[d,\mathbf{F}] =< s; [t_1,t_2] \times V >,$$

where the expression on the right-hand side may be interpreted as a signed measure in $I \times \Omega$. Moreover, if d and $\mathbf{F}$ are understood as *distributions*, we can write

$$\text{normal trace}_{\partial Q}[d, \mathbf{F}] = - \lim_{\varepsilon \to 0} \int_Q [d(t,x), \mathbf{F}(t,x)] \cdot \nabla_{t,x} h_\varepsilon(\text{dist}[(t,x), \partial Q]) \ \text{d}x \ \text{d}t, \tag{2.5}$$

where

$$h_\varepsilon(x) = h\Big(\frac{x}{\varepsilon}\Big), \ \text{with} \ h_\varepsilon(x) = \left\{ \begin{array}{l} x \text{ if } x \in [0,1], \\ \\ 1 \text{ if } x \geq 1, \end{array} \right.$$

and Q denotes a domain in the space-time.

The main advantage of formula (2.5) is that it requires only *integrability* of the fields d and $\mathbf{F}$. Thus the associated conservation law can be written in a concise form

$$- \lim_{\varepsilon \to 0} \int_Q [d(t,x), \mathbf{F}(t,x)] \cdot \nabla_{t,x} h_\varepsilon(\text{dist}[(t,x), \partial Q]) \ \text{d}x \ \text{d}t \tag{2.6}$$

$$= \lim_{\varepsilon \to 0} < s; h_\varepsilon(\text{dist}[(t,x), \partial Q]) >$$

for any domain $Q \subset I \times \Omega$.

Finally, it is customary to replace (2.6) by a seemingly *stronger* stipulation, namely

$$\int_I \int_\Omega \Big(d(t,x) \partial_t \varphi(t,x) + \mathbf{F}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \ \text{d}x \ \text{d}t + < s; \varphi >= 0$$

for any $\varphi \in C^\infty_c(I \times \Omega)$. Even more precisely, we can incorporate also the boundary values of the fields introducing

Balance law (weak form):

$$\int_0^T \int_\Omega \Big(d(t,x) \partial_t \varphi(t,x) + \mathbf{F}(t,x) \cdot \nabla_x \varphi(t,x) \Big) \ \text{d}x \ \text{d}t + < s; \varphi > \tag{2.7}$$

$$= - \int_\Omega d_0(x) \varphi(0,x) \ \text{d}x + \int_0^T \int_{\partial \Omega} F_b(x) \varphi(t,x) \ \text{d}S_x \ \text{d}t$$

$$\text{for any test function } \varphi \in C^\infty_c([0,T) \times \overline{\Omega}) \quad ,$$

where we have taken $I = (0,T)$.

It is easy to check that (2.7) admits a "classical" formulation:

Balance law (strong form):

$$\partial_t d + \mathrm{div}_x \mathbf{F} = s \text{ in } (0,T) \times \Omega, \ d(0,\cdot) = d_0, \ \mathbf{F} \cdot \mathbf{n}|_{\partial\Omega} = F_b, \tag{2.8}$$

as soon as all quantities in (2.7) are smooth.

Evidently, the classical formulation of a balance law given through (2.8) is more concise than its weak counterpart (2.7). For this reason, the principal equations of mathematical fluid dynamics are usually presented in the strong (differential) form, however, their *interpretation* in this text should be understood in the weak sense specified through (2.7). The reader can consult the papers by Chen and Frid [5], Chen, Torres, and Ziemer [3], [4], Šilhavý [31], [32] for more information concerning the concept of fields with "divergence measure".

2.4 Description of motion, velocity

The motion of a fluid is characterized by a *velocity field* $\mathbf{u}$. In the Eulerian description, the velocity, like other state variables, is a function of the spatial position $x \in \Omega$ and the time t. Velocity describes the *transport of mass* in the fluid. Accordingly, the vector field $\varrho\mathbf{u}$ represents the flux function in the balance law describing the mass conservation. This specific balance or rather conservation law is usually termed *equation of continuity*. Its classical formulation reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0. \tag{2.9}$$

As we have already pointed out in the previous part, it is more natural to consider the weak formulation represented by the integral identity

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x. \tag{2.10}$$

Note that, if satisfied for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$, relation (2.10) includes implicitly the satisfaction of the initial condition $\varrho(0,\cdot) = \varrho_0$ and the no-flux boundary condition $\varrho\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$. In particular, if all terms in (2.10) are integrable, the mapping

$$t \mapsto \varrho(t,\cdot) \text{ is weakly continuous,}$$

meaning

$$\left\{ t \mapsto \int_\Omega \varrho(t,\cdot)\varphi \ \mathrm{d}x \right\} \in C[0,T]$$

for any $\varphi \in C^\infty_c(\overline{\Omega})$. Consequently, as the boundary of the physical domain Ω is assumed impermeable, we deduce that the *total mass* of the fluid is a conserved quantity:

$$\int_\Omega \varrho(t,\cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x = M_0 \text{ for all } t \in [0,T]. \tag{2.11}$$

Similarly, the flux of any *extensive* (additive over subregions) property d contains a *convective* component $d\mathbf{u}$ proportional to the velocity. Conversely, postulating the existence of a vector field $\mathbf{u}$ enjoying this property may be viewed as a proper definition of the velocity. The mass density $\varrho = \varrho(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, together with the velocity field $\mathbf{u} = \mathbf{u}(t,x)$ is a trio of fundamental *state variables* in the theory developed in this paper. The value of the state variables at a fixed instant t is supposed to characterize *completely* the state of the physical system and, if possible, to determine in a unique way its behavior in the future. More complex systems and/or alternative approaches to fluid mechanics may use *extended families* of state variables (see the monograph by Müller and Ruggeri [25]).

In accordance with *Newton's second law*, the flux associated to the momentum vector $\varrho\mathbf{u}$ reads $\varrho\mathbf{u}\otimes\mathbf{u} - \mathbb{T}$, where $\mathbb{T}$ is the Cauchy stress tensor, yielding the force per unit surface that the part of a fluid in contact with an ideal surface element imposes on the part of the fluid on the other side of the same surface element. Fluids are characterized among other materials through *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{2.12}$$

where the symbol $\mathbb{S}$ denotes the *viscous stress tensor*.

In accordance with the general principles delineated in Section 2.3, the *balance of linear momentum* or *equation of motion* reads

$$\int_0^T \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi + p\mathrm{div}_x\varphi\ \mathbb{I} \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.13}$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\mathbf{f}\cdot\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0\cdot\varphi(0,\cdot)\ \mathrm{d}x,$$

or, in the classical form,

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \mathrm{div}_x\mathbb{S} + \varrho\mathbf{f},\ \ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0, \tag{2.14}$$

where $\mathbf{f}$ denotes a driving force.

A proper choice of the test functions in (2.13) is open to discussion. Note that, in contrast with the abstract form of a balance law introduced in (2.7), relation (2.13) contains vector-valued test functions φ. Obviously, the space of test functions should contain $C^\infty_c([0,T)\times\Omega;R^3)$ in order to recover, at least formally, equation (2.14). Moreover, in accordance with the hypothesis of *impermeability of the physical boundary*, we restrict ourselves to the case $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$. Accordingly, taking

$$\varphi \in C^\infty_c([0,T)\times\overline{\Omega};R^3),\ \ \varphi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{2.15}$$

we end up, formally, with the *complete slip* boundary conditions for the velocity field

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0. \tag{2.16}$$

Note that for a viscous fluid, where the tensor $\mathbb{S}$ depends effectively on the velocity gradient, it is more customary to use the *no-slip* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.17}$$

corresponding to the space of test functions $C_c^\infty([0,T) \times \Omega; R^3)$. The reader may consult Málek and Rajagopal [23] for more details concerning the physical background of the boundary conditions for viscous fluids.

2.5 Energy, entropy, Second law of thermodynamics

We focus on *conservative systems* for which the *total energy is a constant of motion.* To simplify the presentation, let us assume that $\mathbf{f} = \nabla_x F$, where $F = F(x)$ is a given potential, defined and differentiable in Ω.

Multiplying, formally, the momentum equation (2.14) by $\mathbf{u}$ we deduce

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho F \right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} - \varrho F\mathbf{u} + p\mathbf{u} \right) - \mathrm{div}_x(\mathbb{S}\mathbf{u}) \tag{2.18}$$

$$= p\mathrm{div}_x\mathbf{u} - \mathbb{S} : \nabla_x \mathbf{u}.$$

The quantity $1/2\varrho|\mathbf{u}|^2 - \varrho F$ represents the kinetic energy of the system; whence (2.18) may be viewed as a balance of mechanical energy. Clearly, this quantity is not, in general, conserved as (2.18) contains a source term. In accordance with (2.16), however, there is no flux of the (mechanical) energy through the boundary, and, in addition, we require the total energy of the system to be conserved. It follows, necessarily, that the "missing" part of the energy in (2.18) is converted to its *internal component* so that the *total energy balance* reads

$$E(t) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x = E_0 \text{ for any } t > 0. \tag{2.19}$$

The missing connection between (2.18 - 2.19) is provided by *Second law of thermodynamics*, specifically, by the entropy balance. Following the general framework introduced in Section 2.3 we write the *entropy balance equation* in the abstract form as

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}_s = \sigma, \tag{2.20}$$

where $\mathbf{q}_s$ is the entropy flux, and σ is the *entropy production rate.* In view of (2.1), it is more convenient to set

$$\mathbf{q}_s = \frac{\mathbf{q}}{\vartheta},$$

where $\mathbf{q}$ represents the internal energy (heat) diffusion flux. Unlike mass, the entropy or internal energy may be transported in systems in a stationary state when $\mathbf{u} \equiv 0$. The transport is provided by *diffusive* transfer of energy that is *irreversible* in time.

Multiplying equation (2.20) by ϑ, we use Gibbs' equation (2.1) to deduce the *internal energy balance*

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \vartheta\sigma + \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} - p\mathrm{div}_x\mathbf{u}. \tag{2.21}$$

As we assume there is no flux of energy through the boundary, we take

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{2.22}$$

Consequently, integrating (2.21) over Ω and comparing the resulting expression with (2.18), (2.19), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\ \mathrm{d}x = \int_\Omega \left(\vartheta\sigma - \mathbb{S} : \nabla_x\mathbf{u} + \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x. \tag{2.23}$$

It is worth-noting that we have arrived at (2.23) under the principal assumption that all quantities in question are regular (smooth). Keeping in mind possible singularities that may appear we assume here and hereafter that the entropy production σ is non-negative satisfying

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big) \geq 0. \tag{2.24}$$

In particular, comparing (2.19), (2.23), (2.24) we arrive at the classical relation

$$\sigma = \frac{1}{\vartheta}\Big(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big)$$

provided all quantities are smooth. Note that positivity of σ is enforced by *Second law of thermodynamics.*

Since any non-negative distribution is in fact a Radon measure, the weak formulation of the *entropy balance equation* takes the form

$$\int_0^T\int_\Omega\left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}x + <\sigma;\varphi> \tag{2.25}$$

$$= -\int_\Omega(\varrho s(\varrho,\vartheta))_0\varphi(0,\cdot)\ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, where $\sigma \in \mathcal{M}^+([0,T]\times\overline{\Omega})$ is a measure satisfying (2.24).

2.6 Constitutive equations

Constitutive equations describe the *material properties* of a specific fluid. In principle, they are expressed in terms of the fundamental state variables and their partial derivatives.

2.6.1 Equations of state

A typical example of a constitutive relation is the *thermal equation of state* relating the pressure p to the state variables ϱ, ϑ. Although frequently used in models of fluids underlying a macroscopic motion, we should always keep in mind that the equation of state refers to the system in thermodynamic equilibrium.

As a model example of the thermal equation of state, we consider to so-called monoatomic gas. A *monoatomic gas* is an idealized gas composed of randomly moving point particles that interact only through elastic collisions. Such a concept is amenable to analysis under the methods of statistical mechanics. A universal equation of state characterizing a monoatomic gas reads (see Eliezer et al. [8]):

$$p(\varrho, \vartheta) = \frac{2}{3}\varrho e(\varrho, \vartheta). \tag{2.26}$$

Combining (2.26) with Gibbs' equation (2.1) we obtain

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \text{ for a certain function } P. \tag{2.27}$$

It is interesting to examine the impact of hypothesis of thermodynamics stability stated in (2.2), (2.3) on the state equation (2.27). To begin, positivity of compressibility leads immediately to

$$P'(Z) > 0 \text{ for any } Z \geq 0. \tag{2.28}$$

As a matter of fact, we deduce (2.28) only for $Z > 0$, however, we shall always assume positive compressibility also for $Z = 0$.

Since the specific heat at constant volume is also positive (cf. (2.3)), we obtain

$$\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} > 0 \text{ for all } Z \geq 0, \tag{2.29}$$

in particular,

$$\frac{P(Z)}{Z^{5/3}} \searrow p_\infty \text{ as } Z \to \infty. \tag{2.30}$$

In accordance with Maxwell's equation (2.4), the specific entropy is given as

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{2.31}$$

with

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0 \tag{2.32}$$

Now, we recall *Third law of thermodynamics* discussed briefly in Section 2.1. By virtue of this principle,

$$\lim_{Z\to\infty} S(Z) = 0, \tag{2.33}$$

in particular, it is plausible to require the specific heat at constant volume to be bounded, specifically in accordance with (2.29),

$$0 < \frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq c \text{ for all } Z \geq 0. \tag{2.34}$$

It is interesting to note that (2.33) rules out the standard *Boyle-Marriot law* of perfect gas

$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta$$

that is not suitable for describing *real* gases for large values of the degeneracy parameter $\varrho/\vartheta^{3/2}$. Accordingly, we make a realistic assumption that the gas or at least one of its components (electron gas) behaves like a *Fermi gas* in the degenerate area $\varrho/\vartheta^{3/2} >> 1$, specifically, $p_\infty > 0$ in (2.30) (see Eliezer at al. [8]).

In models describing gases under large temperature regime, it is convenient to consider also the effect of thermal radiation. The simplest, but certainly not optimal way is to add the so-called thermal pressure $p_R = a/3\vartheta^4$, with $a > 0$. A prototype example of the pressure in a real gas then reads

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\varrho, \vartheta), \text{ with } p_M(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \ p_R = \frac{a}{3}\vartheta^4. \tag{2.35}$$

2.6.2 Diffusion flux, transport coefficients

Diffusion in continuum mechanics is an *irreversible process.* The diffusive fluxes $\mathbb{S}$ and $\mathbf{q}$, appearing in the entropy production (2.24), are responsible for an irreversible transfer of the mechanical energy into heat and its trend to attain a spatially homogeneous equilibrium state. Accordingly,

$$\mathbb{S} : \nabla_x \mathbf{u} \geq 0, \ \mathbf{q} \cdot \nabla_x \vartheta \geq 0 \tag{2.36}$$

for any *physically admissible* fluid in motion.

In this text, we suppose a very simple dependence of the fluxes $\mathbb{S}$, $\mathbf{q}$ on the affinities $\nabla_x \mathbf{u}$, $\nabla_x \vartheta$, namely the linear one. More specifically, we assume that the viscous stress $\mathbb{S}$ is given by

Newton's rheological law:

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{2.37}$$

with the *shear viscosity coefficient* μ and the *bulk viscosity coefficient* η.

Analogously, the heat flux $\mathbf{q}$ obeys

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{2.38}$$

where κ is the *heat conductivity coefficient.*

In accordance with *Second law of thermodynamics*, the *transport coefficients* μ, η, and κ must be non-negative. We focus on viscous and heat conducting fluids therefore we always assume that both μ and κ are strictly positive.

2.7 Navier-Stokes-Fourier system

We introduce a model problem of an energetically isolated fluid system based on the physical principles and constitutive assumptions discussed in the preceding text.

2.7.1 Classical formulation

• We are given the thermodynamic functions: the pressure $p = p(\varrho, \vartheta)$, the specific internal energy $e = e(\varrho, \vartheta)$, and the specific entropy $s = s(\varrho, \vartheta)$ satisfying Gibbs' equation (2.1), together with hypothesis of thermodynamic stability (2.2), (2.3).
• The fluid occupies a bounded spatial domain $\Omega \subset R^3$ and is mechanically and thermally insulated, in particular, the total mass M and the total energy E of the fluid are constants of motion:

$$\frac{\mathrm{d}}{\mathrm{dt}} M = 0, \ M(t) = \int_\Omega \varrho(t, \cdot) \ \mathrm{d}x \tag{2.39}$$

$$\frac{\mathrm{d}}{\mathrm{dt}} E = 0, \ E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) (t, \cdot) \ \mathrm{d}x \tag{2.40}$$

provided the fluid is driven by a potential driving force $\mathbf{f} = \nabla_x F(x)$.
• The motion of the fluid is governed by the following principal field equations:

EQUATION OF CONTINUITY:

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0; \tag{2.41}$$

MOMENTUM EQUATION:

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F; \tag{2.42}$$

ENTROPY EQUATION:

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{2.43}$$

• In accordance with (2.39), (2.40), the system of equations (2.41 - 2.43) is supplemented by the *no-slip* boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.44}$$

or the *complete slip* boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0. \tag{2.45}$$

In addition, the normal component of the heat flux vanishes on the boundary:

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.46}$$

• The viscous stress $\mathbb{S}$ is determined through Newton's law (2.37), while the heat flux $\mathbf{q}$ obeys Fourier's law (2.38).

2.7.2 Renormalization of the mass transport

A weak formulation of the equation of continuity has been introduced in (2.10). For purposes of future analysis, however, we need also its "renormalized" version originally introduced by DiPerna and Lions [7]. To this end, multiply (2.41) on $b'(\varrho)$, where b is a (nonlinear) function, to obtain

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0. \tag{2.47}$$

Obviously, equations (2.41), (2.47) are completely *equivalent* as soon as all quantities are smooth enough.

Motivated by (2.47), we introduce a renormalized variant of the *weak formulation* (2.10) reading

$$\int_0^T \int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\mathbf{u}\varphi \ \mathrm{d}x \ \mathrm{d}t \tag{2.48}$$

$$= -\int_\Omega b(\varrho_0)\varphi(0,\cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. Note that (2.48) implicitly includes the initial distribution of the density $\varrho = \varrho_0$ and the satisfaction of the impermeability condition $\mathbf{u} \cdot \mathbf{n} = 0$.

Since the density ϱ may be only an integrable function, we must pay attention to a proper choice of the functions b. Accordingly, we assume that

$$b \in C[0,\infty), \ b' \in C_c[0,\infty), \tag{2.49}$$

in other words, b becomes constant for sufficiently large values of its argument. The renormalized equation (2.48) therefore requires ϱ, $\varrho\mathbf{u}$, $\mathrm{div}_x\mathbf{u}$ to be at least integrable functions in $[0,T) \times \Omega$.

Although the concept of renormalized solutions shares certain common features with the *entropy solutions* to non-linear conservation laws introduced by Kružkov [17], its proper nature is, in fact, rather different as equation (2.10) is *linear* with respect to ϱ and the velocity field $\mathbf{u}$ is typically more regular than in the case of hyperbolic conservation laws. It can be shown that any *weak solution* satisfying (2.10) is a renormalized solution in the sense of (2.48) provided ϱ, $\mathbf{u}$ belong to suitable Lebesgue spaces of integrable functions.

Lemma 2.1 [DiPerna, Lions [7]]
Let $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, $\mathbf{u} \in L^q(0,T;W^{1,q}(\Omega;R^3))$ satisfy (2.10),

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1.$$

Then ϱ, $\mathbf{u}$ is a renormalized solution specified in (2.48).

It turns out that Lemma 2.1 is not strong enough to render the class of renormalized solutions *stable* with respect to the natural energy norm. Indeed the velocity $\mathbf{u}$ in the Navier-Stokes-Fourier system is known to belong, in general, only to the class $L^2(0,T;W^{1,2}(\Omega;R^3))$, while $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, with $\gamma \in [1,5/3]$. As the renormalized equation plays a crucial role in the study of density oscillations, we establish a new criterion for its validity applicable in a more general setting. Following [10, Chapter 6.4] we introduce *oscillations defect measure* $\mathbf{osc}_q[\varrho_n \to \varrho]$ associated to a sequence

$$\varrho_n \to \varrho \text{ weakly in } L^1(Q),$$

$$\mathbf{osc}_q[\varrho_n \to \varrho](Q) = \sup_{k\geq 1}\left(\limsup_{n\to\infty}\int_Q |T_k(\varrho_n) - T_k(\varrho)|^q \ \mathrm{d}y\right), \tag{2.50}$$

where $T_k(\varrho) = \min\{k, \varrho\}$ are cut-off functions.

We report the following result.

Lemma 2.2 [9, Proposition 2.4]
Let ϱ_n, $\mathbf{u}_n$ be a sequence of renormalized solutions of the equation of continuity in the sense specified in (2.48) such that

$$\varrho_n \to \varrho \textit{ weakly in } L^1((0,T)\times\Omega),$$

$$\mathbf{u}_n \to \mathbf{u} \textit{ weakly in } L^q(0,T;L^q(\Omega)),\ \nabla_x\mathbf{u}_n \to \nabla_x\mathbf{u} \textit{ weakly in } L^q(0,T;L^q(\Omega;R^3)),$$

$$\mathbf{osc}_p[\varrho_n \to \varrho]((0,T)\times\Omega) < \infty,$$

where

$$\frac{1}{p} + \frac{1}{q} < 1.$$

Then ϱ, $\mathbf{u}$ is also a renormalized solution.

In the context of the Navier-Stokes-Fourier system, a typical velocity field $\mathbf{u}$ belongs to the class $L^2(0,T;W^{1,2}(\Omega;R^3))$. In addition, it can be shown that

$$\mathbf{osc}_{\gamma+1}[\varrho_n \to \varrho]((0,T)\times\Omega) < \infty$$

for any sequence of weak solutions $\{\varrho_n\}_{n=1}^\infty$, with $\gamma > 1$. In particular, Lemma 2.2 can be used in order to show *weak sequential stability* of the class of renormalized solutions in the context of the Navier-Stokes-Fourier system.

2.7.3 Weak formulation

Our next goal is to reformulate the Navier-Stokes-Fourier system in the framework of weak solutions introduced in Section 2.3.

• Similarly to the classical formulation we suppose that $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ are given functions satisfying Gibbs' equation (2.1) and hypothesis of thermodynamic stability (2.2), (2.3).
• The state of the fluid system at a given instant $t \in (0, T)$ and a spatial position $x \in \Omega \subset R^3$ is determined through the state variables $\varrho = \varrho(t, x)$, $\vartheta = \vartheta(t, x)$, and $\mathbf{u} = \mathbf{u}(t, x)$. The density ϱ is a non-negative measurable function, the absolute temperature ϑ is a measurable function satisfying $\vartheta(t, x) > 0$ for a.a. $(t, x) \in (0, T) \times \Omega$.

In addition, we assume that the system is mechanically and thermally insulated, the total mass is a constant of motion,

$$M(t) = \int_\Omega \varrho(t, \cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x = M_0 \text{ for a.a. } t \in (0, T), \tag{2.51}$$

and so is the total energy

$$E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x \tag{2.52}$$

$$= \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) - \varrho_0 F \right) \ \mathrm{d}x \text{ for a.a. } t \in (0, T).$$

• The time evolution of the system is governed by the following system of equations (integral identities):

CONSERVATION OF MASS (RENORMALIZED):

$$\int_0^T \int_\Omega \Big(b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho)\varrho \Big) \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{2.53}$$

$$= - \int_\Omega b(\varrho_0) \varphi(0, \cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$ and any b satisfying (2.49) and/or $b(\varrho) = \varrho$.

Balance of momentum (weak):

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.54}$$

$$\int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi - \varrho \nabla_x F \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$.
If the complete slip boundary conditions (2.45) are imposed, the space of admissible test functions must be extended to $C_c^\infty([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial \Omega} = 0$.

Entropy balance (weak):

$$\int_0^T \int_\Omega \Big(\varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q} \cdot \nabla_x \varphi}{\vartheta} \Big) \ \mathrm{d}x \ \mathrm{d}t + < \sigma; \varphi > \tag{2.55}$$

$$= - \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0, \cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3)$, where the *entropy production rate* $\sigma \in \mathcal{M}^+([0,T] \times \overline{\Omega})$ satisfies

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{2.56}$$

• The viscous stress $\mathbb{S}$ is determined by Newton's rheological law (2.37), the heat flux $\mathbf{q}$ satisfies Fourier's law (2.38).

2.8 Existence of global-in-time solutions

A rigorous proof of global-in-time weak solutions to the Navier-Stokes-Fourier system for given initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ requires further mostly technical hypotheses stated in the next section.

2.8.1 Hypotheses

The hypotheses listed below are by no means optimal. The interested reader may consult [12, Chapter 3] for possible improvements.

[**H1**] The initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ satisfy:

$$\varrho_0, \vartheta_0 \in L^\infty(\Omega), \ \mathbf{u}_0 \in L^\infty(\Omega; R^3),$$

$$\varrho_0(x) \geq 0, \vartheta(x) > 0 \text{ for a.a. } x \in \Omega.$$

[**H2**] The potential of the driving force F belongs to $W^{1,\infty}(\Omega)$.

[**H3**] The pressure $p = p(\varrho, \vartheta)$ is given by

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{2.57}$$

where

$$P \in C^1[0,\infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.58}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leq c \text{ for all } Z > 0, \tag{2.59}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.60}$$

Moreover, in accordance with Gibbs' equation (2.1), the specific internal energy e obeys

$$e(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{2.61}$$

and

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.62}$$

with

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{2.63}$$

[**H4**] The transport coefficients μ, η, and κ are continuously differentiable functions of the temperature ϑ satisfying

$$\mu \in W^{1,\infty}[0,\infty), \ 0 < \underline{\mu}(1+\vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta^\alpha), \tag{2.64}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta^\alpha), \tag{2.65}$$

where

$$1/2 \leq \alpha \leq 1; \tag{2.66}$$

and

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3). \tag{2.67}$$

2.8.2 Existence result

The following result was proved in [12, Chapter 3.3, Theorem 3.1].

Global-in-time weak solutions:

Theorem 2.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}, \nu > 0$. Suppose that the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ satisfy hypothesis* **[H1]** *and that the driving force potential F obeys* **[H2]***. Furthermore, let the thermodynamic functions p, e, and s be as in* **[H3]***, while the transport coefficients μ, η, and κ satisfy* **[H4]***.*
Then the Navier-Stokes-Fourier system specified in Section 2.7.3 admits a weak solution ϱ, ϑ, and $\mathbf{u}$ belonging to the class:

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathbf{u} \in L^2(0,T;W^{1,q}(\Omega;R^3)),\ q = \frac{8}{5-\alpha},$$

where the parameter α has been introduced in hypotheses (2.64 - 2.66).

The hypotheses concerning smoothness of the boundary of the spatial domain Ω can be relaxed (see Kukučka [18], Poul [26]).

3 Long-time behavior

The mathematical subject called *dynamical system* is completely characterized by its *state* and the rules called the *dynamics* for determining the state at a given future time given the present state. The dynamics of energetically insulated fluid systems considered in this text is determined by the Navier-Stokes-Fourier system of equations introduced in the preceding chapter. In an attempt to predict the long-time behavior of these systems we quickly encounter several difficulties when trying to apply this approach to the *weak* solutions. To begin, these solutions are not known to be uniquely determined by the initial or other external data as the driving force. Strangely enough, this fact does not prevent us completely from obtaining certain qualitative information on the dynamics. It is easy to realize that the standard concepts of *absorbing set*, *invariant set*, or *global attractor* do not really need uniqueness or even existence of some solution semigroup. All the relevant statements concerning the long-time dynamics can be formulated and rigorously verified in certain cases discussed in the present chapter, without the classical concept of a well-posed problem. In addition, the strength of the forthcoming results is underlined but the fact that they apply to a considerably vast class of the weak solutions introduced in Section 2.7.3.

3.1 Equilibrium states

We identify the equilibrium solutions to the energetically insulated fluid systems, and clarify the following commonly accepted but otherwise rather vague statements:

- equilibrium solutions minimize the entropy production;
- equilibrium solutions maximize the total entropy of the system in the class of all admissible states;
- all solutions to the evolutionary system driven by a conservative time-independent external force tend to an equilibrium for large time.

The leading physical principles to be used in the forthcoming analysis are Gibbs' equation stated in (2.1), together with hypothesis of thermodynamic stability specified in (2.2), (2.3). Moreover, in order to fix ideas, we impose the no-slip boundary condition for the velocity

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{3.1}$$

Integrating entropy equation (2.55) over Ω, meaning taking spatially homogeneous test functions, and adding the resulting expression to the total energy balance (2.52), we deduce *total dissipation balance* in the form

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta) - \varrho F \right)(\tau,\cdot)\ \mathrm{d}x + \overline{\vartheta}\sigma\Big[[0,\tau]\times\overline{\Omega}\Big] \\ &\quad = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) - \overline{\vartheta}\varrho_0 s(\varrho_0,\vartheta_0) - \varrho_0 F \right)\ \mathrm{d}x \end{aligned} \tag{3.2}$$

for a.a. $\tau \in [0,T]$ and any positive constant $\overline{\vartheta}$.

It follows from (3.2) that equilibrium (time independent) solutions minimize trivially the entropy production rate, namely $\sigma \equiv 0$. Given the specific forms of the stress tensor $\mathbb{S}$ and the heat flux $\mathbf{q}$, relation (2.56) gives rise to

$$\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big) = 0, \ \text{and } \nabla_x\vartheta = 0 \tag{3.3}$$

for any equilibrium state. In particular, as $\mathbf{u}$ vanishes on the boundary (cf. (3.1)), a direct application of the standard Korn's inequality yields

$$\mathbf{u} \equiv 0 \text{ for any equilibrium state,} \tag{3.4}$$

in other words, the set of equilibrium states is formed by static solutions.

Accordingly, any equilibrium solution $\tilde\varrho$, $\tilde\vartheta$ satisfies

$$\nabla_x p(\tilde\varrho,\tilde\vartheta) = \tilde\varrho\nabla_x F,\ \tilde\vartheta = \text{const} > 0 \text{ in } \Omega.$$

Moreover, the static states must be identified through their mass M_0,

$$M_0 = \int_\Omega \tilde\varrho\ \mathrm{d}x,$$

that is a conserved quantity (cf. (2.51)), and the limit

$$D_\infty[\overline{\vartheta}] = \lim_{\tau\to\infty}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta) - \varrho F \right)(\tau,\cdot)\ \mathrm{d}x.$$

3.2 Static states

Consider a solution $\tilde{\varrho}$, $\tilde{\vartheta}$ of the *static problem*

$$\nabla_x p(\tilde{\varrho},\tilde{\vartheta}) = \tilde{\varrho}\nabla_x F, \ \ \tilde{\varrho} \geq 0, \ \ \tilde{\vartheta} = \text{const} > 0 \text{ in } \Omega, \tag{3.5}$$

satisfying the constraints

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho},\tilde{\vartheta}) - \overline{\vartheta}\tilde{\varrho} s(\tilde{\varrho},\tilde{\vartheta}) - \tilde{\varrho} F \Big) \ \mathrm{d}x = D_\infty[\overline{\vartheta}]. \tag{3.6}$$

3.2.1 Positivity of the static density distribution

Strict *positivity* of the static density $\tilde{\varrho}$ plays a crucial role in the analysis of the static problem. In what follows, we assume, in addition to hypothesis of thermodynamic stability (2.2), that p is a continuously differentiable function, and

$$\lim_{\varrho \to 0} \frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0 \text{ for any fixed } \vartheta > 0. \tag{3.7}$$

Given a positive constant $\tilde{\vartheta}$, equation (3.5) admits only strictly positive solutions $\tilde{\varrho}$ on condition that $\nabla_x F$ is bounded and p satisfies (3.7). Indeed $\tilde{\varrho}$ satisfies

$$\frac{\partial p(\tilde{\varrho},\tilde{\vartheta})}{\partial \varrho} \nabla_x \tilde{\varrho} = \tilde{\varrho} \nabla_x F$$

on any component of Ω, on which $\tilde{\varrho}$ is positive. In other words,

$$\mathcal{P}(\tilde{\varrho},\tilde{\vartheta}) = F + c_{\tilde{\varrho},\tilde{\vartheta}} \text{ in } \Omega, \tag{3.8}$$

where $c_{\tilde{\varrho},\tilde{\vartheta}}$ is a constant, and

$$\frac{\partial \mathcal{P}(\varrho,\tilde{\vartheta})}{\partial \varrho} = \frac{1}{\varrho}\frac{\partial p(\varrho,\tilde{\vartheta})}{\partial \varrho}. \tag{3.9}$$

Consequently, as the right-hand side of (3.8) is bounded in Ω, while the left-hand side tends to infinity for $\tilde{\varrho}$ close to zero, we conclude that $\tilde{\varrho}$ remains bounded below away from zero.

3.2.2 Helmholtz function

Given $\overline{\vartheta} > 0$, we introduce *Helmholtz function*

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta). \tag{3.10}$$

It follows from Gibbs' relation (2.1) that

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho,\overline{\vartheta})}{\partial \varrho}, \tag{3.11}$$

and

$$\frac{\partial H_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta}(\vartheta - \overline{\vartheta})\frac{\partial e(\varrho,\vartheta)}{\partial \vartheta}. \tag{3.12}$$

Consequently, hypothesis of thermodynamic stability (2.2), (2.3) implies that

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ is a strictly convex function;
- $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ is decreasing if $\vartheta < \overline{\vartheta}$ and increasing whenever $\vartheta > \overline{\vartheta}$ for any fixed ϱ.

In addition, the Helmholtz function $H_{\overline{vt}}$ enjoys certain *coercivity properties.* More specifically, for any $\tilde{\varrho}$ such that

$$0 < \underline{\varrho} < \tilde{\varrho} < \overline{\varrho}$$

there exists a positive constant $\Lambda = \Lambda(\underline{\varrho}, \overline{\varrho}, \overline{\vartheta})$ such that

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \tag{3.13}$$

$$\geq \Lambda \left\{ \begin{array}{l} |\varrho - \tilde{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \text{ if } \underline{\varrho} < \varrho < \overline{\varrho},\ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta}, \\ \\ \varrho e(\varrho, \vartheta) + \overline{\vartheta} |s(\varrho, \vartheta)| + 1 \text{ otherwise} \end{array} \right.$$

(see [12, Chapter 3, Proposition 3.2]).

Relations (3.9), (3.11) imply that the functions $\mathcal{P}$ and $\partial_\varrho H_{\tilde{\vartheta}}(\varrho, \tilde{\vartheta})$ differ by a constant (possibly depending on $\tilde{\vartheta}$), in other words, we may replace (3.8) by

$$\frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} = F + c_{\tilde{\varrho}, \tilde{\vartheta}} \text{ in } \Omega \tag{3.14}$$

whenever $\tilde{\varrho} = \tilde{\varrho}(x)$, $\tilde{\vartheta} > 0$ is a solution of static problem (3.5). Consequently, by virtue of (3.11), we may infer from (3.14) that the static solutions may not necessarily exist if the pressure p is a *sublinear* function of ϱ. On the other hand, it follows from (3.14) the the static density $\tilde{\varrho}$ enjoys the same differentiability properties as the potential F.

3.2.3 Principle of maximal entropy

As a direct consequence of relation (3.14), we deduce that the static solutions minimize the entropy among all states of the system having the same mass and total energy. Indeed let $\tilde{\varrho} = \tilde{\varrho}(x) > 0$, $\tilde{\vartheta} = \overline{\vartheta} > 0$ be a solution of problem (3.5), and let $\varrho = \varrho(x) \geq 0$, $\vartheta = \vartheta(x) > 0$ be a couple of functions such that

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = \int_\Omega \varrho \, \mathrm{d}x, \ \int_\Omega \Big(\varrho e(\varrho, \vartheta) - \varrho F \Big) \, \mathrm{d}x = \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \Big) \, \mathrm{d}x. \tag{3.15}$$

It follows from (3.14), (3.15) that

$$\overline{\vartheta} \int_\Omega \Big(\tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) - \varrho s(\varrho, \vartheta) \Big) \, \mathrm{d}x = \int_\Omega \Big(H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \Big) \, \mathrm{d}x + \int_\Omega (\tilde{\varrho} - \varrho) F \, \mathrm{d}x$$

$$= \int_\Omega \Big(H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta}) \Big) \, \mathrm{d}x.$$

Thus, in view of the coercivity properties of Helmholtz function $H_{\overline{\vartheta}}$, we conclude that

- the static solution $\tilde{\varrho}$, $\overline{\vartheta}$ maximizes the total entropy functional

$$(\varrho, \vartheta) \mapsto \int_\Omega \varrho s(\varrho, \vartheta)\ \mathrm{d}x$$

among all admissible states of the system (having the same mass and total energy);

- if

$$\int_\Omega \varrho s(\varrho, \vartheta)\ \mathrm{d}x = \int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta})\ \mathrm{d}x$$

then, necessarily, $\varrho \equiv \tilde{\varrho}$, $\vartheta \equiv \overline{\vartheta}$, in particular, there is at most one static solution with prescribed mass and energy.

Summarizing the previous discussion we obtain:

STATIC SOLUTIONS:

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, and s are continuously differentiable in $(0, \infty)^2$, and that they satisfy Gibbs' equation (2.1), hypothesis of thermodynamic stability (2.2), (2.3), together with condition (3.7). Let $F \in W^{1,\infty}(\Omega)$.*
Then for given constants $M_0 > 0$, E_0, there is at most one solution $\tilde{\varrho}$, $\tilde{\vartheta}$ of static problem (3.5) in the class of locally Lipschitz functions subjected to the constraints

$$\int_\Omega \tilde{\varrho}\ \mathrm{d}x = M_0,\ \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \Big)\ \mathrm{d}x = E_0. \tag{3.16}$$

In addition, $\tilde{\varrho}$ is strictly positive in Ω, and, moreover,

$$\int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta})\ \mathrm{d}x \geq \int_\Omega \varrho s(\varrho, \vartheta)\ \mathrm{d}x$$

for any couple $\varrho \geq 0$, $\vartheta > 0$ of measurable functions satisfying (3.16).

3.3 Conservative systems, attractors

The large time behavior of solutions to the energetically isolated Navier-Stokes-Fourier system is completely determined by *Second law of thermodynamics*. We shall see that all *global* trajectories approach an equilibrium state uniquely determined by the total mass and energy that are constants of motion. Moreover, the set of equilibria is an *attractor* for all trajectories emanating from the states of uniformly bounded mass and energy. This means that all these trajectories tend to the set of equilibria *uniformly* with growing time. This is clearly equivalent to *asymptotic compactness* of bounded trajectories in the associated energy space. Such a property is, however, far from being obvious as the density oscillations governed by the equation of continuity may and indeed do propagate in time. The hypothetical presence of density oscillations represents the main stumbling block of the weak stability and is closely related to the problem of global existence. Fortunately, the density oscillations are damped uniformly with growing time, as their amplitude is intimately related to the changes in the pressure.

As stated in (2.51), (2.52), the total mass

$$M_0 = \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x$$

as well as the total energy

$$E_0 = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F \right)(t,\cdot)\ \mathrm{d}x$$

are constants of motion. Moreover, in accordance with (2.55), (2.56) we may assume that

$$\int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)\ \mathrm{d}x \geq S_0$$

where S_0 represents the “initial” entropy of the system. Revoking hypotheses of Theorem 2.1, we have

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}$$

and assume that

$$S_0 > M_0 s_\infty,\ s_\infty = \lim_{Z\to\infty} S(Z) \geq -\infty. \tag{3.17}$$

As already pointed out, our aim is to show that the set of equilibria is an attractor for all trajectories emanating from a set of bounded total mass and energy. This means that the distance of all trajectories tends to zero *uniformly* with growing time. As we will see below, this is practically the only situation when the energetically insulated Navier-Stokes-Fourier system possesses an attractor. In a way, such a conclusion can be viewed as the most pessimistic scenario dictated by *Second law of thermodynamics* (the interested reader may consult the book by Prigogine [27] for more general aspects of the problem).

The following result was proved in [13, Chapter 5, Theorem 5.1].

GLOBAL ATTRACTOR:

Theorem 3.2 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Assume that the hypotheses of Theorem 2.1 are satisfied. Let $M_0 > 0$, E_0, S_0 be given, with S_0 satisfying (3.17).*
Then for any $\varepsilon > 0$, there exists a time $T = T(\varepsilon)$ such that

$$\left\{ \begin{array}{c} \|(\varrho\mathbf{u})(t,\cdot)\|_{L^{5/4}(\Omega;R^3)} \leq \varepsilon, \\ \\ \|\varrho(t,\cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} \leq \varepsilon, \\ \\ \|\vartheta(t,\cdot) - \overline{\vartheta}\|_{L^4(\Omega)} \leq \varepsilon \end{array} \right\} \text{for a.a. } t > T(\varepsilon)$$

for any weak solution $\{\varrho, \mathbf{u}, \vartheta\}$ of the Navier-Stokes-Fourier system defined on $(0,\infty) \times \Omega$ and satisfying

$$\left\{ \begin{array}{c} \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M_0, \\ \\ \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)(t,\cdot)\ \mathrm{d}x < E_0, \\ \\ \mathrm{ess}\liminf_{t\to 0} \int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)(t,0)\ \mathrm{d}x > S_0, \end{array} \right\} \tag{3.18}$$

where $\tilde{\varrho}$, $\overline{\vartheta}$ is a solution of the static problem (3.5) determined uniquely by the condition

$$\int_\Omega \tilde{\varrho}\ \mathrm{d}x = \int_\Omega \varrho\ \mathrm{d}x,$$

$$\int_\Omega \left(\tilde{\varrho}e(\tilde{\varrho},\overline{\vartheta}) - \tilde{\varrho}F\right)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\ \mathrm{d}x$$

3.4 Systems driven by a non-conservative force

In the light of Theorem 3.2, it is natural to ask what happens if the fluid system is driven by a *non-conservative* driving force and/or if sources of heat are present. Strictly speaking, such a situation is not covered by the existence result stated in Theorem 2.1, however, it can be shown (cf. [12, Chapter 3.3, Theorem 3.1]) that the conclusion of Theorem 2.1 remains valid if, for instance,

$\nabla_x F$ in equation (2.54) is replaced by a general vector function $\mathbf{f} = \mathbf{f}(t,x)$,

$$\mathbf{f} \in L^\infty((0,T)\times\Omega; R^3).$$

Accordingly, the total energy balance (2.52) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x. \tag{3.19}$$

We discus first the simpler case $\mathbf{f} = \mathbf{f}(x)$ independent of t. We report the following result ([13, Chapter 5.2, Theorem 5.2]).

Theorem 3.3 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Under the hypotheses of Theorem 2.1, let $\{\varrho,\vartheta,\mathbf{u}\}$ be a weak solution of the Navier-Stokes-Fourier system driven by an external force $\mathbf{f} = \mathbf{f}(x)$ on the time interval $[T_0,\infty)$, where $\mathbf{f} \not\equiv \nabla_x F$.*
Then

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \to \infty \ \textit{as}\ t\to\infty. \tag{3.20}$$

Thus the total energy of the fluid system becomes ultimately unbounded if subjected to a non-conservative driving force. This a bit surprising result remains valid even for "genuinely" time dependent driving forces. As a matter of fact, the total energy remains bounded only if the driving force approaches asymptotically a gradient. A more precise statement is provided by the following theorem (see [13, Chapter 5.2, Theorem 5.2]).

Theorem 3.4 *In addition to the hypotheses of Theorem 3.3, assume that $\mathbf{f} = \mathbf{f}(t,x)$, $\mathbf{f} \in L^\infty((0,T)\times\Omega;R^3)$.*
The either

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \to \infty \ \textit{as}\ t\to\infty$$

or

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\ \mathrm{d}x \le E_\infty \ \textit{for a.a.}\ t > T_0$$

for a certain constant E_∞. Moreover, in the latter case, each sequence $\tau_n \to \infty$ contains a subsequence (not relabeled) such that

$$\mathbf{f}(\tau_n+\cdot,\cdot) \to \nabla_x F \ \textit{weakly-(*) in}\ L^\infty((0,1)\times\Omega;R^3)$$

for a certain $F = F(x)$, $F \in W^{1,\infty}(\Omega)$ that, in general, may depend on the choice of $\{\tau_n\}_{n=1}^\infty$.

Theorem 3.4 has several interesting corollaries. In particular, the total energy of an energetically insulated fluid system driven by a *bounded* external force cannot oscillate between a finite value and infinity: The situation

$$\limsup_{t\to\infty} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(t,\cdot)\ \mathrm{d}x = \infty,$$

and

$$\liminf_{t\to\infty} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(t,\cdot)\ \mathrm{d}x < \infty$$

is prohibited by Theorem 3.4.

Second observation is that a system driven by a *time-periodic* driving force $\mathbf{f}$ possesses a periodic solution only if $\mathbf{f} = \nabla_x F(x)$, in which case the corresponding solution is a static one. The same conclusion holds also in the framework of quasi(almost)-periodic driving forces.

3.4.1 Highly oscillating driving force

In light of the arguments presented in the previous section, it may seem that *any* time-dependent driving force imposed on the energetically insulated Navier-Stokes-Fourier system produces a "grow-up" of the total energy for large values of time. Such a conclusion, however, is obviously false, and simple example of driving forces that stabilize quickly to a conservative form (or simply vanish) can be easily constructed. A somewhat less trivial example is provided by rapidly oscillating driving forces. Here, "rapidly oscillating" refers to the time variable, however, analogous examples when the force oscillates in the spatial variable can also be constructed. The main conclusion asserts that some *rapidly oscillating* external forces may, rather surprisingly, stabilize the system. The following result was proved in [13, Chapter 5.3, Theorem 5.3].

Theorem 3.5 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. In addition to the hypotheses of Theorem 2.1, assume that the driving force takes the form*

$$\mathbf{f}(t,x) = \omega(t^\beta)\mathbf{w}(x),\ t>0,\ x\in\Omega,$$

where $\mathbf{w} \in W^{1,\infty}(\Omega)$, $\mathbf{w} \neq 0$, and

$$\omega \in L^\infty(R),\ \omega \neq 0,\ \sup_{\tau>0} \left| \int_0^\tau \omega(t)\ \mathrm{d}t \right| < \infty,$$

are given functions.

Then for all $\beta > 2$ any global-in-time weak solution of the Navier-Stokes-Fourier system satisfies

$$\varrho\mathbf{u}(t,\cdot) \to 0 \ \textit{in}\ L^{5/4}(\Omega;R^3)\ \textit{as}\ t\to\infty,$$

$$\vartheta(t,\cdot) \to \overline{\vartheta}\ \textit{in}\ L^4(\Omega)\ \textit{as}\ t\to\infty,$$

and

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty,$$

where ϱ_s, ϑ_s *are positive constants,*

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho \, \mathrm{d}x.$$

4 Scale analysis

The extreme generality of the complete *Navier-Stokes-Fourier system* whereby the equations describe the entire spectrum of possible fluid motions - including sound waves, cyclone waves in the atmosphere, models of gaseous stars in astrophysics - constitutes a serious defect of these equations from the point of view of applications. Eliminating unwanted or unimportant modes of motion, and building in the essential balances between flow fields, allow the investigator to better focus on a particular class of phenomena and to potentially achieve a deeper understanding of the problem. Scaling and asymptotic analysis play an important role in this approach. By scaling the equations, meaning by choosing appropriately the system of the reference units, the parameters determining the behavior of the system become explicit. Asymptotic analysis provides a useful tool in the situations when certain of these parameters called *characteristic numbers* vanish or become infinite.

For physical systems related to our model problem we identify four fundamental dimensions: Time, Length, Mass, and Temperature. Each physical quantity that appears in the Navier-Stokes-Fourier equations can be measured in units expressed as a product of the fundamental ones.

As a matter of fact, the *Navier-Stokes-Fourier system* in the standard form introduced in Section 2.7 does not reveal anything more than the balance laws of certain quantities characterizing the instantaneous state of a fluid. In order to get a somewhat deeper insight into the structure of possible solutions, we identify the *characteristic values* of relevant physical quantities: the *reference time* $T_{\rm ref}$, the *reference length* $L_{\rm ref}$, the *reference density* $\varrho_{\rm ref}$, the *reference temperature* $\vartheta_{\rm ref}$, together with the *reference velocity* $U_{\rm ref}$, and the characteristic values of other composed quantities $p_{\rm ref}$, $e_{\rm ref}$, $\mu_{\rm ref}$, $\eta_{\rm ref}$, $\kappa_{\rm ref}$, and the source term $\nabla_x F_{\rm ref}$. Introducing new independent and dependent variables $X' = X/X_{\rm ref}$ and omitting the primes in the resulting equations, we arrive at the following scaled system:

Scaled Navier-Stokes-Fourier system:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p = \frac{1}{\mathrm{Re}}\mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2}\varrho \nabla_x F, \tag{4.2}$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}}\mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \sigma, \tag{4.3}$$

together with the associated total energy balance

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho F\Big)\mathrm{d}x = 0, \tag{4.4}$$

with

$$\sigma \geq \frac{1}{\vartheta}\Big(\frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}}\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{4.5}$$

and the associated boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{4.6}$$

(cf. Klein et al. [16]).

Note that relation (4.5) requires satisfaction of a natural compatibility condition

$$p_{\mathrm{ref}} = \varrho_{\mathrm{ref}} e_{\mathrm{ref}}. \tag{4.7}$$

As a result, we obtain a sample of dimensionless *characteristic numbers* listed below.

△ Symbol	△ Definition	△ Name
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}}U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}}U_{\mathrm{ref}}L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}}f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}}L_{\mathrm{ref}}U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}}\kappa_{\mathrm{ref}})$	Péclet number

The set of the chosen characteristic numbers is not unique, however, the maximal number of *independent* ones can be determined by means of Buckingham's $\Pi-$theorem (see Curtis et al. [6]).

4.1 From compressible to incompressible fluids

In many real world applications, such as atmosphere-ocean flows, fluid flows in engineering devices and astrophysics, velocities are small compared with the speed of sound proportional to $1/\sqrt{\mathrm{Ma}}$ in the *scaled Navier-Stokes-Fourier system*. This observation has a significant impact on both exact solutions to the governing equations and their numerical approximations. Physically, in the limit of vanishing flow velocity or infinitely fast speed of sound propagation, the elastic features of the fluid become negligible and sound-wave propagation insignificant. The low Mach number regime is particularly interesting when accompanied simultaneously with smallness of other dimensionless parameters such as *Froude*, *Reynolds*, and/or *Péclet numbers*. When the Mach number Ma approaches zero, the pressure is almost constant while the speed of sound tends to infinity. If, simultaneously, the temperature tends to a constant, the fluid is driven to *incompressibility*. If, in addition, *Froude number* is small, specifically if $\mathrm{Fr} \approx \sqrt{\mathrm{Ma}}$, a formal asymptotic expansion produces a well-known model - the *Oberbeck-Boussinesq approximation* - probably the most widely used simplification in numerous problems in fluid dynamics (cf. Zeytounian [35], [34]). An important consequence of the heating process is the appearance of a driving force in the target system, the size of which is proportional to the temperature.

To be more specific, we take $\mathrm{Ma} = \varepsilon$, $\mathrm{Fr} = \sqrt{\varepsilon}$ and keep all other characteristic numbers of order unity, obtaining, at least formally,

$$
\begin{aligned}
\varrho &= \overline{\varrho} + \varepsilon \varrho^{(1)} + \varepsilon^2 \varrho^{(2)} + \dots, \\
\mathbf{u} &= \mathbf{U} + \varepsilon \mathbf{u}^{(1)} + \varepsilon^2 \mathbf{u}^{(2)} + \dots, \\
\vartheta &= \overline{\vartheta} + \varepsilon \vartheta^{(1)} + \varepsilon^2 \vartheta^{(2)} + \dots
\end{aligned}
\tag{4.8}
$$

Regrouping the scaled system with respect to powers of ε, we get, again formally comparing terms of the same order,

$$
\nabla_x p(\overline{\varrho}, \overline{\vartheta}) = 0. \tag{4.9}
$$

Of course, relation (4.9) *does not* automatically imply that both $\overline{\varrho}$ and $\overline{\vartheta}$ must be constant; however, since we are primarily interested in solutions defined on large time intervals, the necessary uniform estimates on the velocity field have to be obtained from the dissipation equation (3.2) introduced in Section 3.1. In particular, the entropy production rate $\sigma = \sigma_\varepsilon$ is to be kept small of order $\varepsilon^2 \approx \mathrm{Ma}^2$. Consequently, as seen from (4.5), the quantity $\mathbf{q} \cdot \nabla_x \vartheta / \vartheta^2$ vanishes in the asymptotic limit $\varepsilon \to 0$. As $\mathbf{q}$ is given through Fourier's law (2.38), it is therefore natural to assume that $\overline{\vartheta}$ is a positive constant; whence, in agreement with (4.9),

$$
\overline{\varrho} = \text{const in } \Omega
$$

as soon as the pressure is a strictly monotone function of ϱ. The fact that the density ϱ and the temperature ϑ will be always considered in a vicinity of a *thermodynamic equilibrium* $(\overline{\varrho}, \overline{\vartheta})$ (cf. Section 3.1) is an inevitable hypothesis in our approach to singular limits based on the concept of *weak solution* and energy estimates "in-the-large".

Finally, neglecting all terms of order ε and higher in (4.1 - 4.4), we arrive at the following system of equations:

OBERBECK-BOUSSINESQ APPROXIMATION:

$$\mathrm{div}_x \mathbf{U} = 0, \tag{4.10}$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U}\otimes\mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x \left(\mu(\overline{\vartheta})(\nabla_x \mathbf{U} + \nabla_x^T \mathbf{U})\right) + r\nabla_x F, \tag{4.11}$$

$$\overline{\varrho} c_p(\overline{\varrho},\overline{\vartheta})\Big(\partial_t \Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big) - \mathrm{div}_x(G\mathbf{U}) - \mathrm{div}_x(\kappa(\overline{\vartheta})\nabla_x\Theta) = 0, \tag{4.12}$$

where

$$G = \overline{\varrho}\ \overline{\vartheta}\alpha(\overline{\varrho},\overline{\vartheta})F, \tag{4.13}$$

and

$$r + \overline{\varrho}\alpha(\overline{\varrho},\overline{\vartheta})\Theta = 0. \tag{4.14}$$

The quantity r can be identified with $\varrho^{(1)}$ modulo a multiple of F, while $\Theta = \vartheta^{(1)}$. The *specific heat at constant pressure* c_p is evaluated by means of the standard thermodynamic relation

$$c_p(\varrho,\vartheta) = \frac{\partial e}{\partial\vartheta}(\varrho,\vartheta) + \alpha(\varrho,\vartheta)\frac{\vartheta}{\varrho}\frac{\partial p}{\partial\vartheta}(\varrho,\vartheta), \tag{4.15}$$

where the *coefficient of thermal expansion* α reads

$$\alpha(\varrho,\vartheta) = \frac{1}{\varrho}\frac{\partial_\vartheta p}{\partial_\varrho p}(\varrho,\vartheta). \tag{4.16}$$

A fundamental issue is a proper choice of the initial data for the limit system. Note that, in order to obtain a non-trivial dynamics, it is necessary to consider general $\varrho^{(1)}$, $\vartheta^{(1)}$, in particular, the initial values $\varrho^{(1)}(0,\cdot)$, $\vartheta^{(1)}(0,\cdot)$ must be allowed to be large. According to the standard terminology, such a stipulation corresponds to the so-called *ill-prepared initial data* in contrast with the *well-prepared data* for which

$$\frac{\varrho(0,\cdot)-\overline{\varrho}}{\varepsilon} \approx \varrho_0^{(1)},\ \frac{\vartheta(0,\cdot)-\overline{\vartheta}}{\varepsilon} \approx \vartheta_0^{(1)} \text{ provided } \varepsilon\to 0,$$

where $\varrho_0^{(1)}$, $\vartheta_1^{(1)}$ are related to F through

$$\frac{\partial p}{\partial\varrho}(\overline{\varrho},\overline{\vartheta})\varrho_0^{(1)} + \frac{\partial p}{\partial\vartheta}(\overline{\varrho},\overline{\vartheta})\vartheta_0^{(1)} = \overline{\varrho}F.$$

4.1.1 Low Mach number limit

Motivated by the previous discussion, we consider a scaled *Navier-Stokes-Fourier* system in the form:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{4.17}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{4.18}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_{\mathbf{x}} \vartheta)}{\vartheta} \right) = \sigma_\varepsilon, \tag{4.19}$$

supplemented with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \ \mathrm{d}x = 0, \tag{4.20}$$

where the viscous stress tensor $\mathbb{S}$ is given by Newton's rheological law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathbb{I} \mathrm{div}_x \mathbf{u} \right) + \eta(\vartheta) \mathbb{I} \ \mathrm{div}_x \mathbf{u}, \tag{4.21}$$

the heat flux $\mathbf{q}(\vartheta, \nabla_x \vartheta)$ obeys Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \tag{4.22}$$

and the entropy production rate σ_ε satisfies

$$\sigma_\varepsilon \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right) \geq 0. \tag{4.23}$$

The system is supplemented with conservative boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial \Omega_\varepsilon} = 0, \tag{4.24}$$

$$\mathbf{q} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0. \tag{4.25}$$

Finally, the initial state of the fluid system is determined by the following conditions:

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^1_{0,\varepsilon}, \ \vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^1_{0,\varepsilon}, \tag{4.26}$$

where

$$\overline{\varrho}, \ \overline{\vartheta} > 0, \ \int_{\Omega_\varepsilon} \varrho^1_{0,\varepsilon} \ \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^1_{0,\varepsilon} \ \mathrm{d}x = 0 \text{ for all } \varepsilon > 0, \tag{4.27}$$

and

$$\{\varrho^1_{0,\varepsilon}\}_{\varepsilon > 0}, \ \{\vartheta^1_{0,\varepsilon}\}_{\varepsilon > 0} \text{ are bounded in } L^2 \cap L^\infty(\Omega_\varepsilon). \tag{4.28}$$

In addition, we suppose

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \tag{4.29}$$

where

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon > 0} \text{ is bounded in } L^2 \cap L^\infty(\Omega_\varepsilon; R^3). \tag{4.30}$$

The family of *bounded* domains Ω_ε is chosen to "mimick" the behavior of the fluid in a fictitious large (unbounded) domain Ω. Pursuing the philosophy that any real physical space is always bounded but possibly "large" with respect to the speed of sound in the medium, we consider a family of *bounded* domains $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ such that $\Omega_\varepsilon \approx \Omega$ in a certain sense as $\varepsilon \to 0$. More specifically, we suppose that

$$\Omega \subset R^3 \text{ is an unbounded domain with a compact smooth boundary } \partial\Omega, \tag{4.31}$$

and set

$$\Omega_\varepsilon = B_{r(\varepsilon)} \cap \Omega, \tag{4.32}$$

where $B_{r(\varepsilon)}$ is a ball centered at zero with a radius $r(\varepsilon)$, with $r(\varepsilon) \to \infty$.

Our goal will be:

• to establish uniform bounds on the family of solution $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (4.17 - 4.26) independent of the parameter $\varepsilon \to 0$;

• to show strong (pointwise a.a.) convergence

$$\left\{ \begin{array}{c} \varrho_\varepsilon \to \overline{\varrho} \\ \\ \vartheta_\varepsilon \to \overline{\vartheta} \end{array} \right\} \text{a.a. in } (0,T) \times \Omega, \tag{4.33}$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ a.a. in } (0,T) \times \Omega \tag{4.34}$$

at least for suitable subsequences. In other words, the convergence imposed on the initial data through (4.26 - 4.30) "propagates" in time. As we shall see, this is not surprising for ϱ_ε, ϑ_ε, but far less obvious for the velocity $\mathbf{u}_\varepsilon$.

With (4.33), (4.34) at hand, it is relatively easy to identify the limit system (4.10 - 4.14). The details can be found in [12, Chapter 5].

4.1.2 Stability of static equilibria in the low Mach number limit

As already observed in Section 3.2, any weak solution $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}$ of the Navier-Stokes-Fourier system (4.17 - 4.20) satisfies the *total dissipation balance*

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big[H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big]\right)(\tau,\cdot)\ \mathrm{d}x \tag{4.35}$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon\Big[[0,\tau]\times\overline{\Omega}_\varepsilon\Big] =$$

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}\Big[H_{\overline{\vartheta}}(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho_{0,\varepsilon} - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big]\right)\ \mathrm{d}x$$

for a.a. $\tau \in [0,T]$, with the Helmholtz function $H_{\overline{\vartheta}}$ introduced in (3.10).

Relation (4.35), together with the structural properties of the function $H_{\overline{\vartheta}}$ established in (3.13), can be used to deduce uniform bounds independent of ε.

To this end, it is convenient to introduce the *essential* and *residual* parts of a function h as

$$h = [h]_{\rm ess} + [h]_{\rm res},\ [h]_{\rm ess} = \Psi(\varrho_\varepsilon, \vartheta_\varepsilon)h,\ [h]_{\rm res} = \Big(1 - \Psi(\varrho_\varepsilon, \vartheta_\varepsilon)\Big)h,$$

where

$$\Psi \in C_c^\infty(0,\infty)^2,\ 0 \leq \Psi \leq 1,$$

$$\Psi \equiv 1 \text{ in an open neighborhood of the point } [\overline{\varrho}, \overline{\vartheta}].$$

In addition, for the sake of simplicity, we assume that the viscosity coefficient μ obeys *Chapman's law*

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta) \tag{4.36}$$

under the given scaling (cf. (2.64)). The remaining hypotheses (2.65 - 2.67) remain unchanged.

The total dissipation balance (4.35), together with the hypotheses (4.26 - 4.30) imposed on the initial data, givee rise to the following estimates:

$$\text{ess} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{4.37}$$

$$\text{ess} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\rm res} \right\|_{L^{5/4}(\Omega_\varepsilon)} \leq c, \tag{4.38}$$

$$\text{ess} \sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{4.39}$$

$$\text{ess} \sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\rm ess} \right\|_{L^4(\Omega_\varepsilon)} \leq c, \tag{4.40}$$

$$\text{ess} \sup_{t\in(0,T)} \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega_\varepsilon;R^3)} \leq c, \tag{4.41}$$

and

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c \tag{4.42}$$

where the generic constant c is independent of ε.

In addition, as a direct consequence of (4.23), the previously established bounds, the structural properties of the transport coefficients, and Korn's and Poincare's inequalities, we obtain

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}\Omega_\varepsilon;R^3)} \ {\rm d}t \leq c, \tag{4.43}$$

and

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \ {\rm d}t \leq c \tag{4.44}$$

(see [12, Chapter 5.2] for details).

The uniform bounds (4.37 - 4.40) reflect *stability* of the static state $\overline{\varrho}, \overline{\vartheta}$ in the low Mach number regime. No matter how large the initial velocity distribution is, the fluid density ϱ and the temperature ϑ remain close to the static equilibrium provided this was the case at the initial instant $t = 0$. In particular, we deduce immediately the pointwise convergence claimed in (4.33). A similar result for the velocity field $\mathbf{u}_\varepsilon$ is less trivial and, as a matter of fact, not always true depending on the geometry of the spatial domain Ω. This issue will be discussed in detail in the remaining part of this text.

4.2 Acoustic waves

As already mentioned many times, the strong convergence of the velocity (4.34) is is intimately related to propagation and attenuation of acoustic waves. As a matter of fact, (4.34) is not expected to hold on *bounded* domains with acoustically hard boundary, where large amplitude rapidly oscillating waves are generated in the limit $\varepsilon \to 0$ (see, for instance, Lions and Masmoudi [22], or Schochet [30]). Accordingly, for (4.34) to hold it is necessary that the target domain Ω be *unbounded* (cf. hypotheses (4.31), (4.32)), more specifically, the two closely related properties must be satisfied:

- the point spectrum of the associated wave operator must be empty;
- the *local* acoustic energy decays in time (cf. Morawetz [24], Walker [33]).

4.2.1 Lighthill's acoustic equation

The forthcoming analysis primarily rests on the approach proposed by Lighthill [21], where the original Navier-Stokes-Fourier system is rewritten in the form of a wave equation with a source term usually called *Lighthill's tensor*.

We begin by introducing a "time lifting" Σ_ε of the measure σ_ε through formula

$$< \Sigma_\varepsilon; \varphi >=< \sigma_\varepsilon; I[\varphi] >,$$

where we have set

$$< \Sigma_\varepsilon; \varphi >=< \sigma_\varepsilon; I[\varphi] >, \ I[\varphi](t,x) = \int_0^t \varphi(z,x) \, \mathrm{d}z \text{ for any } \varphi \in L^1(0,T;C(\overline{\Omega}_\varepsilon)). \tag{4.45}$$

It is easy to check that Σ_ε can be identified with an abstract function $\Sigma_\varepsilon \in L^\infty_{\rm weak}(0,T;\mathcal{M}^+(\overline{\Omega}_\varepsilon))$, where where

$$< \Sigma_\varepsilon(\tau), \varphi >= \lim_{\delta \to 0+} < \sigma_\varepsilon, \psi_\delta \varphi >,$$

with

$$\psi_\delta(t) = \begin{cases} 0 \text{ for } t \in [0,\tau), \\ \\ \frac{1}{\delta}(t-\tau), \ \text{ for } t \in (\tau, \tau+\delta), \\ \\ 1 \text{ for } t \geq \tau + \delta, \end{cases}$$

in particular, the measure Σ_ε is well-defined for *any* $\tau \in [0,T)$, and the mapping $\tau \mapsto \Sigma_\varepsilon$ is non-increasing in the sense of measures. Here the subscript in $L^\infty_{\rm weak}$ means "weakly measurable".

Lighthill's idea [21] is to rewrite the Navier-Stokes-Fourier system (4.17 - 4.19) in the form:

$$\varepsilon\partial_t Z_\varepsilon + {\rm div}_x \mathbf{V}_\varepsilon = \varepsilon {\rm div}_x \mathbf{F}^1_\varepsilon, \tag{4.46}$$

$$\varepsilon\partial_t \mathbf{V}_\varepsilon + \omega\nabla_x Z_\varepsilon = \varepsilon\Big({\rm div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2\omega}\nabla_x\Sigma_\varepsilon\Big), \tag{4.47}$$

supplemented with the homogeneous Neumann boundary conditions

$$\mathbf{V}_\varepsilon\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{4.48}$$

where

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega}\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right) + \frac{A}{\varepsilon\omega}\Sigma_\varepsilon,\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon, \tag{4.49}$$

$$\mathbf{F}^1_\varepsilon = \frac{A}{\omega}\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\right)\mathbf{u}_\varepsilon + \frac{A}{\omega}\frac{\kappa\nabla_x\vartheta_\varepsilon}{\varepsilon\vartheta_\varepsilon}, \tag{4.50}$$

$$\mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon, \tag{4.51}$$

and

$$F^3_\varepsilon = \omega\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\right) + A\varrho_\varepsilon\left(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right). \tag{4.52}$$

Here the constants A and ω has to be chosen to eliminate the first order term in the (formal) asymptotic expansion of the forcing term (4.52) expressed in terms of the quantities $(\varrho_\varepsilon - \overline{\varrho})/\varepsilon$, $(\vartheta_\varepsilon - \overline{\vartheta})/\varepsilon$, more specifically,

$$A\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ \omega + A\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}. \tag{4.53}$$

Note that the *wave speed* ω is strictly positive as a direct consequence of *hypothesis of thermodynamic stability* introduced in (2.2), (2.3).

System (4.46), (4.47) can be viewed as a variant of *Lighthill's acoustic analogy* supplemented with the so-called *acoustically hard boundary condition* (4.48) (cf. Lighthill [20]). We assume that equations (4.46), (4.47) as well as the boundary condition (4.48) are satisfied in a weak sense, more precisely, the integral identity

Lighthill's acoustic equation:

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon Z_\varepsilon \partial_t \varphi + \mathbf{V}_\varepsilon \cdot \nabla_x \varphi \Big] \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} \varepsilon \mathbf{F}^1_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{4.54}$$

holds for any test function $\varphi \in C^\infty_c((0,T) \times \overline{\Omega}_\varepsilon)$, and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \varphi + \omega Z_\varepsilon \mathrm{div}_x \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t \tag{4.55}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \Big(\varepsilon \mathbb{F}^2_\varepsilon : \nabla_x \varphi + \varepsilon F^3_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t + \frac{A}{\varepsilon\omega} < \Sigma_\varepsilon ; \mathrm{div}_x \varphi >$$

is satisfied for any

$$\varphi \in C^\infty_c((0,T) \times \overline{\Omega}_\varepsilon; R^3), \ \varphi \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

4.2.2 Regularization and finite speed of propagation

Our ultimate goal is to show the strong (pointiwise a.a.) convergence of the velocities $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ claimed in (4.34). To this end, it is convenient to consider the acoustic equation (4.54), (4.55) directly on the *unbounded* domain Ω rather than Ω_ε. In addition to (4.31), (4.32), we suppose that $\Omega_\varepsilon = \Omega \cap B_{r(\varepsilon)}$, where

$$\lim_{\varepsilon \to 0} \varepsilon r(\varepsilon) = \infty. \tag{4.56}$$

As a matter of fact, the balls $B_{r(\varepsilon)}$ in the definition of Ω_ε may be replaced by general bounded domains $\tilde{B}_\varepsilon$, namely

$$\Omega_\varepsilon = \Omega \cap \tilde{B}_\varepsilon, \ \text{with} \ B_{r(\varepsilon)} \subset \tilde{B}_\varepsilon.$$

Hypothesis (4.56) means the distance to $\partial B_{r(\varepsilon)}$ dominates the speed of sound proportional to $1/\varepsilon$. In particular, the acoustic waves cannot reach the outer boundary $\partial B_{r(\varepsilon)}$ and return to a fixed compact set $K \subset \Omega$ within the time interval $(0,T)$. Since the pointwise convergence of the velocities is a *local* property, we may therefore replace Ω_ε by Ω. In fact, the convergence result stated in (4.34) is not optimal with respect to the *space* variable, where the velocity fields enjoy higher regularity, however, the main issue here is to eliminate fast oscillations of acoustic waves in *time*.

Next, we claim that for (4.34) to hold it is enough to show

$$\Big[t \mapsto \int_\Omega \mathbf{u}_\varepsilon(t,\cdot) \cdot \mathbf{w} \ \mathrm{d}x \Big] \to \Big[t \mapsto \int_\Omega \mathbf{U}(t,\cdot) \cdot \mathbf{w} \ \mathrm{d}x \Big] \ \text{in} \ L^1(0,T) \tag{4.57}$$

for any fixed $\mathbf{w} \in C_c^\infty(K; R^3)$, $K \subset \Omega$ a given ball. Indeed, by virtue of (4.43), we may infer that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),$$

extending $\mathbf{u}_\varepsilon$ outside Ω_ε. As $W^{1,2}(\Omega; R^3)$ is compactly imbedded into $L^2(K)$ for any bounded K, it is easy to see that (4.57) yields (4.34).

Finally, since

$$[\mathbf{u}_\varepsilon]_{\rm res} \to 0 \text{ in, say, } L^1((0,T) \times K),$$

it is enough to show (4.57) with $\mathbf{u}_\varepsilon$ replaced by $[\mathbf{u}_\varepsilon]_{\rm ess}$, which is equivalent to

$$\left[t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t, \cdot) \cdot \mathbf{w} \ {\rm d}x \right] \to \left[t \mapsto \int_\Omega \mathbf{V}(t, \cdot) \cdot \mathbf{w} \ {\rm d}x \right] \text{ in } L^1(0,T) \tag{4.58}$$

for any fixed $\mathbf{w} \in C_c^\infty(K; R^3)$, where $\mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$ appears in the acoustic equation (4.54), (4.55), and $\mathbf{V} = \overline{\varrho}\mathbf{U}$.

Since our task has been reduced to showing (4.58), we may assume, with help of a simple approximation, that all quantities in (4.54), (4.55) are smooth in Ω_ε. Moreover, system (4.54), (4.55) admits a finite speed of propagation of order $\sqrt{\omega}/\varepsilon$. This can be easily seen multiplying equation (4.54) by Z_ε, taking the scalar product of (4.55) with $\mathbf{V}_\varepsilon$, and integrating the resulting expression over the set

$$\left\{ (t,x) \ \Big| t \in [0, \tau], \ x \in \Omega_\varepsilon, \ |x| < r - \frac{\sqrt{\omega}}{\varepsilon} t \right\}.$$

Thus we can replace (4.54), (4.55) by a regularized problem (see [11] for details) :

Show that the family

$$\left[t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t, \cdot) \cdot \mathbf{w} \ {\rm d}x \right] \textit{ is precompact in } L^1(0,T) \tag{4.59}$$

for any $\mathbf{w} \in C_c^\infty(K; R^3)$, $K \subset \overline{K} \subset \Omega$ *a bounded ball, provided that*

$$\varepsilon \partial_t Z_\varepsilon + {\rm div}_x \mathbf{V}_\varepsilon = \varepsilon {\rm div}_x \mathbf{F}_\varepsilon^1 \textit{ in } (0,T) \times \Omega, \tag{4.60}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon {\rm div}_x \mathbb{F}_\varepsilon^2 \textit{ in } (0,T) \times \Omega, \tag{4.61}$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{4.62}$$

$$Z_\varepsilon(0, \cdot) = Z_{0,\varepsilon}, \ \mathbf{V}_\varepsilon(0, \cdot) = \mathbf{V}_{0,\varepsilon} \textit{ in } \Omega, \tag{4.63}$$

where

$$Z_{0,\varepsilon} = Z_{0,\varepsilon}^1 + Z_{0,\varepsilon}^2, \ Z_{0,\varepsilon}^i \in C_c^\infty(\Omega), \ i = 1, 2,$$

$$\mathbf{V}_{0,\varepsilon} = \mathbf{V}_{0,\varepsilon}^1 + \mathbf{V}_{0,\varepsilon}^2, \ \mathbf{V}_{0,\varepsilon}^i \in C_c^\infty(\Omega; R^3), \ i = 1, 2,$$

and

$$\mathbf{F}_\varepsilon^1 = \mathbf{F}_\varepsilon^{1,1} + \mathbf{F}_\varepsilon^{1,2}, \ \mathbf{F}_\varepsilon^{1,i} \in C_c^\infty((0,T) \times \Omega; R^3), \ i = 1, 2,$$

$$\mathbb{F}^2_\varepsilon = \mathbb{F}^{2,1}_\varepsilon + \mathbb{F}^{2,2}_\varepsilon,\ \mathbb{F}^{2,i}_\varepsilon \in C^\infty_c((0,T)\times\Omega; R^{3\times 3}),\ i = 1,2,$$

with

$$\{Z^1_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1(\Omega),\ \{Z^2_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega), \tag{4.64}$$

$$\{\mathbf{V}^1_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^1(\Omega;R^3),\ \{\mathbf{V}^2_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega;R^3), \tag{4.65}$$

$$\left\{ \begin{array}{l} \{\mathbf{F}^{1,1}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^1(\Omega;R^3)), \\ \\ \{\mathbf{F}^{1,2}_{\varepsilon,0}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega;R^3)), \end{array} \right\} \tag{4.66}$$

$$\left\{ \begin{array}{l} \{\mathbb{F}^{2,1}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^1(\Omega;R^{3\times 3})), \\ \\ \{\mathbb{F}^{2,2}_{\varepsilon,0}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega;R^{3\times 3})). \end{array} \right\} \tag{4.67}$$

4.2.3 Compactness of the solenoidal part

Consider $\psi \in W^{1,2}\cap W^{1,\infty}(\Omega;R^3)$, $\mathrm{div}_x\psi = 0$, $\psi\cdot\mathbf{n}|_{\partial\Omega} = 0$. Multiplying equation (4.61) on ψ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{V}_\varepsilon\cdot\psi\ \mathrm{d}x = -\int_\Omega \mathbb{F}^2_\varepsilon : \nabla_x\psi\ \mathrm{d}x,\ \int_\Omega \mathbf{V}_\varepsilon(0,\cdot)\cdot\psi\ \mathrm{d}x = \int_\Omega \mathbf{V}_{0,\varepsilon}\cdot\psi\ \mathrm{d}x,$$

in particular the family

$$\left[t\mapsto \int_\Omega \mathbf{V}_\varepsilon\cdot\psi\ \mathrm{d}x\right] \text{ is precompact in } C[0,T]. \tag{4.68}$$

Relation (4.68) may be viewed as (weak) precompactness of the *solenoidal* component of the vector field $\mathbf{V}_\varepsilon$.

4.2.4 Abstract variational formulation

Our aim is to rewrite system (4.60), (4.61) in terms of an abstract differential operator

$$\Delta_N,\ \Delta_N[v] = \Delta v,\ \nabla_x v\cdot\mathbf{n}|_{\partial\Omega} = 0,\ v(x)\to 0 \text{ as } |x|\to\infty,$$

with

$$\mathcal{D}(\Delta_N) = \{w\in L^2(\Omega)\ |\ w\in W^{2,2}(\Omega),\ \nabla_x w\cdot\mathbf{n}|_{\partial\Omega} = 0\}.$$

It can be shown that $-\Delta_N$ is a self-adjoint, non-negative operator in $L^2(\Omega)$, with an absolutely continuous spectrum $[0,\infty)$. Moreover, Δ_N satisfies the *limiting absorption principle*

$$\sup_{\lambda\in C, 0<\alpha\le \mathrm{Re}[\lambda]\le\beta<\infty,\ \mathrm{Im}[\lambda]\neq 0} \left\|\mathcal{V}\circ(-\Delta_N-\lambda)^{-1}\circ\mathcal{V}\right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \le c_{\alpha,\beta}, \tag{4.69}$$

where

$$\mathcal{V}(x) = (1 + |x|^2)^{-\frac{s}{2}},\ s > 1$$

(see Leis [19]).

We denote by $\{P_\lambda\}_{\lambda\in[0,\infty)}$ the spectral resolution associated to the operator $-\Delta_N$ in the Hilbert space $L^2(\Omega)$. Accordingly, we can define, at least formally, $G(-\Delta_N)[v]$ by duality as

$$< G(-\Delta_N)[v]; \varphi >=< v; G(-\Delta_N)[\varphi] >$$

$$= \int_0^\infty G(\lambda)\mathrm{d}\left(\int_\Omega P_\lambda[\varphi]v\ \mathrm{d}x\right) \text{ for } \varphi \in C_c^\infty(\overline{\Omega})$$

as long as the integral on the right-hand side converges.

Introducing the *acoustic potential*

$$\Phi_\varepsilon = \Delta_N^{-1}[\mathrm{div}_x \mathbf{V}_\varepsilon], \tag{4.70}$$

we can rewrite equation (4.60) in the form

$$\varepsilon\partial_t Z_\varepsilon + \Delta_N \Phi_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}_\varepsilon^1, \tag{4.71}$$

while (4.61) reads

$$\varepsilon\partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \Delta_N^{-1} \mathrm{div}_x \mathrm{div}_x \mathbb{F}_\varepsilon^2. \tag{4.72}$$

Next, we claim that the mapping

$$\chi \mapsto \int_\Omega (\mathbb{F}_\varepsilon^{2,1} + \mathbb{F}_\varepsilon^{2,2}) : \nabla_x^2 \Psi\ \mathrm{d}x,$$

where $\Psi = \Delta_N^{-1}\chi$, represents a bounded linear form for

$$\chi \in \mathcal{D}(-\Delta_N) \cap \mathcal{D}\left(\frac{1}{\sqrt{-\Delta_N}}\right), \tag{4.73}$$

the norm of which can be estimated in terms of $\|\mathbb{F}_\varepsilon^{1,1}\|_{L^1(\Omega;R^{3\times3})}$, $\|\mathbb{F}_\varepsilon^{1,2}\|_{L^2(\Omega;R^{3\times3})}$. To this end, it is enough to observe that the function Ψ has two derivatives bounded in $L^2 \cap L^\infty(\Omega)$. Indeed

$$\sqrt{-\Delta_N}\Psi \in L^2(\Omega) \text{ or, equivalently, } \nabla_x \Psi \in L^2(\Omega).$$

Consequently, we have $\Psi \in \mathcal{D}^{1,2}(\Omega)$ and the desired conclusion follows from the standard elliptic regularity estimates. Note that

$$\mathcal{D}(\Delta_N) \subset W^{2,2}(\Omega) \subset C^\nu(\overline{\Omega}) \text{ for a certain } \nu > 0.$$

Similarly, the mapping

$$\chi \mapsto \int_\Omega \mathbf{F}_\varepsilon^1 \cdot \nabla_x \Psi\ \mathrm{d}x,\ \Psi = \Delta_N^{-1/2}\chi$$

represents a bounded linear form on the function space specified in (4.73).

Consequently, the potential Φ_ε may be expressed by means of the standard *Duhamel's formula*:

$$\Phi_\varepsilon(t,\cdot) = \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} + \frac{\mathrm{i}}{2\sqrt{-\Delta_N}}[Z_{0,\varepsilon}]\right] \tag{4.74}$$
$$+\exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\frac{1}{2}\Phi_{0,\varepsilon} - \frac{\mathrm{i}}{2\sqrt{-\Delta_N}}[Z_{0,\varepsilon}]\right]$$
$$+\int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\frac{1}{2}\frac{1}{\Delta_N}\mathrm{div}_x\mathrm{div}_x\mathbb{F}^2_\varepsilon + \frac{\mathrm{i}}{2\sqrt{-\Delta_N}}[\mathrm{div}_x\mathbf{F}^1_\varepsilon]\right]\ \mathrm{d}s$$
$$+\int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\frac{1}{2}\frac{1}{\Delta_N}\mathrm{div}_x\mathrm{div}_x\mathbb{F}^2_\varepsilon - \frac{\mathrm{i}}{2\sqrt{-\Delta_N}}[\mathrm{div}_x\mathbf{F}^1_\varepsilon]\right]\ \mathrm{d}s,$$

or, in accordance with the previous considerations,

$$\Phi_\varepsilon(t,\cdot) \tag{4.75}$$
$$= \exp\left(\pm\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[h^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^2_\varepsilon] \pm \mathrm{i}\left(\Delta_N[h^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^4_\varepsilon]\right)\right]$$
$$+\int_0^t \exp\left(\pm\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[H^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^2_\varepsilon]\right.$$
$$\left.\pm\mathrm{i}\left(\Delta_N[H^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^4_\varepsilon]\right)\right]\ \mathrm{d}s,$$

with

$$\{h^i_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega), \tag{4.76}$$

$$\{H^i_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2((0,T)\times\Omega), \tag{4.77}$$

for $i = 1,\ldots,4$.

4.2.5 An abstract result of Kato

In order to show strong convergence of the gradient component of the velocity field, we revoke the space-time decay estimates for the group $\exp(\mathrm{i}t\sqrt{-\Delta_N})$ obtained by Kato [15] (see also Burq et al. [1]).

Theorem 4.1 [Reed and Simon [29, Theorem XIII.25 and Corollary]]

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H-\lambda\mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda\notin R,\ v\in\mathcal{D}(A^*),\ \|v\|_X=1} \|A\circ R_H[\lambda]\circ A^*[v]\|_X < \infty. \tag{4.78}$$

Then

$$\sup_{w\in X,\ \|w\|_X=1}\frac{\pi}{2}\int_{-\infty}^{\infty}\|A\exp(-\mathrm{i}tH)[w]\|^2_X\ \mathrm{d}t \le \Gamma^2.$$

We intend to apply Theorem 4.1 to

$$X = L^2(\Omega),\ H = \sqrt{-\Delta_N},\ A[v] = \varphi G(-\Delta_N)[v],\ v \in X,$$

where

$$G \in C_c^\infty(0,\infty),\ \varphi \in C_c^\infty(\Omega) \text{ are given functions.}$$

To begin, we have to verify hypothesis (4.78). Since

$$A \circ R_H[\lambda] \circ A^* = \varphi G(-\Delta_N) \frac{1}{\sqrt{-\Delta_N} - \lambda} G(-\Delta_N)\varphi,$$

it is enough to consider the values of the parameter λ belonging to a *bounded* set Q of the complex plane, namely

$$\lambda \in Q = \{z \in C \mid \mathrm{Re}[z] \in [a,b],\ 0 < |\mathrm{Im}[z]| < d\},$$

where

$$0 < a < b < \infty,\ \mathrm{supp}[G] \subset (a^2, b^2). \text{ and } d > 0.$$

Indeed if $\lambda \notin Q$, then

$$G(-\Delta_N) \frac{1}{\sqrt{-\Delta_N} - \lambda} G(-\Delta_N)$$

is a bounded linear operator in $L^2(\Omega)$, with a norm bounded in terms of the parameters a, b, d.

Next, we have

$$A \circ R_H[\lambda] \circ A^* = \varphi \frac{M(-\Delta_N, \lambda)}{(-\Delta_N) - \lambda^2} \varphi,$$

with

$$M(-\Delta_N, \lambda) = G(-\Delta_N)(\sqrt{-\Delta_N} + \lambda) G(-\Delta_N)$$

- a bounded linear operator in $L^2(\Omega)$ as soon as $\lambda \in Q$.

Finally, we may write

$$\left\| \varphi \circ \frac{M(-\Delta_N, \lambda)}{(-\Delta_N) - \lambda^2} \circ \varphi \right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \tag{4.79}$$

$$= \sup_{\|v\|_{L^2(\Omega)}, \|w\|_{L^2(\Omega)} \le 1} \int_\Omega \varphi \frac{M(-\Delta_N, \lambda)}{(-\Delta_N) - \lambda^2} [\varphi v] w \ \mathrm{d}x$$

$$= \sup_{\|v\|_{L^2(\Omega)}, \|w\|_{L^2(\Omega)} \le 1} \int_\Omega \frac{M(-\Delta_N, \lambda)}{\sqrt{(-\Delta_N) - \lambda^2}} [\varphi v] \frac{1}{\sqrt{(-\Delta_N) - \lambda^2}} [\varphi w] \ \mathrm{d}x$$

$$\le c \left\| \frac{1}{\sqrt{(-\Delta_N) - \lambda^2}} \circ \varphi \right\|^2_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \quad \text{provided } \lambda \in Q.$$

On the other hand,

$$\left\| \frac{1}{\sqrt{(-\Delta_N) - \lambda^2}} \circ \varphi \right\|^2_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]} \tag{4.80}$$

$$= \sup_{\|v\|_{L^2(\Omega)} \leq 1} \int_\Omega \frac{1}{\sqrt{(-\Delta_N) - \lambda^2}}[\varphi v] \frac{1}{\sqrt{(-\Delta_N) - \lambda^2}}[\varphi v] \ \mathrm{d}x$$

$$= \sup_{\|v\|_{L^2(\Omega)} \leq 1} \int_\Omega \varphi \frac{1}{(-\Delta_N) - \lambda^2}[\varphi v] v \ \mathrm{d}x$$

$$= \left\| \varphi \circ \frac{1}{(-\Delta_N) - \lambda^2} \circ \varphi \right\|_{\mathcal{L}[L^2(\Omega);L^2(\Omega)]};$$

whence hypothesis (4.78) is satisfied as a direct consequence of the *limiting absorption principle* stated in (4.69).

4.2.6 Space-time decay estimates

Going back to formula (4.75) we can apply Theorem 4.1 to obtain

$$\int_0^T \left\| \varphi \Delta_N G(-\Delta_N) \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} \sqrt{-\Delta_N} \right) [h^1_\varepsilon] \right\|^2_{L^2(\Omega)} \mathrm{d}t \tag{4.81}$$

$$\leq \varepsilon c_1 \int_{-\infty}^{\infty} \left\| \varphi \Delta_N G(-\Delta_N) \exp\left(\pm \mathrm{i} t \sqrt{-\Delta_N} \right) [h^1_\varepsilon] \right\|^2_{L^2(\Omega)} \mathrm{d}t \leq \varepsilon c_2 \|h^1_\varepsilon\|^2_{L^2(\Omega)},$$

and, similarly,

$$\int_0^T \left\| \varphi \frac{G(-\Delta_N)}{\sqrt{-\Delta_N}} \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} \sqrt{-\Delta_N} \right) [h^2_\varepsilon] \right\|^2_{L^2(\Omega)} \mathrm{d}t \tag{4.82}$$

$$\leq \varepsilon c_1 \int_{-\infty}^{\infty} \left\| \varphi \frac{G(-\Delta_N)}{\sqrt{-\Delta_N}} \exp\left(\pm \mathrm{i} t \sqrt{-\Delta_N} \right) [h^2_\varepsilon] \right\|^2_{L^2(\Omega)} \mathrm{d}t \leq \varepsilon c_2 \|h^2_\varepsilon\|^2_{L^2(\Omega)}.$$

Analogously, we get

$$\int_0^T \int_0^t \left\| \varphi \Delta_N G(-\Delta_N) \exp\left(\pm \mathrm{i} \frac{t-s}{\varepsilon} \sqrt{-\Delta_N} \right) [H^1_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \mathrm{d}s \ \mathrm{d}t \tag{4.83}$$

$$\leq \varepsilon \int_0^T \int_{-\infty}^{\infty} \left\| \varphi \Delta_N G(-\Delta_N) \exp\left(\pm \mathrm{i} \left(t - \frac{s}{\varepsilon} \right) \sqrt{-\Delta_N} \right) [H^1_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \mathrm{d}t \ \mathrm{d}s$$

$$\leq \varepsilon c_1 \int_0^T \left\| \exp\left(\pm \mathrm{i} \frac{s}{\varepsilon} \sqrt{-\Delta_N} \right) [H^1_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \mathrm{d}s = \varepsilon c_1 \int_0^T \left\| H^1_\varepsilon \right\|^2_{L^2(\Omega)} \mathrm{d}s;$$

and, finally,

$$\int_0^T \int_0^t \left\| \varphi \frac{G(-\Delta_N)}{\sqrt{\Delta_N}} \exp\left(\pm \mathrm{i} \frac{t-s}{\varepsilon} \sqrt{-\Delta_N} \right) [H^2_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}s \ \mathrm{d}t \tag{4.84}$$

$$\leq \varepsilon \int_0^T \int_{-\infty}^{\infty} \left\| \varphi \frac{G(-\Delta_N)}{\sqrt{\Delta_N}} \exp\left(\pm \mathrm{i} \left(t - \frac{s}{\varepsilon} \right) \sqrt{-\Delta_N} \right) [H^2_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}t \ \mathrm{d}s$$

$$\leq \varepsilon c_1 \int_0^T \left\| \exp\left(\pm \mathrm{i} \frac{s}{\varepsilon} \sqrt{-\Delta_N} \right) [H^2_\varepsilon(s,\cdot)] \right\|^2_{L^2(\Omega)} \ \mathrm{d}s = \varepsilon c_1 \int_0^T \left\| H^2_\varepsilon \right\|^2_{L^2(\Omega)} \ \mathrm{d}s.$$

Similar estimates can be established for the terms in (4.75) containing h^i_ε, H^i_ε, $i = 3, 4$.

Combining relations (4.81 - 4.84) we may infer that

$$\|G(-\Delta_N)[\Phi_\varepsilon]\|^2_{L^2((0,T)\times K} \leq \varepsilon c(K, G) \tag{4.85}$$

for any compact $K \subset \Omega$, and any $G \in C^\infty_c(0,\infty)$.

4.2.7 Compactness of the gradient part, conclusion

We have

$$\int_\Omega \Phi_\varepsilon \mathrm{div}_x \mathbf{w} \ \mathrm{d}x = \int_\Omega G(-\Delta_N)[\Phi_\varepsilon] \mathrm{div}_x \mathbf{w} \ \mathrm{d}x + \int_\Omega [\mathrm{Id} - G(-\Delta_N)][\Phi_\varepsilon] \mathrm{div}_x \mathbf{w} \ \mathrm{d}x$$

$$= \int_\Omega G(-\Delta_N)[\Phi_\varepsilon] \mathrm{div}_x \mathbf{w} \ \mathrm{d}x + \int_\Omega \Phi_\varepsilon [\mathrm{Id} - G(-\Delta_N)][\mathrm{div}_x \mathbf{w}] \ \mathrm{d}x,$$

where, by virtue of (4.85), the former integral on the right-had side tends to zero in $L^2(0,T)$ as $\varepsilon \to 0$ for any fixed $\mathbf{w} \in C^\infty_c(\Omega; R^3)$, $G \in C^\infty_c(0,\infty)$.

On the other hand, it is a routine matter to check that for $G \approx 1$, the quantity $\Delta^\alpha_N \left[\frac{\mathrm{Id} - G(-\Delta_N)}{\sqrt{-\Delta_N}} \right] [\mathrm{div}_x \mathbf{w}]$, $\alpha = 0, 1$, will be small in $L^2(\Omega)$. Seeing that Φ_ε is determined by (4.75), we therefore conclude that

$$\int_\Omega \Phi_\varepsilon \mathrm{div}_x \mathbf{w} \ \mathrm{d}x \to 0 \text{ for any } \mathbf{w} \in C^\infty_c(\Omega; R^3). \tag{4.86}$$

Relations (4.68), (4.70), together with (4.86), yield the desired conclusion (4.58).

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