Hyperbolic Systems of Conservation Laws

in One Space Dimension

II - Solutions to the Cauchy problem

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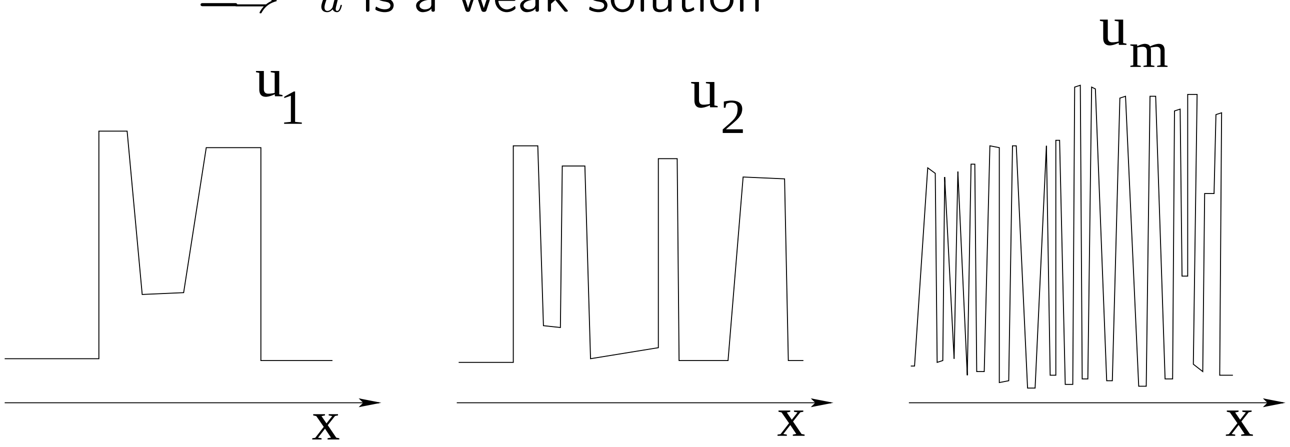
Global Solutions to the Cauchy Problem

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

- Construct a sequence of approximate solutions u_m
- Show that (a subsequence) converges: $u_m \to u$ in $\mathbf{L}^1_{loc}$

$\Longrightarrow$ u is a weak solution



To prevent oscillations, one needs an a-priori bound on the total variation (J. Glimm, 1965)

Building block: the Riemann Problem

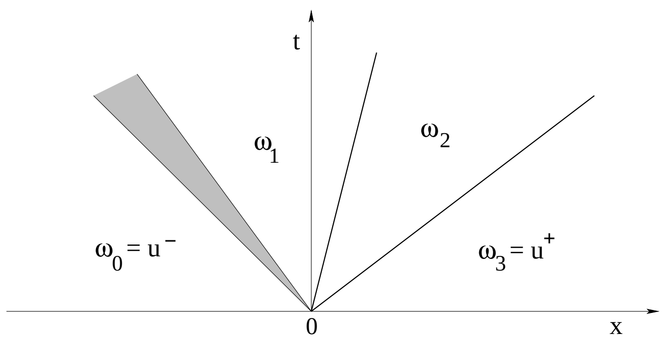
$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

B. Riemann 1860: 2×2 system of isentropic gas dynamics

P. Lax 1957: $n \times n$ systems (+ special assumptions)

T. P. Liu 1975 $n \times n$ systems (generic case)

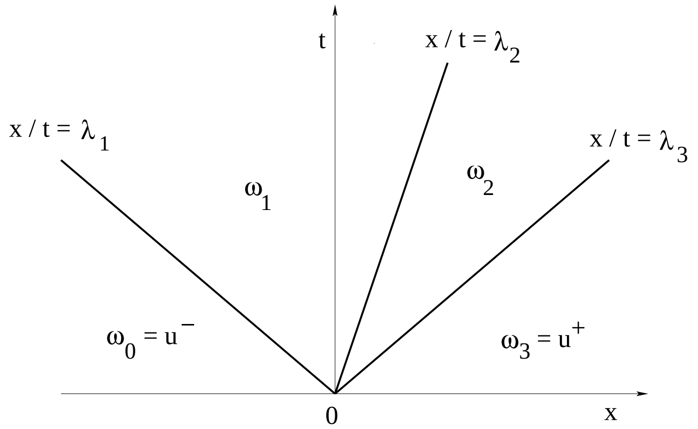
S. Bianchini 2003 (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)



invariant w.r.t. rescaling symmetry: $u^\theta(t,x) \doteq u(\theta t, \theta x)$ for all $\theta > 0$

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$



$$u^+ - u^- \;=\; \sum_{j=1}^{n} c_j r_j \qquad\qquad \text{(sum of eigenvectors of } A\text{)}$$

intermediate states : $\quad \omega_i \doteq u^- + \sum_{j \le i} c_j r_j$

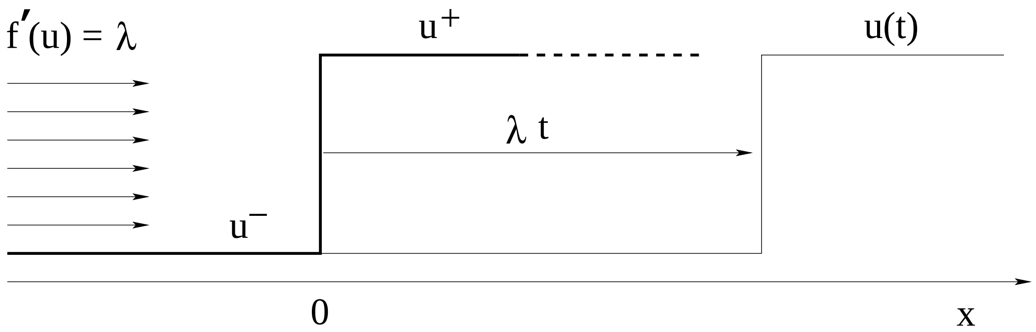
i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

Scalar Conservation Law

$$u_t + f(u)_x = 0 \qquad\qquad u \in I\!R$$

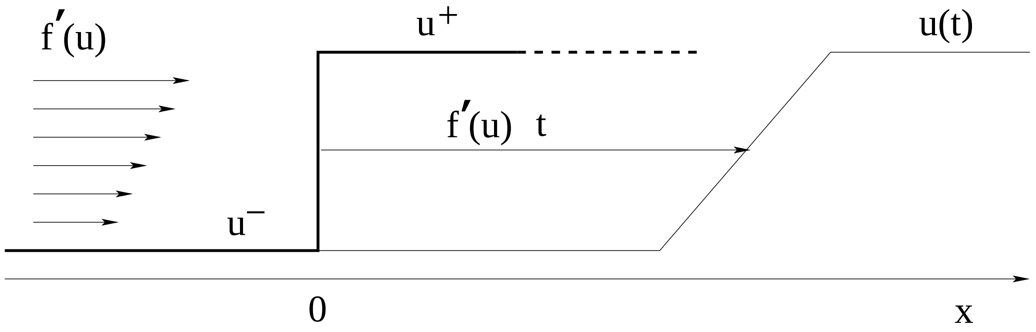
CASE 1: Linear flux. $f(u) = \lambda u$.

Jump travels with speed λ **(contact discontinuity)**

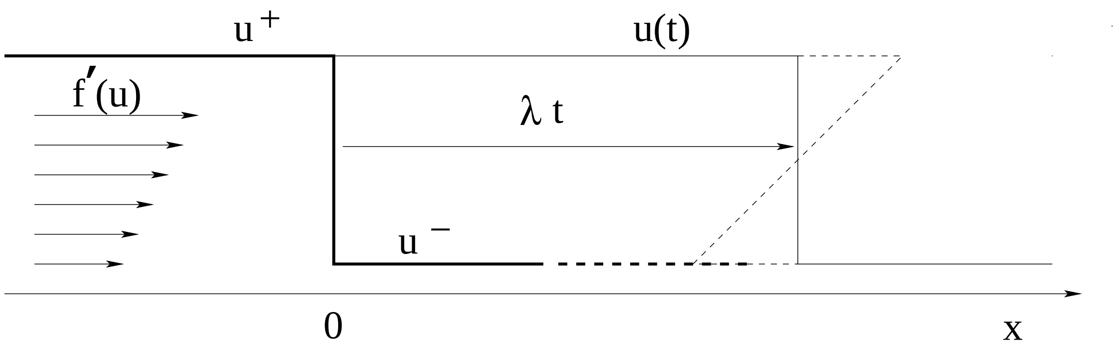


CASE 2: the flux f is convex, so that $u \mapsto f'(u)$ is increasing.

$u^+ > u^- \qquad \Longrightarrow \qquad$ **centered rarefaction wave**



$u^+ < u^- \qquad \Longrightarrow \qquad$ **stable shock**



$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

A class of nonlinear hyperbolic systems

$$u_t + f(u)_x \ = \ 0$$

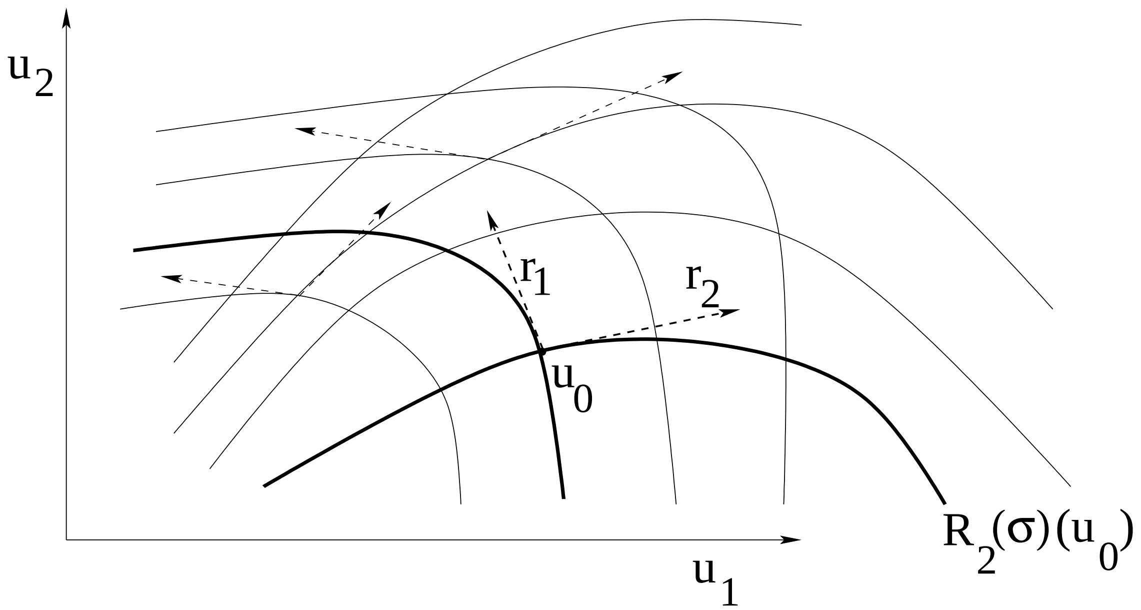
$$A(u) = Df(u) \qquad\qquad A(u)r_i(u) \ = \ \lambda_i(u)r_i(u)$$

Assumption (H) (P.Lax, 1957): Each i-th characteristic field is

- either **genuinely nonlinear**, so that $\nabla\lambda_i \cdot r_i > 0$ for all u
- or **linearly degenerate**, so that $\nabla\lambda_i \cdot r_i = 0$ for all u

genuinely nonlinear $\Longrightarrow$ characteristic speed $\lambda_i(u)$ is strictly increasing along integral curves of the eigenvectors r_i

linearly degenerate $\Longrightarrow$ characteristic speed $\lambda_i(u)$ is constant along integral curves of the eigenvectors r_i



Shock and Rarefaction curves

$$u_t + f(u)_x = 0 \qquad\qquad A(u) = Df(u)$$

i-rarefaction curve through u_0: $\quad \sigma \mapsto R_i(\sigma)(u_0)$

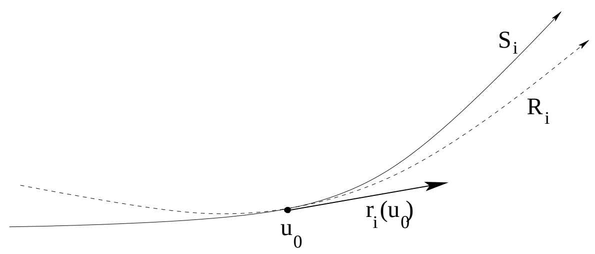
= integral curve of the field of eigenvectors r_i through u_0

$$\frac{du}{d\sigma} = r_i(u), \qquad\qquad u(0) = u_0$$

i-shock curve through u_0: $\quad \sigma \mapsto S_i(\sigma)(u_0)$

= set of points u connected to u_0 by an i-shock, so that

$u - u_0$ is an i-eigenvector of the averaged matrix $A(u, u_0)$



Elementary waves

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

CASE 1 **(Centered rarefaction wave).** Let the i-th field be genuinely nonlinear.

Assume $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$, so that $\lambda_i(u^+) > \lambda_i(u^-)$.

Then

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < t\lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x = t\lambda_i(s), \quad s \in [0,\sigma], \\ u^+ & \text{if} \quad x > t\lambda_i(u^+), \end{cases}$$

is a weak solution (piecewise smooth) of the Riemann problem.

Indeed, the previous construction achieves:

• u is constant along rays $\{x/t = \text{constant}\}$, hence $tu_t + xu_x = 0$

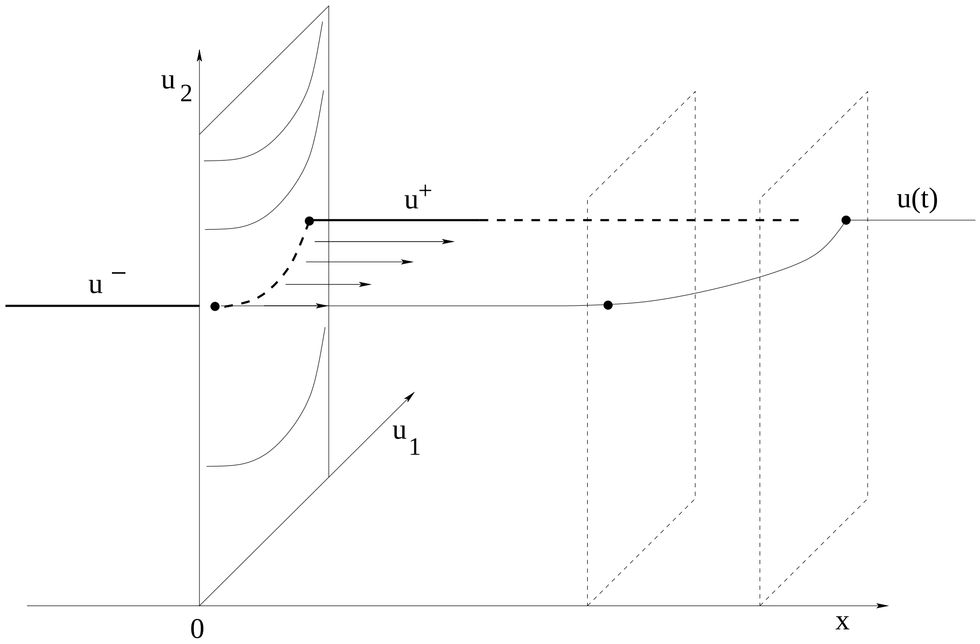
• $\lambda_i(u(t,x)) = x/t$

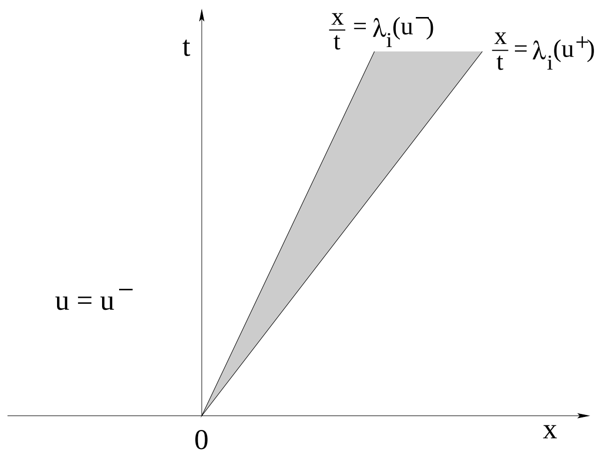
• u_t, u_x are parallel to $r_i(u)$, hence they are i-eigenvectors of $A(u)$

Therefore, for $\lambda_i(u^-) < x/t < \lambda_i(u^+)$ one has

$$0 \;=\; u_t + \frac{x}{t}u_x \;=\; u_t + \lambda_i(u)u_x \;=\; u_t + A(u)u_x$$

A centered rarefaction wave





CASE 2 **(Shock or contact discontinuity).** Assume that

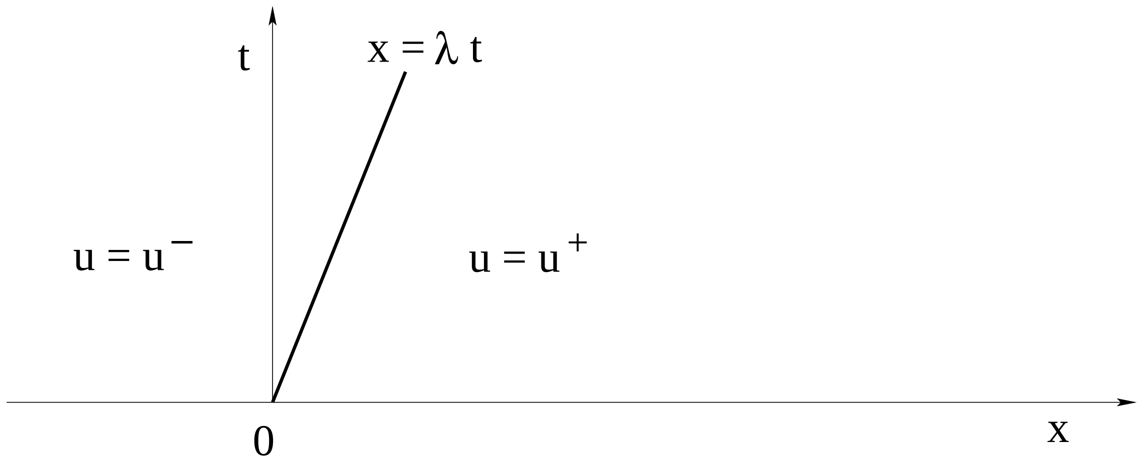
$u^+ = S_i(\sigma)(u^-)$ for some i, σ. Let $\lambda = \lambda_i(u^-, u^+)$ be the shock speed.

Then the function

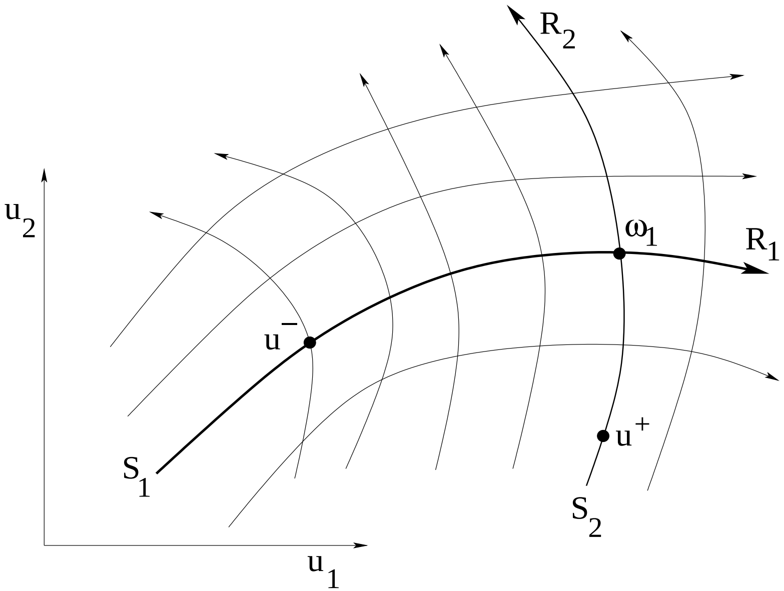
$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

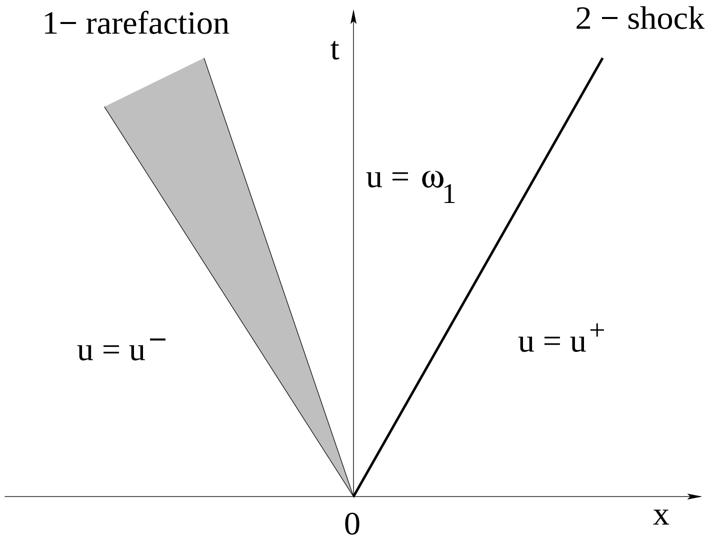
is a weak solution to the Riemann problem.

In the genuinely nonlinear case, this shock is admissible (Lax condition, Liu condition) iff $\sigma < 0$.



Solution to a 2 x 2 Riemann problem





Solution of the general Riemann problem (P. Lax, 1957)

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

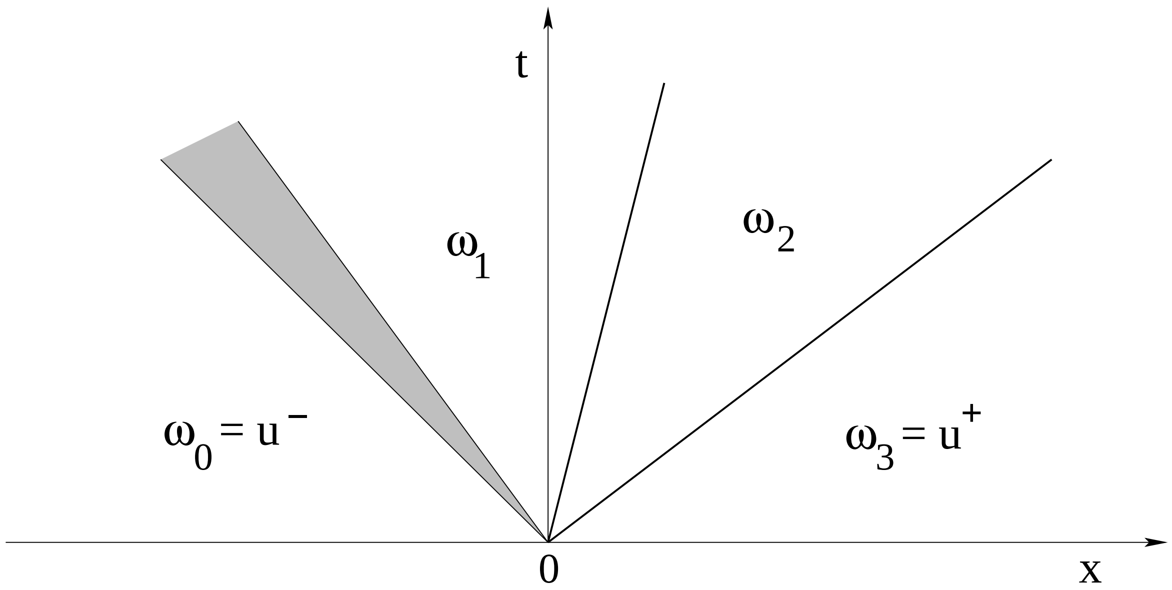
Find states $u^- = \omega_0, \omega_1, \ldots \omega_n$ such that

$$\omega_0 = u^- \qquad\qquad \omega_n = u^+$$

and every couple ω_{i-1}, ω_i are connected by an elementary wave (shock or rarefaction)

$$\begin{cases} \text{either} \quad \omega_i = R_i(\sigma_i)(\omega_{i-1}) & \sigma_i \geq 0 \\ \qquad \text{or} \quad \omega_i = S_i(\sigma_i)(\omega_{i-1}) & \sigma_i < 0 \end{cases}$$

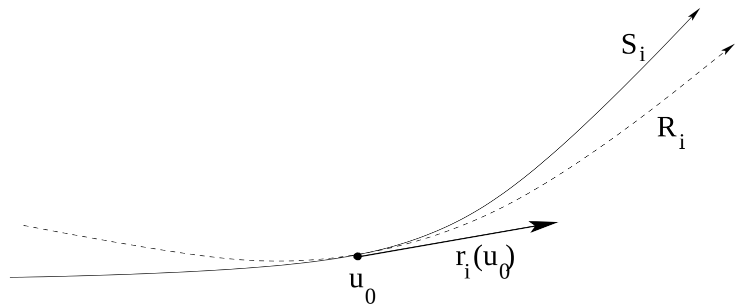
General solution of the Riemann problem: concatenation of elementary waves



Assume: $|u^+ - u^-|$ small

Implicit function theorem $\Longrightarrow$ existence, uniqueness of the intermediate states $\omega_0, \omega_1, \ldots, \omega_n$

$$\Psi_i(\sigma)(u) = \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0 \\ S_i(\sigma)(u) & \text{if } \sigma < 0 \end{cases}$$



$$(\sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto \Psi_n(\sigma_n) \circ \cdots \circ \Psi_2(\sigma_2) \circ \Psi_1(\sigma_1)(u^-)$$

Jacobian matrix at the origin: $J \doteq \Big(r_1(u^-) \Big| r_2(u^-) \Big| \cdots \Big| r_n(u^-) \Big)$

always has full rank

Global solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm 1965). *Assume:*

- *system is strictly hyperbolic*

- *each characteristic field is either linearly degenerate or genuinely nonlinear*

Then there exists a constant $\delta > 0$ *such that, for every initial condition* $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ *with*

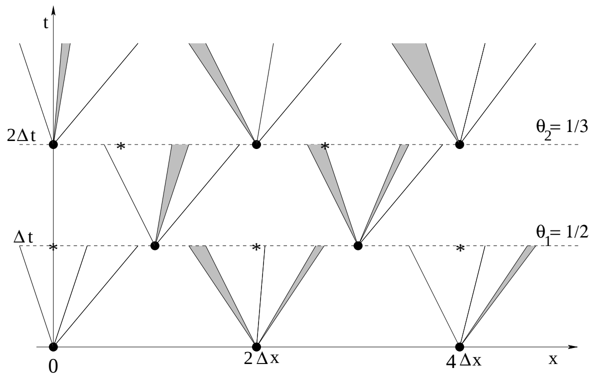
$$\text{Tot.Var.}(\bar{u}) \leq \delta,$$

the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ *defined for all* $t \geq 0$.

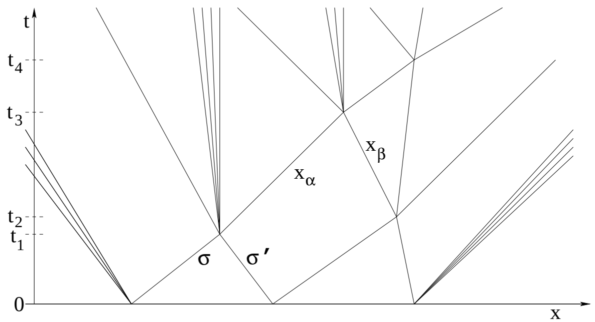
Construction of a sequence of approximate solutions:

by piecing together solutions of Riemann problems

- on a fixed grid in t-x plane (Glimm scheme)



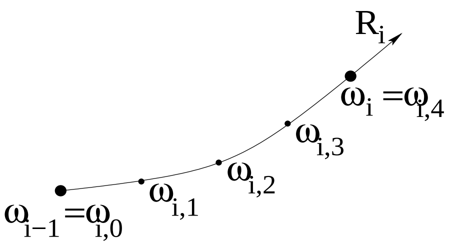
- at points where fronts interact (front tracking)

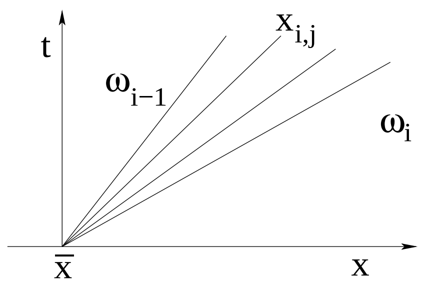


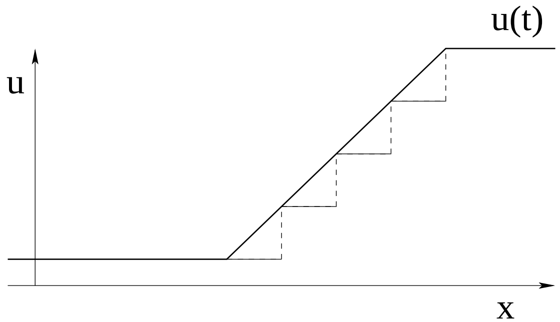
Piecewise constant approximate solution to a Riemann problem

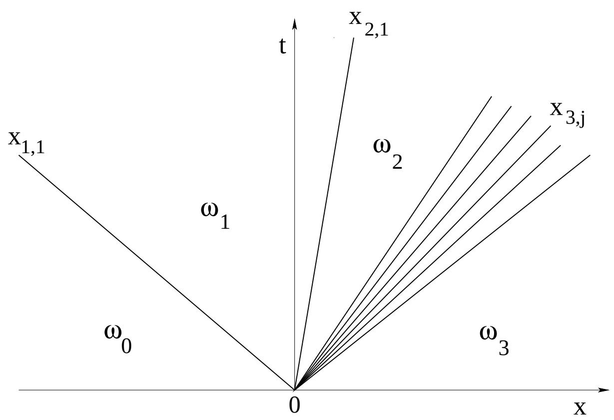
replace centered rarefaction waves

with piecewise constant rarefaction fans

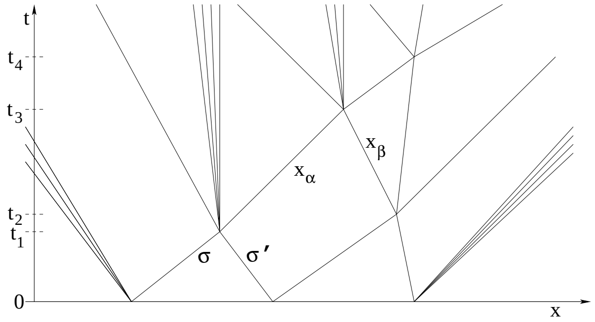








Front Tracking Approximations



• Approximate the initial data $\bar{u}$ with a piecewise constant function.

• Construct a piecewise constant approximate solution to each Riemann problem at $t = 0$

• at each time t_j where two fronts interact, construct a piecewise constant approximate solution to the new Riemann problem ...

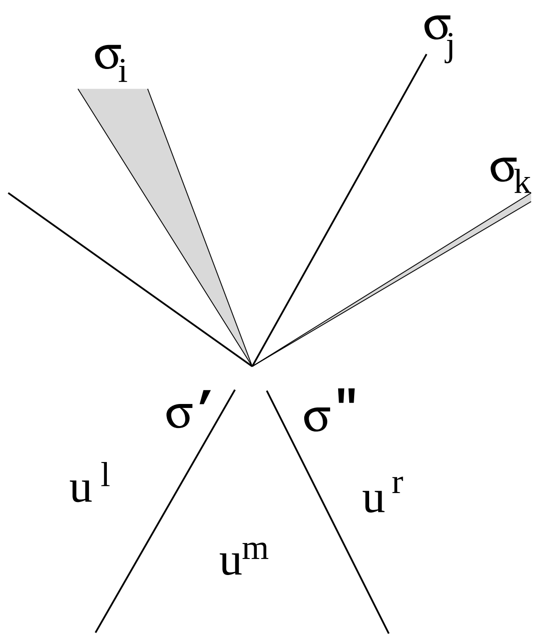
NEED TO CHECK:

- total variation remains small

- number of wave fronts remains finite

Interaction Estimates

GOAL: estimate the strengths of the waves in the solution of a Riemann problem, depending on the strengths of the two interacting waves σ', σ''



Incoming: a j-wave of strength σ' and an i-wave of strength σ''

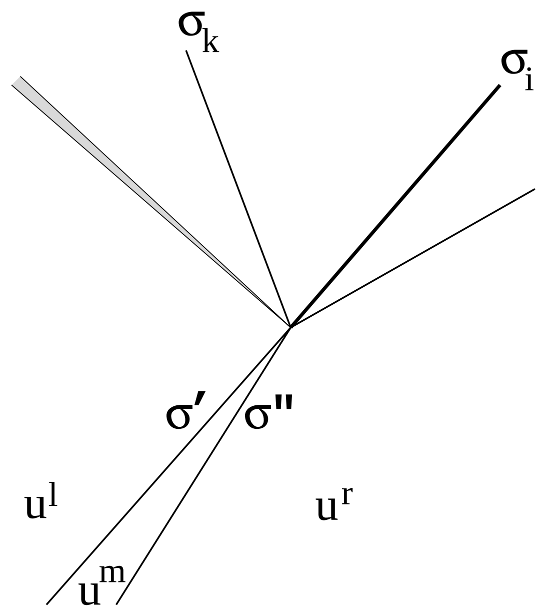
Outgoing: waves of strengths $\sigma_1, \ldots, \sigma_n$. Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| + \sum_{k \neq i,j} |\sigma_k| \;=\; \mathcal{O}(1) \cdot |\sigma' \sigma''|$$

Incoming: two i-waves of strengths σ' and σ''

Outgoing: waves of strengths $\sigma_1, \ldots, \sigma_n$. Then

$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| \;=\; \mathcal{O}(1) \cdot |\sigma' \sigma''| (|\sigma'| + |\sigma''|)$$

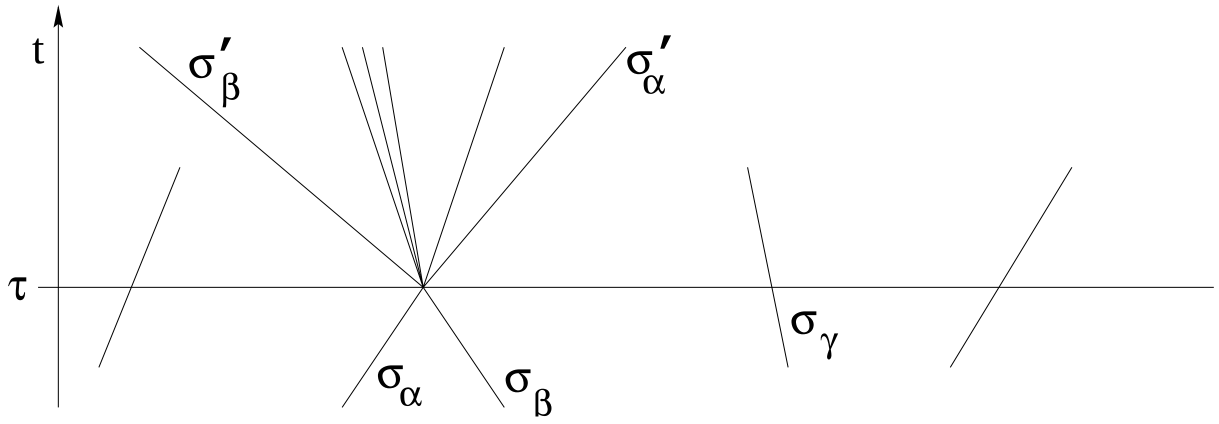


Glimm Functionals

Total strength of waves: $$V(t) \doteq \sum_{\alpha} |\sigma_\alpha|$$

Wave interaction potential: $$Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|$$

$\mathcal{A} \doteq$ couples of *approaching* wave fronts



Changes in V, Q at time τ when the fronts $\sigma_\alpha, \sigma_\beta$ interact:

$$\Delta V(\tau) \;=\; \mathcal{O}(1)|\sigma_\alpha \sigma_\beta|$$

$$\Delta Q(\tau) \;=\; -|\sigma_\alpha \sigma_\beta| + \mathcal{O}(1) \cdot V(\tau-)|\sigma_\alpha \sigma_\beta|$$

Choosing a constant C_0 large enough, the map

$$t \mapsto V(t) + C_0 Q(t)$$

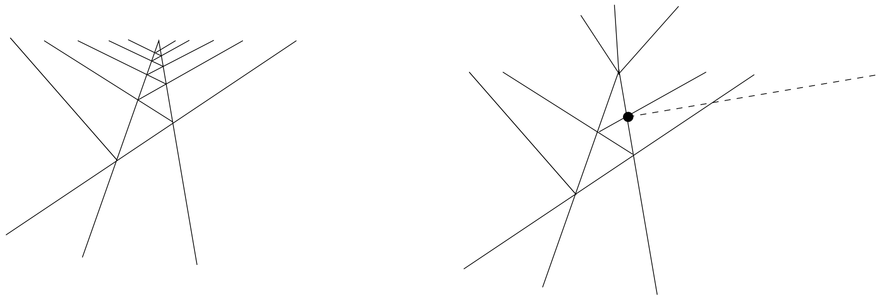
is nonincreasing, as long as V remains small

Total variation initially small $\Longrightarrow$ global BV bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \;\leq\; V(t) \;\leq\; V(0) + C_0 Q(0)$$

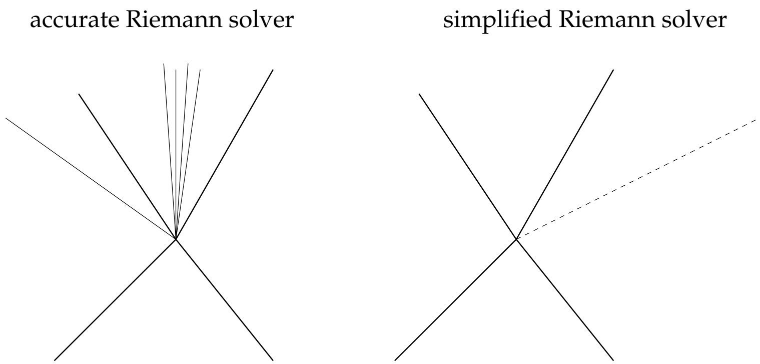
Front tracking approximations can be constructed for all $t \geq 0$

Keeping finite the number of wave fronts



At each interaction point, the **Accurate Riemann Solver** yields a solution, possibly introducing several new fronts

Number of fronts can become infinite in finite time



Need: a **Simplified Riemann Solver**, producing only one *“non-physical”* front

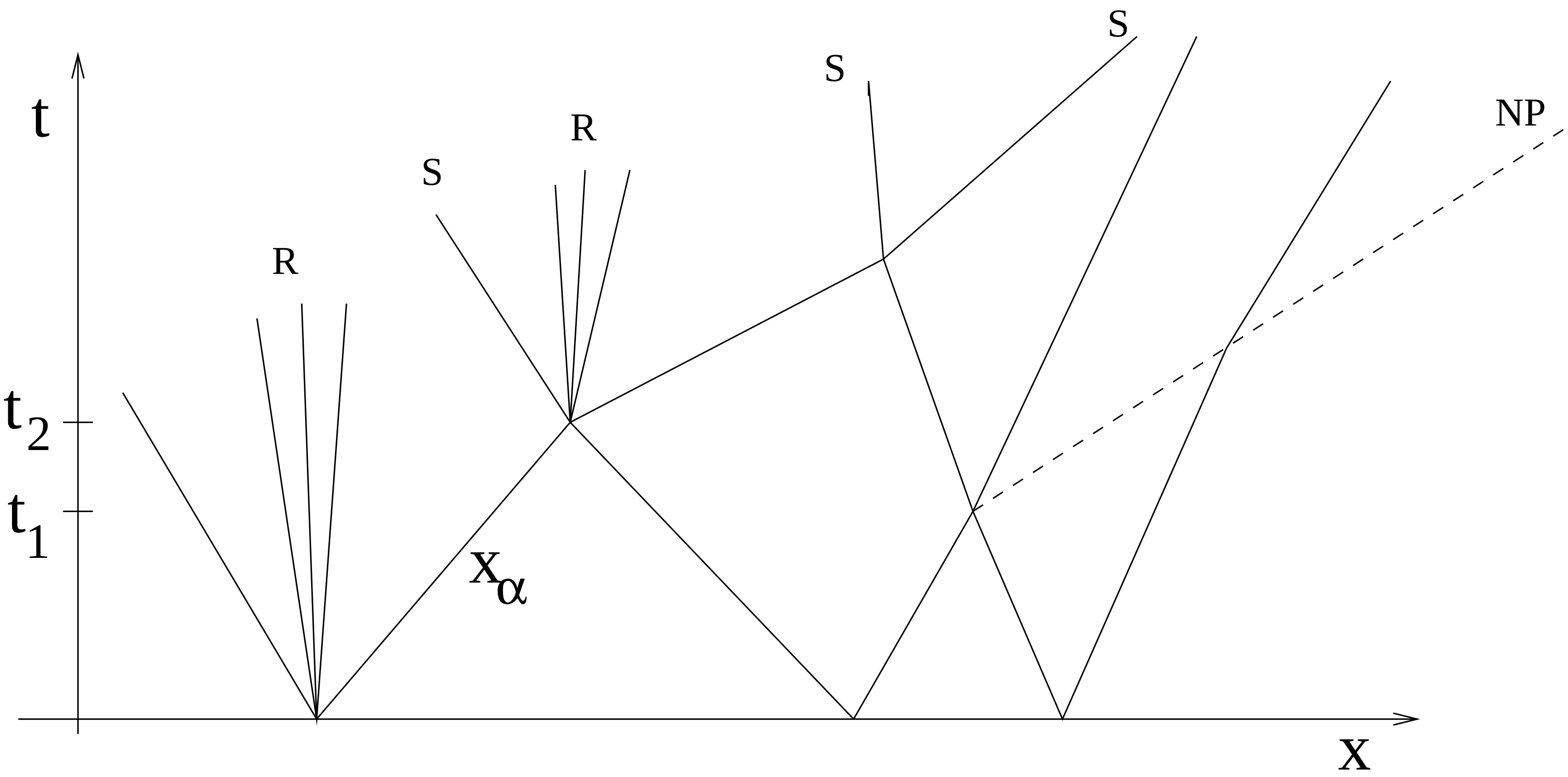
A sequence of approximate solutions

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

let $\varepsilon_\nu \to 0$ as $\nu \to \infty$

$(u_\nu)_{\nu\geq 1}$ sequence of approximate front tracking solutions

- initial data satisfy $\|u_\nu(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} \leq \varepsilon_\nu$
- all shock fronts in u_ν are entropy-admissible
- each rarefaction front in u_ν has strength $\leq \varepsilon_\nu$
- at each time $t \geq 0$, the total strength of all non-physical fronts in $u_\nu(t,\cdot)$ is $\leq \varepsilon_\nu$


t
t_2
t_1
R
S
R
S
S
NP
x_α
x

Existence of a convergent subsequence

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C$$

$$\begin{aligned} \|u_\nu(t) - u_\nu(s)\|_{\mathbf{L}^1} &\leq (t-s)\cdot [\text{total strength of all wave fronts}]\cdot[\text{maximum speed}] \\ &\leq L\cdot(t-s) \end{aligned}$$

Helly's compactness theorem $\Longrightarrow$ a subsequence converges

$$u_\nu \to u \qquad\qquad \text{in } \mathbf{L}^1_{loc}$$

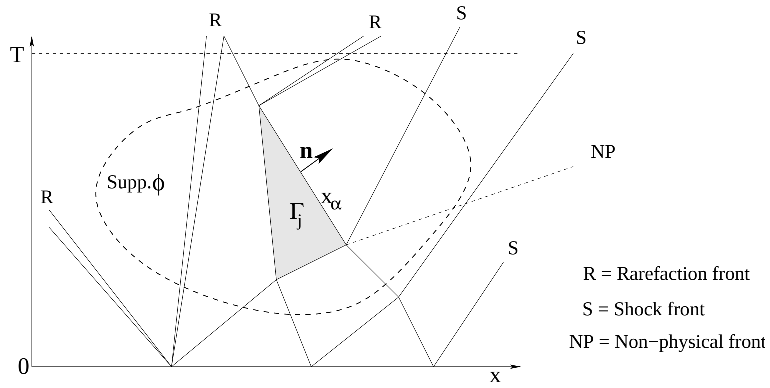
Claim: $u = \lim_{\nu\to\infty} u_\nu$ is a weak solution

$$\iint \{\phi_t u + \phi_x f(u)\}\, dxdt = 0 \qquad \phi \in \mathcal{C}^1_c\Big(]0,\infty[\times I\!R\Big)$$

Need to show:

$$\lim_{\nu\to\infty} \iint \Big\{\phi_t u_\nu + \phi_x f(u_\nu)\Big\}\, dxdt = 0$$

$$\int_0^\infty \int_{-\infty}^\infty \Big\{ \phi_t(t,x) u_\nu(t,x) + \phi_x(t,x) f(u_\nu(t,x)) \Big\}\, dxdt \qquad (*)$$



Assume $\phi(t,x) = 0$ outside the strip $[0,T] \times I\!R$. Define

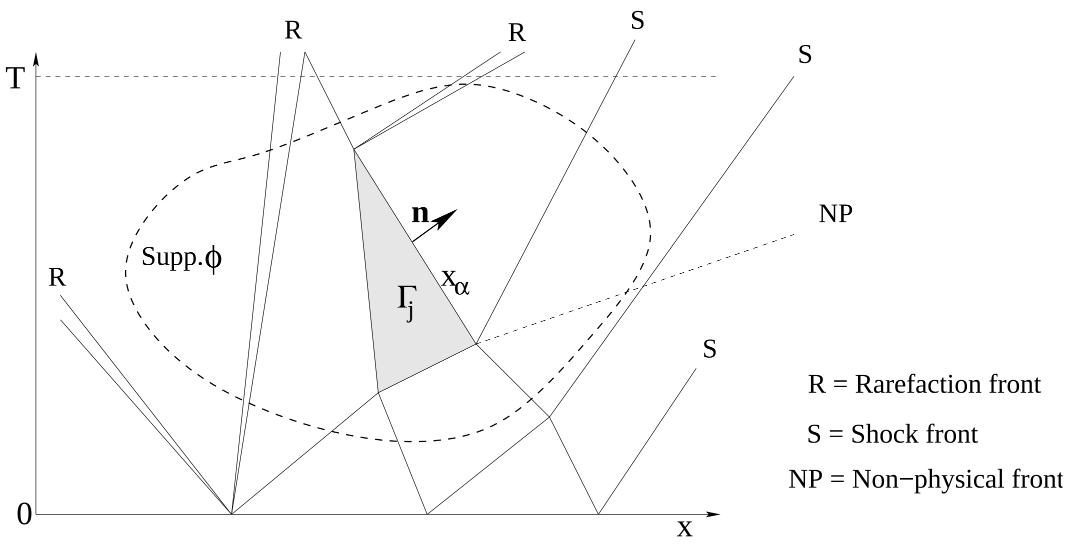
$$\Delta u_\nu(t,x_\alpha) \doteq u_\nu(t,x_\alpha+) - u_\nu(t,x_\alpha-)$$

$$\Delta f(u_\nu(t,x_\alpha)) \doteq f(u_\nu(t,x_\alpha+)) - f(u_\nu(t,x_\alpha-))$$

$$\Phi_\nu \doteq (\phi \cdot u_\nu,\ \phi \cdot f(u_\nu)).$$

Use the divergence theorem on each polygonal domain Γ_j where u_ν is constant:

$$(*) \ = \ \sum_j \iint_{\Gamma_j} \text{div}\ \Phi_\nu(t,x)\ dxdt \ = \ \sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n}\ d\sigma$$



$$\limsup_{\nu\to\infty}\left|\sum_j \int_{\partial\Gamma_j} \Phi_\nu \cdot \mathbf{n}\ d\sigma\right|$$

$$\leq \limsup_{\nu\to\infty}\left|\sum_{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}} \Big[\dot{x}_\alpha(t)\cdot\Delta u_\nu(t,x_\alpha) - \Delta f(u_\nu(t,x_\alpha))\Big]\phi(t,x_\alpha(t))\right|$$

$$\leq \limsup_{\nu\to\infty}\left\{\mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{R}}\varepsilon_\nu|\sigma_\alpha| \ + \ \mathcal{O}(1)\cdot\sum_{\alpha\in\mathcal{NP}}|\sigma_\alpha|\right\}\cdot\max_{t,x}|\phi(t,x)|$$

$$= 0$$

The Glimm scheme

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

Assume: all characteristic speeds satisfy $\lambda_i(u) \in [0,1]$

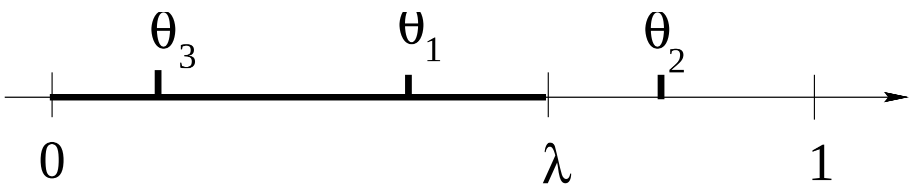
This is not restrictive. If $\lambda_i(u) \in [-M, M]$, simply change coordinates:

$$y = x + Mt, \qquad\qquad \tau = 2Mt$$

Choose:

- a grid in the t-x plane with step size $\Delta t = \Delta x$
- a sequence of numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over $[0,1]$

$$\lim_{N\to\infty} \frac{\#\{j \;;\; 1 \le j \le N, \quad \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad\qquad \text{for each } \lambda \in [0,1].$$

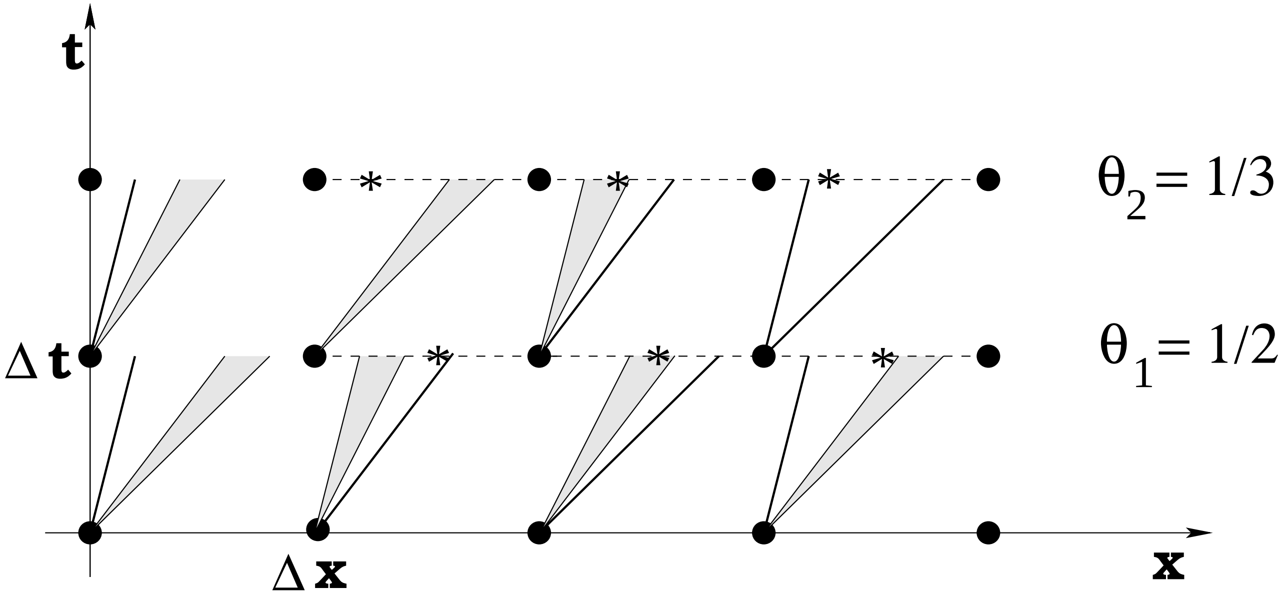


Glimm approximations

Grid points : $x_j = j \cdot \Delta x, \qquad t_k = k \cdot \Delta t$

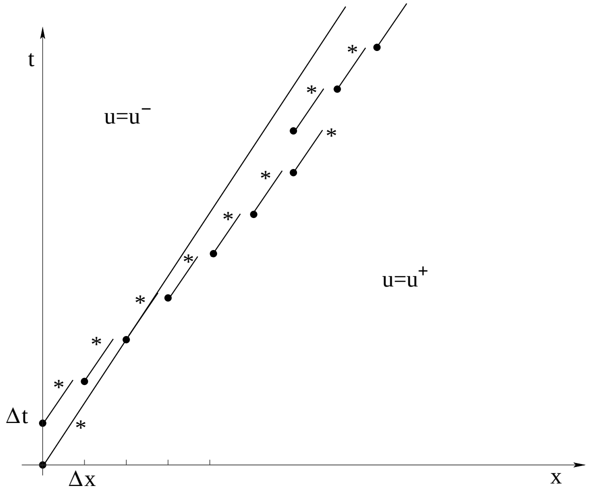
• for each $k \geq 0$, $u(t_k, \cdot)$ is piecewise constant, with jumps at the points x_j. The Riemann problems are solved exactly, for $t_k \leq t < t_{k+1}$

• at time t_{k+1} the solution is again approximated by a piecewise constant function, by a sampling technique



Example: Glimm's scheme applied to a solution containing a single shock

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t, \\ u^- & \text{if} \quad x < \lambda t. \end{cases}$$



Fix $T > 0$, take $\Delta x = \Delta t = T/N$

$$x(T) = \#\{j \; ; \quad 1 \le j \le N, \quad \theta_j \in [0, \lambda] \; \} \cdot \Delta t$$

$$= \frac{\#\{j \; ; \quad 1 \le j \le N, \quad \theta_j \in [0, \lambda] \; \}}{N} \cdot T \quad \to \quad \lambda T \qquad\qquad \text{as} \quad N \to \infty$$

Rate of convergence for Glimm approximations

Random sampling at points determined by the equidistributed sequence $(\theta_k)_{k\geq 1}$

$$\lim_{N\to\infty} \frac{\#\{j\ ;\quad 1\leq j\leq N,\quad \theta_j\in[0,\lambda]\ \}}{N} = \lambda \qquad\qquad \text{for each } \lambda\in[0,1].$$

Need fast convergence to uniform distribution. Achieved by choosing:

$$\theta_1 = 0.1\,,\qquad \ldots\qquad ,\qquad \theta_{759} = 0.957\,,\qquad \ldots\qquad ,\qquad \theta_{39022} = 0.22093\,,\qquad \ldots$$

Error Estimate:
$$\lim_{\Delta x\to 0} \frac{\|u^{\text{Glimm}}(T,\cdot) - u^{\text{exact}}(T,\cdot)\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot|\ln \Delta x|} \;=\; 0$$

(A.Bressan & A.Marson, 1998)