On the Approximation of Conservation Laws by Vanishing Viscosity

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Joint work with Alberto Bressan, PSU

Introduction

Consider the Cauchy problem

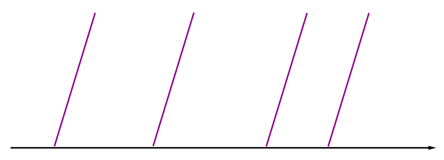
$$\begin{cases} u_t + f(u)_x = 0, & u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{1}$$

and its viscous approximation

$$\begin{cases} u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}, \\ u^\varepsilon(0,x) = \bar{u}(x). \end{cases} \tag{2}$$

Assume that the system is strictly hyperbolic, f is C^2, and $TV(\bar{u}) < \mu \ll 1$.
Then $u^\varepsilon \to u$ strongly in L^1_{loc} as $\varepsilon \to 0^+$, [1].

A similar convergence result was obtained by Goodman and Xin, [4], for the special case in which the solution u of (1) contains a finite number of noninteracting entropic shocks.



When u contains only one shock, $\xi(t)$, connecting (u^-, u^+), the convergence estimate in [4] is written as

$$\sup_{\substack{0 \le t \le T \\ |x - \xi(t)| \ge h}} |u(t,x) - u^\varepsilon(t,x)| \le C_h \varepsilon, \qquad \forall h > 0.$$

This result was obtained using a singular perturbation approach. An approximate solution of (2) was constructed by matching the following asymptotic expansions

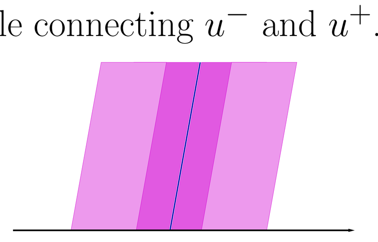
- the Outer Expansion, valid where u is smooth,

$$u^\varepsilon = u + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots + \varepsilon^k v_k + \mathcal{O}(\varepsilon^{k+1}), \tag{3}$$

- the Inner Expansion, in a neighborhood of size $\mathcal{O}(\varepsilon)$ around the discontinuity,

$$u^\varepsilon = U(t,\eta) + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots + \varepsilon^k U_k + \mathcal{O}(\varepsilon^{k+1}),$$

  where $U(t,\eta)$ is the viscous shock profile connecting u^- and u^+.



More recently, Shen and Xu proved the following result, [5]. Let u be the solution of (1) in the scalar case, $n = 1$. Assume that u is piecewise smooth, then the Outer Expansion (3) is valid far from the discontinuities, even if the solution u contains interacting shocks.
Moreover, the position of the discontinuities can be carefully estimated. From this theoretical result the authors also obtained a numerical method with high order convergence.

The main result, [Bressan-D., 2007]

Consider the scalar Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, & u(t,x) \in \mathbb{R}, \\ u(0,x) = \bar{u}(x). \end{cases}$$

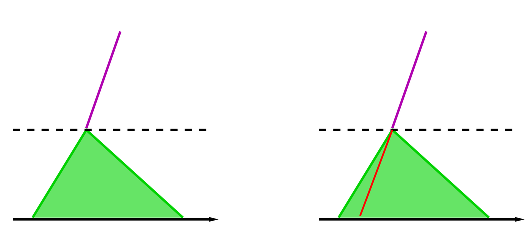
Let $f \in C^2$, and $f''(u) \ge k > 0$, $\forall u$, and assume

- for $t < \tau$: $u(t)$ contains arbitrarily many shocks,
- for $t > \tau$: only one isolated shock in $x = \xi(t)$ remains.

Choose $\tilde{u}(0,x)$ in a suitable way:

$$\tilde{u}(0,x) = u(0,x), \qquad x \notin \left[x^-(0),\, x^+(0)\right],$$

$$\int_{x^-(0)}^{x^+(0)} \tilde{u}(0,x)\,dx = \int_{x^-(0)}^{x^+(0)} u(0,x)\,dx\,.$$



Theorem

Let u^ε, $\tilde{u}^\varepsilon$ solve

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx},$$

with initial data

$$u^\varepsilon(x,0) = u(x,0)\,, \qquad \tilde{u}^\varepsilon(x,0) = \tilde{u}(x,0)\,.$$

Then

$$\lim_{\varepsilon \to 0+} \varepsilon^{-k} \cdot \left\| u^\varepsilon - \tilde{u}^\varepsilon \right\|_{C^\nu(\Omega)} = 0, \qquad \forall k, \nu \ge 0,$$

over every compact domain $\Omega \subset\subset \left\{(t,x);\ \ t > \tau\,,\ x \in \mathbb{R}\right\}$.

Roughly speaking: for $t > \tau$ any singular perturbation expansion of $\tilde{u}^\varepsilon$ is valid for u^ε as well.

Outline of the proof

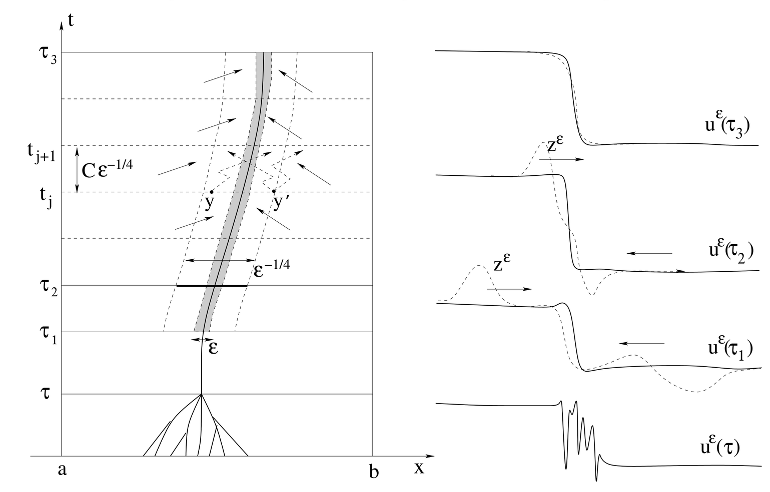
The result in [5] provides the desired estimate far from the location of the hyperbolic shock. In a neighborhood of the discontinuity we use a homotopy method. Consider the Cauchy problem

$$\begin{cases} u^{\varepsilon,\theta}_t + f(u^{\varepsilon,\theta})_x = \varepsilon u^{\varepsilon,\theta}_{xx}, & \theta \in [0,1], \\ u^{\varepsilon,\theta}(0,x) = \theta u^\varepsilon(0,x) + (1-\theta)\,\tilde{u}^\varepsilon(0,x)\,, \end{cases}$$

and the infinitesimal perturbation

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial\theta} u^{\varepsilon,\theta}.$$

The proof of the theorem is obtained by a careful study on the decay of $z^{\varepsilon,\theta}$.



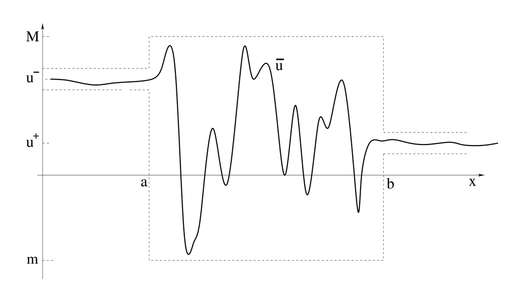
Formation of a Viscous Shock [Bressan-D., 2006]

A fundamental step in the proof above is to show that after a finite time all solutions $u^{\varepsilon,\theta}$ contain a viscous shock profile. The formation of a viscous shock profile, starting from a rather general class of initial data was the main focus of our fist work, [2].

We consider a uniformly parabolic scalar equation with possibly non convex flux

$$u_t + f(u)_x = u_{xx},$$

and an initial condition of the form



Then: there exists a constant C such that $\forall \delta_0 > 0$

$$\left\| u(t,\cdot) - \phi(t, \cdot - c(t)) \right\|_{L^\infty} \le C\delta_0\,, \qquad t > T.$$

where ϕ is the viscous shock profile connecting (u^-, u^+). The time needed for the viscous shock to appear can be estimated in terms of the initial condition, $T = T(m, M, b-a, u^- - u^+)$.

Define the curve γ, [1],

$$\gamma(t,x) = \begin{pmatrix} u(t,x) \\ w(t,u) = f\big(u(t,x)\big) - u_x(t,x) \end{pmatrix},$$

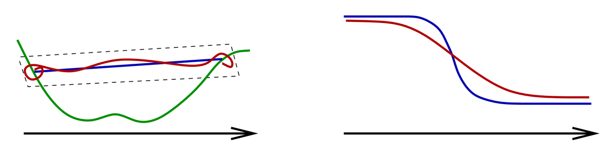
From the conservation law we obtain an equation for w

$$u_t + f(u)_x = u_{xx} \;\Rightarrow\; w_t = \big(w - f(u)\big)^2 w_{uu}\,.$$

The proof of the theorem consists of the study of the evolution of the curve γ in the $u - w$ plane and relies on comparison arguments with upper and lower solutions. Observe that in the $u - w$ plane the curve $\tilde{\gamma}$ corresponding to a viscous shock profile, ϕ, is the portion of a straight line.

Lemma

$$\left| f\big(u(x)\big) - u_x(x) - \tilde{w}\big(u(x)\big) \right| \le \delta \;\Rightarrow\; \left\| u - \phi(\cdot - c) \right\|_{L^\infty(\mathbb{R})} \le K_0\,\delta.$$



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