LECTURE 17

MAPPING DATA

JUNE 25, 2009

$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ covering of X

$C(X, \mathcal{U}) =$ Simp cplx

Vertex set $= \{A\}$

$\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ k simplex $\Leftrightarrow$ $U_{\alpha_0} \cap \dots \cap U_{\alpha_k} \neq \emptyset$

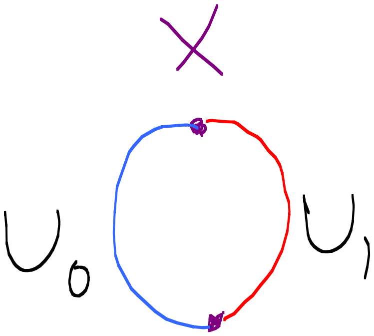
"Mayer Vietoris Blow up"

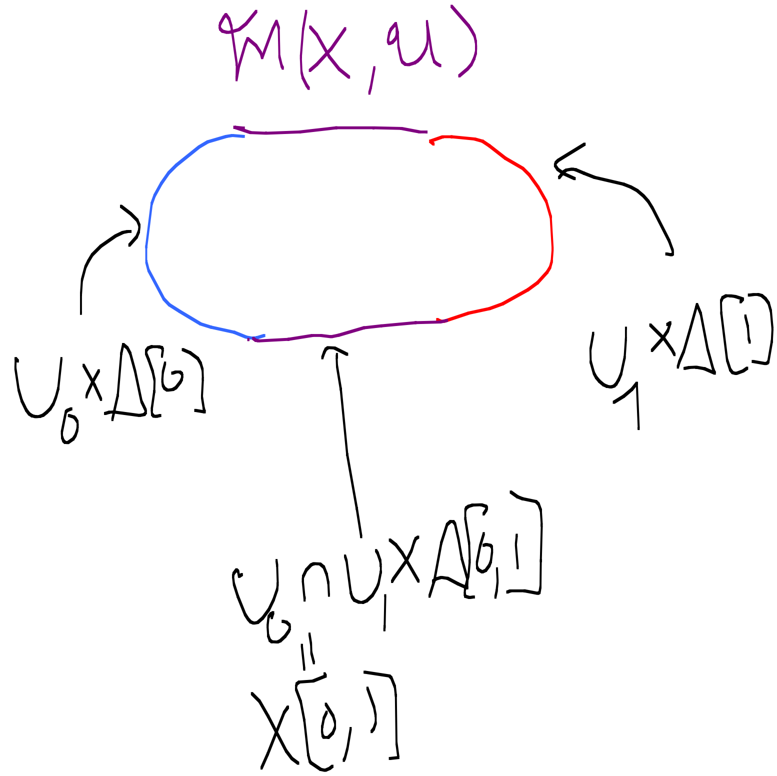
$$\mathfrak{M}(X,\mathfrak{U}) \subseteq X \times \Delta[A]$$

$\emptyset \neq S \subseteq A$, $\Delta[S]$ = faces spanned by S,

$$X[S] = \bigcap_{s \in S} U_{\cancel{\alpha} s}$$

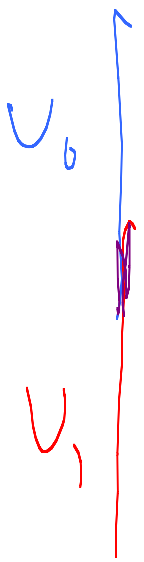
$$\mathfrak{M}(X,\mathfrak{U}) = \bigcup_{\emptyset \neq S \subseteq A} X[S] \times \Delta[S]$$

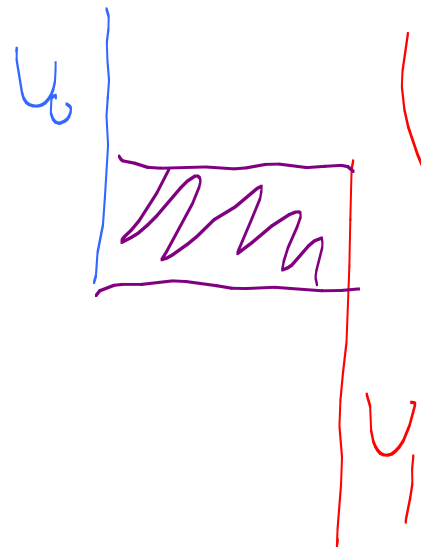




$N(\mathfrak{U}) =$ 

$\not\simeq S^1$





$\amalg$: disjoint union

L.E.S OF PAIR $(M(U), U_0 \amalg U_1)$

$$\mathfrak{M}(\mathcal{U}) / U_0 \amalg U_1 \simeq \sum U_0 \cap U_1$$

$$\cdots \to H_\lambda(U_0 \amalg U_1) \longrightarrow H_\lambda(\mathfrak{M}) \longrightarrow H_\lambda\left(\sum U_0 \cap U_1\right) \to \cdots$$

$$\cong$$

$$\cdots \to H_\lambda(U_0) \oplus H_\lambda(U_1) \to \underbrace{H_\lambda(\mathfrak{M})}_{\cong H_\lambda(X)} \longrightarrow H_{\lambda-1}(U_0 \cap U_1) \to \cdots$$

Thm: For reasonable X & $\mathcal{U}$,

$$X \simeq \mathfrak{M}(\mathcal{U})$$

Pf $\mathfrak{M}(\mathcal{U}) \xrightarrow{\pi_X} X$ (projection on X factor)

If $\mathcal{U}$ is paranumerable (admits a partition of unity subordinate to $\mathcal{U}$)

$\{\varphi_\alpha\}$ is such a partition of 1

$$X \longrightarrow X \times \Delta[A]$$

$(\mathrm{id}, \varphi_0, \varphi_1, \dots, \varphi_n)$ is a map from X to $X \times \Delta[A]$ which lands in $m(\mathcal{U})$

Produces homotopy inverse to π_X

Note: $\exists$ map

want

$$\begin{tikzcd}
\mathfrak{M}(\mathfrak{A}) \arrow[rr, "\pi\Delta[A]"] \arrow[dr, dashed] & & \check{C}(\mathfrak{A}) \\
& \underline{\underline{\mathfrak{M}^{\Pi\diamond}(\mathfrak{A})}} \arrow[ur, dashed] &
\end{tikzcd}$$

object of interest

Let X be a simplicial cplx

By a <u>simplex functor</u> on X, we'll mean

an assignment $\sigma \longrightarrow F(\sigma)$ ← space

simplex of X

so that to every $\tau \leq \sigma$ is associated a

cont map $F(\sigma) \longrightarrow F(\tau)$ $(F(\tau \leq \sigma))$

s.t. $F(\tau \leq \sigma) \circ F(\sigma \leq \xi) = F(\tau, \xi)$

Example: X a space, and $\mathcal{U}$ is a covering.

then $S \to X[S]$ is a simplicial functor on $C(\mathcal{U})$

$S \subseteq S'$, $X[S'] \hookrightarrow X[S]$

Example: For any simplicial cplx, the constant functor w/ value $\{pt\}$

Example: $S \to \pi_0 X[S]$, ~~is~~ a simplicial functor on $C(\mathcal{U})$

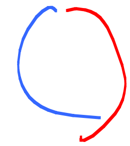
$$X[S] \longrightarrow \pi_0 X[S] \longrightarrow \{pt\}$$

Given any simplicial functor F on simplicial cplx X, form

$$\underline{\text{hocolim}}\, F = \coprod_{\sigma \in \Sigma(X)} F(\sigma) \times \Delta[\sigma] \Big/ \cong \qquad \tau \leq \sigma$$

$$F(\sigma) \times \Delta[\sigma] \longleftarrow F(\sigma) \times \Delta[\tau] \longrightarrow F(\tau) \times \Delta[\tau]$$

$$\begin{array}{lll}
\mathfrak{M}(\mathcal{U}) \simeq \underset{\longrightarrow}{\mathrm{hocolim}}\, X[-] & & \simeq S^1 \\
\qquad\qquad\qquad\qquad\quad \downarrow & & \\
\mathfrak{M}^{\pi_0}(\mathcal{U}) \simeq \underset{\longrightarrow}{\mathrm{hocolim}}\, \pi_0 X[-] & & \simeq S^1 \\
\qquad\qquad\qquad\qquad\quad \downarrow & & \\
\check{C}(\mathcal{U}) \simeq \underset{\longrightarrow}{\mathrm{hocolim}}\, \{pt\} & & \simeq [0,1]
\end{array}$$



<u>Ingredients:</u> Coverings, needed a way to form π_0

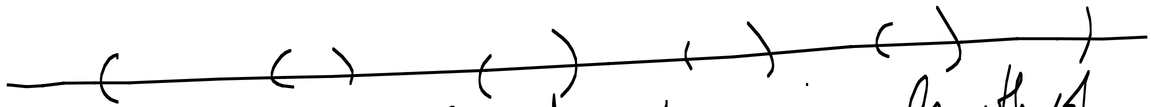
<u>Point Clouds:</u> Coverings make sense

$\pi_0 \rightsquigarrow$ clustering

Given a covering & clustering scheme, can construct a simplicial cplx

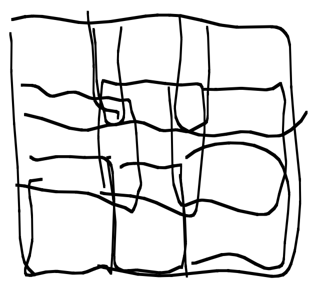
How to produce coverings?

$\mathbb{R}$

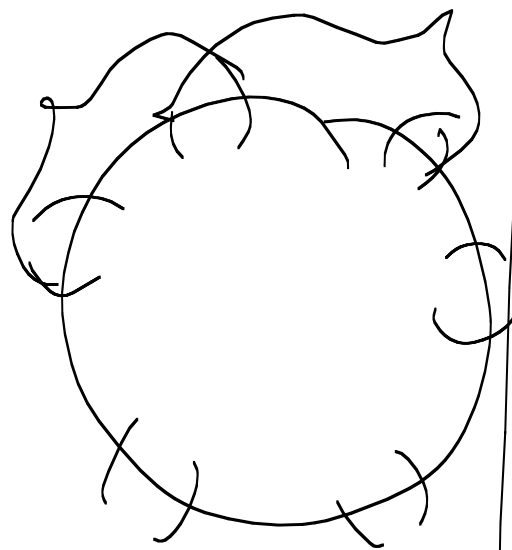


parametrized family of coverings, length of intervals + degree of overlap are params

$\mathbb{R}^2$



S^1



Suppose our pt cloud X is equipped w/ reference map to ~~onto~~ a foundation metric space (i.e. one with a known family of coverings)

$X \xrightarrow{\rho} Z \qquad \mathcal{U} = \{V_\alpha\}_{\alpha \in A}$ covering of Z

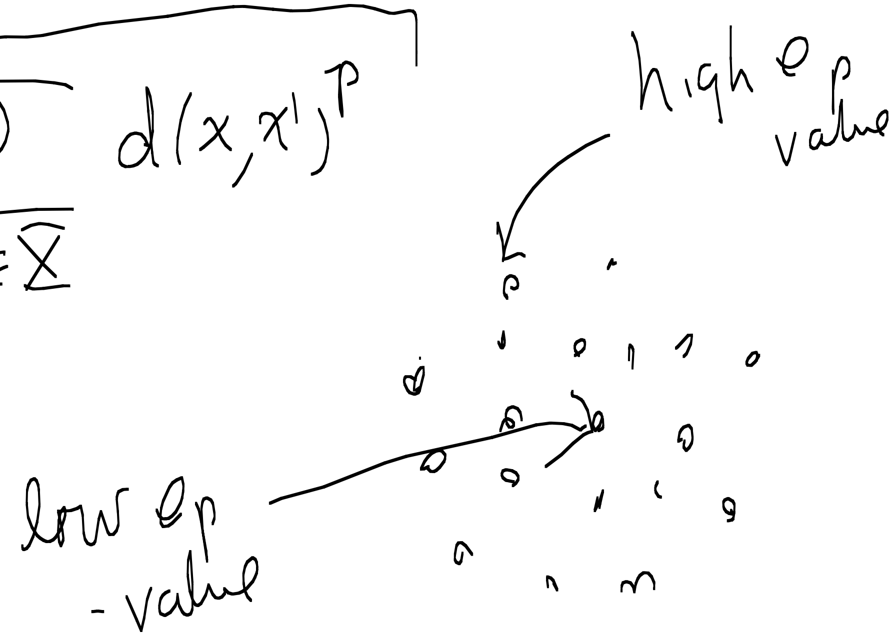
$\{\rho^{-1} V_\alpha\}_{\alpha \in A}$ becomes a covering of X

Reference Maps

(a) Let $\delta : X \to \mathbb{R}$ be any density estimator

(b) "Eccentricity" or "Data depth". param p

$$e_p(x) = \sqrt[p]{\sum_{x' \in \bar{X}} d(x,x')^p}$$



(One could use (δ, e_p) as an $\mathbb{R}^2$-valued "filter" = reference map)

(c) Reference map to S^1 could be T (mod 24 hrs)

(d) <u>Any user defined function</u>

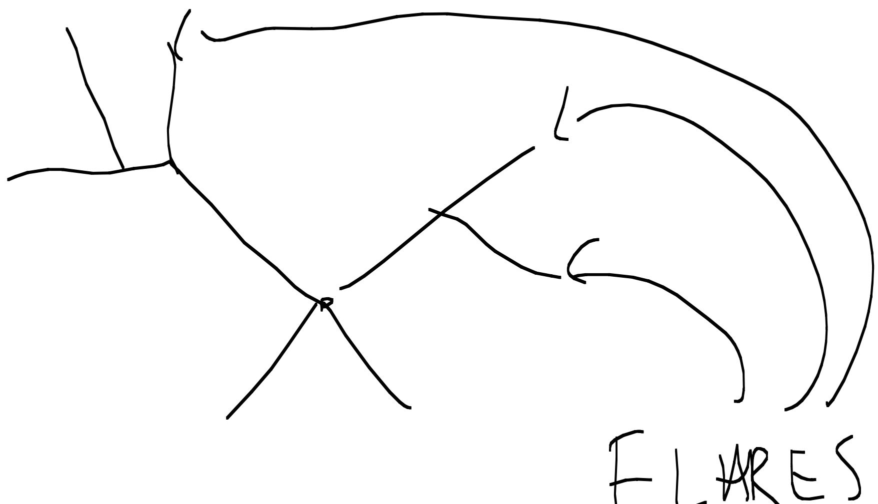
Call the Construction

Mapper

Choosing Scale Parameters + Filters is time consuming

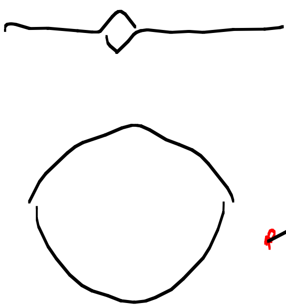
Can we automate it?

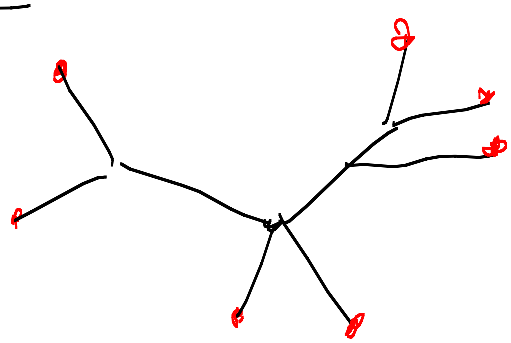
Homology can identify "interesting" values



TYPICAL OUTPUT

SUPPOSE WE CAN IDENTIFY "BOUNDARY" OF AN OUTPUT





BOUNDARY $= \partial X$

$H_*(X, \partial X)$ COUNTS FLARES

SCALE SPACE (METHOD FOR CHOOSING
SOME SCALE VALUES)

(JOINT W: G. SINGH, F. MEMOLI)

SUPPOSE THAT CLUSTERING METHOD IS SINGLE LINKAGE, BUT THAT WE PERMIT VARIABLE CHOICES OVER α's

FOR EACH α, PRODUCE A β_0-BARCODE

WANT TO MAKE A CHOICE

$$\alpha \longrightarrow \varepsilon_\alpha \longleftarrow \text{parameter choice on } \rho^{-1}U_\alpha$$

For each α, we obtain a family of intervals (stability intervals)

STABILITY INTERVALS ARE INTERVALS OF CONSTANCY FOR #(CLUSTERS)

$\alpha \longrightarrow I_\alpha$ (stability interval for $\rho^{-1}(U_\alpha)$)

HEURISTIC – IF $U_\alpha \cap U_{\alpha'} \neq \emptyset$ WANT $I_\alpha \cap I_{\alpha'}$ to be non-empty (LARGE)

HAVE A SIMPLICIAL COMPLEX OVER $\check{C}(\mathcal{U})$, whose vertices are pairs (α, I_α)

$\{(\alpha_0, I_0), \ldots, (\alpha_k, I_k)\}$ SPANS A SIMPLEX

IF $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k} \neq \emptyset$

$I_0 \cap \quad \cap I_k \neq \emptyset$

$SS(\mathcal{U})$

$$\begin{tikzcd} SS(\mathfrak{A}) \arrow[d, shift right, "\pi"'] \\ \check{C}(\mathfrak{A}) \arrow[u, shift right, "s"'] \end{tikzcd}$$

π forgets I_α

Scale shove amounts to a section of π

$$\pi \circ s = id_{\check{C}(\mathfrak{A})}$$

Can cook up objective functions based on degree of overlap of stability intervals, and can frequently optimize such measures

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Multiscale/Multiresolution

Z $\qquad$ $\mathcal{U}, \mathcal{U}'$ coverings of Z

reference space

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}, \quad \mathcal{U}' = \{U'_\beta\}_{\beta \in B}$$

A map of coverings $\mathcal{U} \longrightarrow \mathcal{U}'$ is a set map

$$\theta : A \to B \text{ s.t. } U_\alpha \subseteq U'_{\theta(\alpha)} \quad \forall \alpha \in A$$

Given a map of coverings, we obtain maps of the corresponding Mapper Constructions. Studying "persistence" of features (loops, flares) over such maps helps to confirm significance of features