Gasdynamic regularity and nondegeneracy: some classifying remarks

LIVIU FLORIN DINU
Institute of Mathematics, Romanian Academy

MARINA ILEANA DINU
Polytechnical University of Bucharest

Burnat type "algebraic" approach

For the multidimensional first order hyperbolic system of a gasdynamic type [whose coefficients only depend on u]

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k}=0,\ \ 1\le i\le n \tag{1}$$

the "algebraic" approach (M. Burnat; see [1]) starts with identifying *dual* pairs of directions $\vec\beta,\vec\kappa$ [we write $\vec\kappa \rightharpoonup \vec\beta$] connecting [via their duality relation] the hodograph [= in the hodograph space H of the entities u] and physical [= in the physical space E of the independent variables] characteristic details. The duality relation at $u^*\in H$ has the form:

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j=0,\ \ 1\le i\le n. \tag{2}$$

Here $\vec\beta$ is an *exceptional* direction [= orthogonal in the physical space E to a characteristic character]. A direction $\vec\kappa$ dual to an exceptional direction $\vec\beta$ is said to be a *hodograph characteristic* direction.

Number and structure of the dual pairs. Examples.

1. For the *one-dimensional* strictly hyperbolic version of system (1) a *finite* number n of dual pairs $\vec\kappa_i \rightharpoonup \vec\beta_i$ consisting in $\vec\kappa_i=\vec R_i$ and $\vec\beta_i=\Theta_i(u)[-\lambda_i(u),1]$, where $\vec R_i$ is a right eigenvector of the $n\times n$ matrix a and λ_i is an eigenvalue of a, are available ($i=1,\dots,n$). Each dual pair associates in this case, at each $u^*\in\mathcal R$ [for a suitable $\mathcal R\subset H$], to a vector $\vec\kappa$ a *single* dual vector $\vec\beta$.

2. For the *two-dimensional* version of (1) corresponding to an isentropic description (in usual notations: c is the sound velocity, v_x, v_y are fluid velocities) an *infinite* number of dual pairs are available at each $u^*\in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec\kappa$ a *single* dual vector $\vec\beta$ (Z. Peradzynski; see [1]).

3. For the *three-dimensional* version of (1) corresponding to an isentropic description an *infinite* number of dual pairs are available at each $u^*\in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec\kappa$ a *finite* [constant, $\neq 1$] number of k independent exceptional dual vectors $\vec\beta_j$, $1\le j\le k$; and therefore has the structure $\vec\kappa \rightharpoonup (\vec\beta_1,\dots,\vec\beta_k)$ (Peradzynski).

Definition (Burnat). A curve $\mathcal C\subset H$ is said to be *characteristic* if it is tangent at each point of it to a characteristic direction $\vec\kappa$. A hypersurface $\mathcal S\subset H$ is said to be *characteristic* if it possesses at least a characteristic system of coordinates.

Genuine nonlinearity. Simple waves solution

In case of an one-dimensional strictly hyperbolic version of (1) any hodograph characteristic curve $\mathcal C\subset\mathcal R\subset H$, of index i, is said to be *genuinely nonlinear (gnl)* if the dual constructive pair $\vec\kappa_i \rightharpoonup \vec\beta_i$ is restricted by

$$\vec\kappa_i(u)\diamond\vec\beta_i(u)\equiv \vec R_i(u)\cdot\mathrm{grad}_u\lambda_i(u)\neq 0\,\mathrm{in}\,\mathcal R \tag*{$(3)_1$}$$

see Example 1. This condition transcribes the requirement $\frac{\mathrm d\vec\beta}{\mathrm d\alpha}\neq 0$ along $\mathcal C$.

Definition. We naturally extend the *gnl* character of a hodograph characteristic curve $\mathcal C$ to the cases corresponding to Examples 2 and 3, by requiring along $\mathcal C$

$$\left|\frac{\mathrm d\vec\beta}{\mathrm d\alpha}\right|\neq 0 \tag*{$(3)_2$}$$

and, respectively,

$$\sum_{\mu=1}^{k}\left|\frac{\mathrm d\vec\beta_\mu}{\mathrm d\alpha}\right|\neq 0. \tag*{$(3)_3$}$$

Definition. A solution of (1) whose hodograph is laid along a *gnl* characteristic curve is said to be a *simple waves solution* (*sws*; here below also called *wave*).

Remark 1. To each of the Examples 1-3 above a *sws* construction corresponds. These constructions show a *nondegenerate* character.

Remark 2 (Burnat). Let $R_1,\dots,R_p$ be *gnl* characteristic coordinates on a given p-dimensional characteristic region $\mathcal R$ of a hodograph hypersurface $\mathcal S$ with the normal $\vec n$. Solutions of the system

$$\frac{\partial u_l}{\partial x_s}=\sum_{k=1}^{p}\eta_k\kappa_{kl}(u)\beta_{ks}(u),\ u\in\mathcal R;\ 1\le l\le n,\ 0\le s\le m;\ \vec\kappa_k\perp\vec n,\ 1\le k\le p \tag{4}$$

appear to concurrently satisfy the system (1). This indicates an "algebraic" importance of the concept of dual pair [see (2)].

Wave-wave "algebraic" regular interactions. Riemann–Burnat invariants

Definition. A solution of (1) whose hodograph is laid on a characteristic hypersurface is said to correspond to a *wave-wave regular interaction* if its hodograph possesses a *gnl* system of coordinates and a set of *Riemann–Burnat invariants* $R(x)$ exists, structuring the dependence on x of the solution u by a *regular* interaction representation

$$u_l=u_l[R_1(x_0,\dots,x_m),\dots,R_p(x_0,\dots,x_m)],\ 1\le l\le n. \tag{5}$$

- We compare (4) and (5) to see that for a wave-wave regular interaction solution of (4) $R_i(x)$ must fulfil an (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s}=\eta_k\beta_{ks}[u(R)],\ 1\le k\le p,\ 0\le s\le m. \tag{6}$$

- Sufficient restrictions for solving (6) are proposed by Z. Peradzynski or E. Ferapontov (see [1]).
- A wave-wave regular interaction reflects the *nondegenerate* nature of the *gnl* hodographs of the interacting simple waves solutions.

Martin type "differential" approach. An unsteady one-dimensional version

A continuous [smooth] strictly adiabatic [anisentropic] flow results behind a shock discontinuity of non-constant continuous [smooth] velocity which penetrates into a region of uniform flow. For such a particular anisentropic flow we seek for solutions which fulfil the (natural) requirement $\frac{\partial p}{\partial t}\frac{\partial\psi}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial\psi}{\partial t}\neq 0$ to select p and ψ as new independent variables in place of x and t, and find (M.H. Martin; see [2])

$$v_x==v_x(p,\psi)=\frac{\partial\xi}{\partial\psi},\ x=x(p,\psi)=\int\left(\frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p\partial\psi}+\frac{1}{\rho}\right)\mathrm d\psi+\left(\frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p^2}\right)\mathrm dp,\ t=t(p,\psi)=\frac{\partial\xi}{\partial p}. \tag{7}$$

where ξ must fulfil the hyperbolic Monge–Ampère equation

$$\frac{\partial^2\xi}{\partial p^2}\frac{\partial^2\xi}{\partial\psi^2}-\left(\frac{\partial^2\xi}{\partial p\partial\psi}\right)^2=-\zeta^2(p,\psi)\equiv\frac{\partial}{\partial p}\left(\frac{1}{\rho}\right)\equiv-\frac{1}{\rho^2c^2} \tag{8}$$

with $\rho=\rho(p,\psi)$ and $c(p,\psi)=\sqrt{\left(\frac{\partial\rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed.

Remark 3. On reversing $(7)_{2,3}$ into $p=p(x,t)$, $\psi=\psi(x,t)$ and carrying this into $(7)_1$ we get a form $p(x,t)$, $v_x(x,t)$, $\psi(x,t)$ of the corresponding anisentropic solution.

Martin linearization. Riemann–Martin invariants

Some "differential" *analogues* of the simple waves solutions or wave-wave regular interaction solutions are available for $\zeta=P(p)\Psi(\psi)$ in (8) *in six cases only* (Martin; see [2]). For each of these six cases to the equation (8) two intermediate integrals $\mathcal F_\pm\left(p,\psi,\xi,\frac{\partial\xi}{\partial p},\frac{\partial\xi}{\partial\psi}\right)$, *linear* in ξ, can be associated for which $\mathcal F_\pm=\mathrm{constant}_\pm=R_\pm$ along a characteristic $\overline{\mathcal C}_\pm$. We have to distiguish, for each of the mentioned six cases, between the circumstances (a) when $R_\pm$ depend on the characteristic $\overline{\mathcal C}_\pm$, and (b) when R_+ or R_- are overall constants.

In the case (a) we may use $R_\pm$ as new independent variables. It can be shown, in this construction (a), that the entities p^{-1}, v_x, ψ^{-1}, t fulfil [Martin] various Euler–Poisson–Darboux *linear* equations to which well-known representations are associated; we present these representations by

$$p=p(R_+,R_-),\ \psi=\psi(R_+,R_-),\ v_x=v_x(R_+,R_-),\ t=t(R_+,R_-),\ x=x(R_+,R_-) \tag{9}$$

where $x(R_+,R_-)$ results by quadratures. Reversing $(9)_{4,5}$ into $R_\pm=R_\pm(x,t)$ will induce a form of solution (9), parallel to (5) [as $R_\pm$ have a characteristic nature]. We call $R_\pm(x,t)$ *Riemann–Martin invariants.*

In the case (b) we notice that a solution $\xi(p,\psi)$ of the *linear* equation $\mathcal F_+\equiv R_+$ or $\mathcal F_-\equiv R_-$ will automatically fulfil (8). We have to follow, in this construction (b), Remark 3 to describe a solution (7); we call such solution a *pseudo simple waves solution.*

Example 4. To $\zeta=\frac{\psi^{\nu-1}}{p^{\nu+1}}$ two intermediate integrals $\mathcal F_\pm\equiv p\frac{\partial\xi}{\partial p}+\psi\frac{\partial\xi}{\partial\psi}-\xi\pm\frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$ correspond. We satisfy $\mathcal F_+\equiv R_+=0$ by $\xi=\frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$, $\nu=-\frac{\gamma-1}{2\gamma}$, integral ν, $\nu\neq 0{,}1$.

Remark 4. A pseudo wave-wave interaction results by glueing together an element of construction (a) and four suitable elements of construction (b).

"Algebraic" approach and "differential" approach: first details of a parallel

In [2] it is computed, at each point of the hodograph (9), the following relation between the Burnat characteristic directions $\vec\kappa$ and the Martin characteristic directions $\vec\mu$

$$\vec\mu_\pm=\left(\frac{\partial p}{\partial R_\pm},\frac{\partial v_x}{\partial R_\pm},\frac{\partial S}{\partial R_\pm}\right)^t=\eta_\mp\vec\kappa_\mp+\eta_0\vec\kappa_0;\ S(R_+,R_-)\equiv F[\psi(R_+,R_-)],\ \eta_\mp=\frac{1}{\Lambda_\mp}\frac{\partial v_x}{\partial R_\pm},\ \eta_0=\frac{\partial S}{\partial R_\pm}.$$

Representation (9) corresponds to an example of hodograph surface which *is not* a Burnat characteristic surface. Still, incidentally and essentially for the anisentropic linearized approach, this representation is associated with an example of hodograph surface for which a characteristic character persists in a Martin sense.

Pseudo simple waves solution

We notice from the details of Example 4 (see [2]) that [in contrast with a *sws*]
- a *pseudo sws* has a two-dimensional hodograph and
- for it none of the characteristic fields in the physical plane x,t is made of straightlines generally.

Martin type "differential" approach. A steady supersonic version

A continuous [smooth] strictly adiabatic [anisentropic] *rotational* flow results behind a curved shock discontinuity from a region of uniform flow ahead. For such a particular anisentropic flow we seek for solutions which fulfil the (natural) requirement $\frac{\partial p}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial\psi}{\partial x}\neq 0$ to select p and ψ as new independent variables in place of x and y, and find [Martin]

$$x=\frac{\partial\eta}{\partial p},\ y=-\frac{\partial\xi}{\partial p},\ v_x=\frac{\partial\xi}{\partial\psi},\ v_y=\frac{\partial\eta}{\partial\psi}, \tag{10}$$

where ξ and η must fulfil the same Monge–Ampère equation

$$4\mathcal F\left[\left(\frac{\partial^2\xi}{\partial p\partial\psi}\right)^2-\frac{\partial^2\xi}{\partial p^2}\frac{\partial^2\xi}{\partial\psi^2}\right]-4\left(\frac{\partial\xi}{\partial\psi}\frac{\partial\mathcal F}{\partial p}\right)\frac{\partial^2\xi}{\partial p\partial\psi}+2\left(\frac{\partial\xi}{\partial\psi}\frac{\partial\mathcal F}{\partial\psi}\right)\frac{\partial^2\xi}{\partial p^2}+\left\{\left(\frac{\partial\mathcal F}{\partial p}\right)^2-2\left[\mathcal F-\left(\frac{\partial\xi}{\partial\psi}\right)^2\right]\frac{\partial^2\mathcal F}{\partial p^2}\right\}=0 \tag{11}$$

with $\rho=\rho(p,\psi)$ and $c(p,\psi)=\sqrt{\left(\frac{\partial\rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed, and we notice that ξ and η are connected by

$$v_x\frac{\partial y}{\partial\psi}-v_y\frac{\partial x}{\partial\psi}=\frac{1}{\rho},\ \ v_x\frac{\partial y}{\partial p}-v_y\frac{\partial x}{\partial p}=0.$$

- On reversing $(10)_{1,2}$ into $p=p(x,y)$, $\psi=\psi(x,y)$ and carrying this into $(10)_{3,4}$ we get a 2D analogue of Remark 3.

- The characteristic directions for (12) in the plane p,ψ are given by

$$\left(\frac{\mathrm dp}{\mathrm d\psi}\right)_\pm=\frac{2\mathcal F\frac{\partial^2\xi}{\partial p\partial\psi}-\frac{\partial\mathcal F}{\partial p}\frac{\partial\xi}{\partial\psi}\pm\sqrt\Delta}{-2\mathcal F\frac{\partial^2\xi}{\partial p^2}}\ ,\ \Delta=\frac{4}{\rho^2c^2}v_y^2\left(V^2-c^2\right)\ ,\ V^2=v_x^2+v_y^2\ .$$

  The steady anisentropic description shows a *hyperbolic* character for a *supersonic* flow.
- Details of the steady Martin linearization are in progress ([3]).

References

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