HOMOTOPY COLIMITS + LIMITS

∞ Unions

$$X_0 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots \qquad \bigcup_{i=0}^{\infty} X_i$$

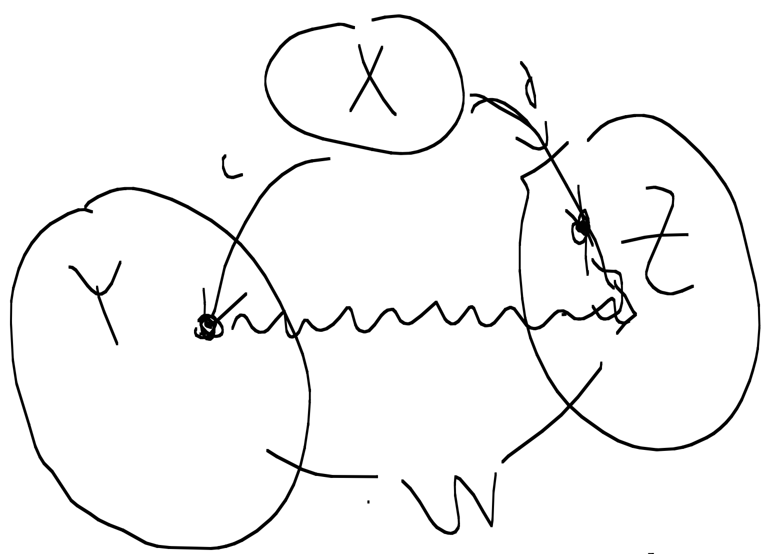
Pushouts

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "j"'] & Y \arrow[d] \\ Z \arrow[r] & ? \end{tikzcd}$$

$$? = \frac{Y \amalg Z}{\sim}$$

$\sim$

$\underset{(\in Y)}{i(x)} \simeq \underset{\in Z}{j(x)}$



If I have $W = Y \cup Z$, $W \cong Y \cup_{Y \cap Z} Z$

simp complex

$P = \text{Pushout}\begin{pmatrix} X \xrightarrow{i} Y \\ \downarrow j \\ Z \end{pmatrix}$ Satisfies

$\varphi : Y \to W$
$\psi : Z \to W$

s.t. $\varphi \circ i = \psi \circ j$, then

$\exists!$ map $\theta : P \to W$

s.t. $\underbrace{Y \to P \xrightarrow{\theta} W}_{=\varphi}$

SIMILARLY FOR ψ.

Suppose we have a group action on X by G

Form X/G = set of orbits, equipped w/ quotient topology

S^2 w/ $\mathbb{Z}_2$ action by antipodal map

$S^2/\mathbb{Z}_2 = \mathbb{RP}^2$

All are examples of <u>colimits</u>.

$\underline{C}$ any category, $F: \underline{C} \longrightarrow \underline{Top}$ be any functor

$$\underset{\underline{C}}{\underrightarrow{\mathrm{colim}}}\, F = \coprod_{c \in \mathrm{ob}\,\underline{C}} F(c) \Big/ \simeq$$

$\simeq$ generated by $\underset{\text{in } \underline{C}}{c \xrightarrow{f} c'}$ $\quad F(c) \ni x \;\; F(c) \xrightarrow{F(f)} F(c')$

$x \simeq F(f)(x)$

What are categories in each of examples?

(a) $\bullet \to \bullet \to \bullet \to \bullet \to \bullet \cdots$ or Poset $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

where there is a unique morphism

$i \longrightarrow j$ iff $i \leq j$.

No morphism $i \to j$ if $i > j$

(b)
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \\ C & \end{tikzcd}$$

A, B, C are objects, id_A, id_B, id_C,

i, j

(c) G a group, regard G as a category $\underline{G}$,
w/ one object $*$, and one morphism for every group element (composition = mult in grp)

What is a functor $\underline{G} \xrightarrow{\Phi} \underline{Top}$?

Consists of a space X $(= \Phi(*))$ together w/ an action of G on X, colimit is the orbit space.

It is desirable to introduce notion of homotopy colimit

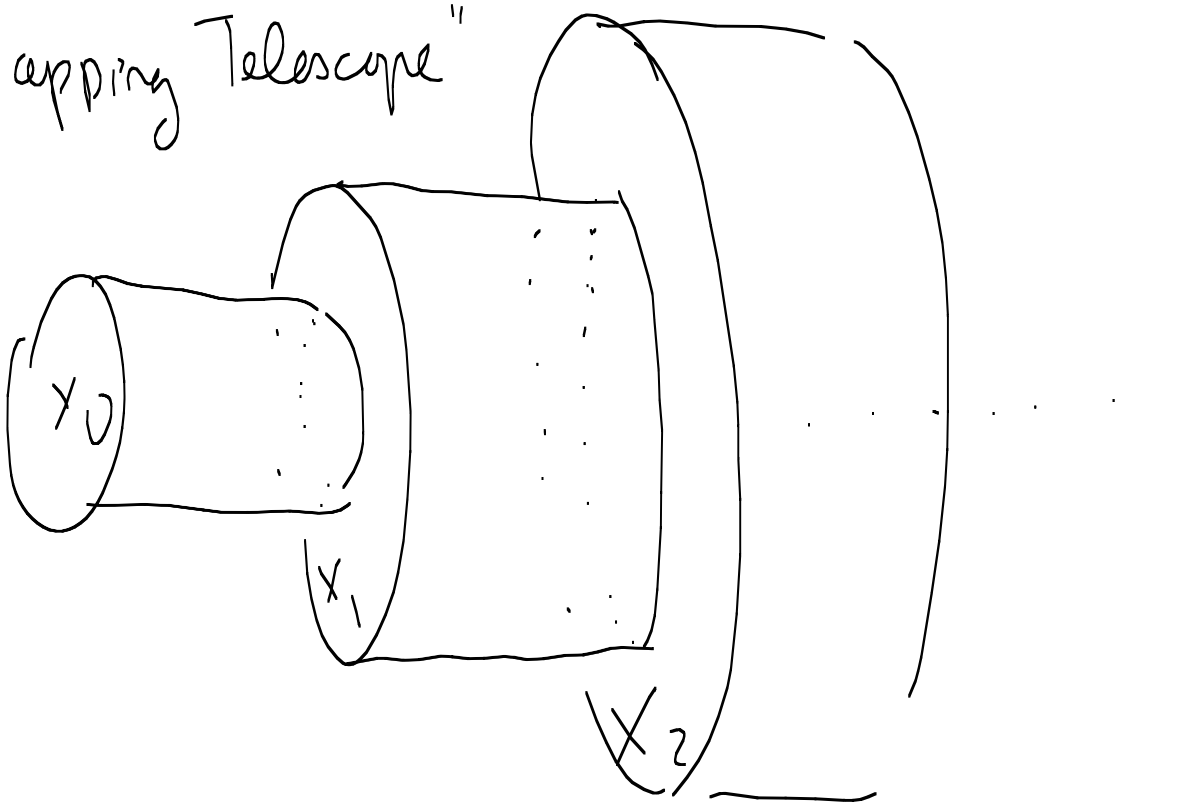
Instead of creating equalities, we create paths between points

(a) $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$

homotopy colimit is $\mathrm{hocolim}\, X_i = \coprod X_i \times [0,1] \big/ \sim$

$$\begin{array}{ccc} X_i \times [0,1] & \longrightarrow & X_{i+1} \times [0,1] \\ \cup & & \cup \\ (x,1) & \rightsquigarrow & (f_i(x), 0) \end{array}$$

"Mapping Telescope"



"Homotopy Pushout" (Double Mapping Cylinder)

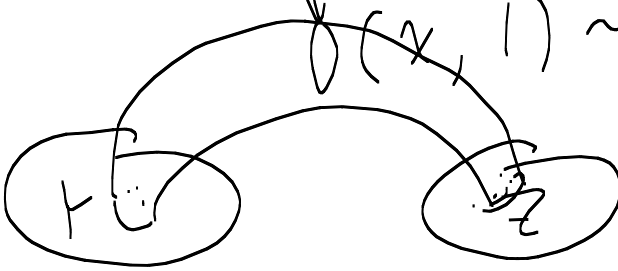
$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array}$$

$$P = X \times [-1,1] \amalg Y \amalg Z / \simeq$$

$X \ni x$

$(x,-1) \rightsquigarrow f(x) \in Z$

$(x,1) \rightsquigarrow g(x) \in Y$



Group Action of G on X

$G = \mathbb{Z}/2 = \{1, T\}$ $\quad$ $\mathbb{Z}/2$ acts freely on S^n by antipodal map

(no fixed pts of T)

$$S^n \subseteq S^{n+1} \subseteq \cdots \qquad EG = \bigcup_n S^n$$

G acts freely on EG, and $EG \simeq *$

Homotopy Colimit is (Homotopy orbit space)

$$\underset{G}{EG \times X}$$

Take the orbit space of diagonal action on $EG \times X$.

G acts by identity on $*$, $\operatorname{hocolim}_G * = EG/G = BG$

$G = \mathbb{Z}/2 \qquad \bigcup S^n/\pm 1 = \bigcup \mathbb{R}P^n = \mathbb{R}P^\infty$

homotopy colimit has an invariance property

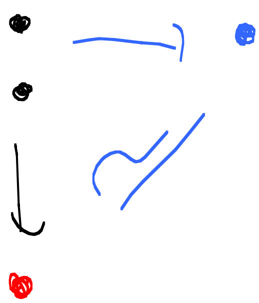
$$\begin{tikzcd} & \underline{\mathrm{Top}} \\ \underline{C} \arrow[ur, "F"] \arrow[dr, "G"'] & \\ & \underline{\mathrm{Top}} \end{tikzcd}$$

$N(c): F(c) \to G(c)$ all c, each $N(c)$ is a htpy equiv.

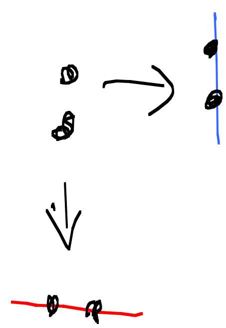
Then there is an induced equivalence

$$\mathrm{hocolim}\, F \xrightarrow{\sim} \mathrm{hocolim}\, G.$$

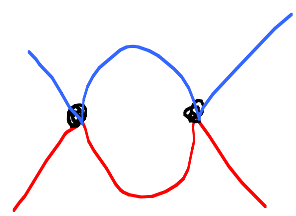
Fails for ordinary colimits



Colimit is one pt



colimit =



LIMITS? (INVERSE LIMIT)

$$\cdots \to X_2 \to X_1 \to X_0$$

$$\cdots \subseteq X_2 \subseteq X_1 \subseteq X_0 \subseteq X$$

LIMIT IS $\bigcap_{i=0}^{\infty} X_i$

PULLBACK

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

"Pullback"

$$= \{ (y, z) \mid f(y) = g(z) \}$$

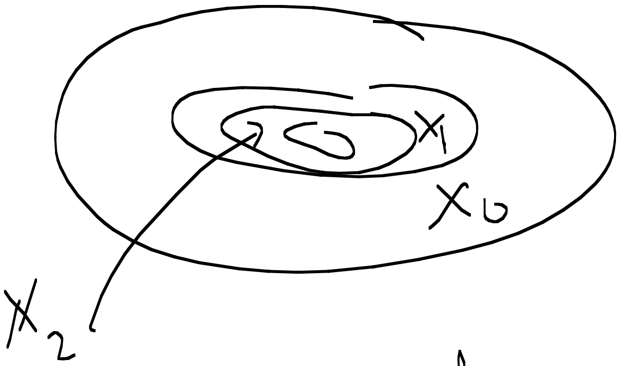
$$\text{"} \; Y \times_X Z \; \text{"}$$

G acts on X

$$\lim_{\underset{G}{\longleftarrow}} X \overset{?}{=}$$

Constraints x s.t. $gx = g'x$ for all g, g'.

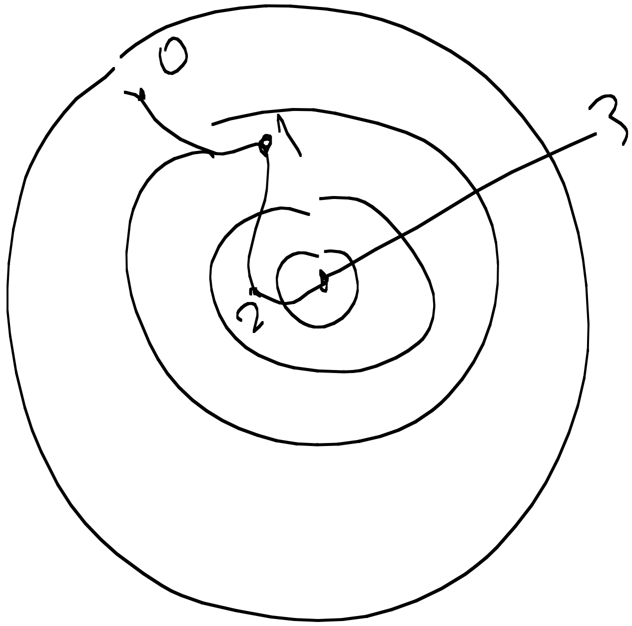
$$= X^G = \{x \in X \mid gx = x \ \forall g \in G\}$$



Homotopy limit is a space of maps (compact open top)

$$f : [0, +\infty) \longrightarrow X$$

$$f([n, +\infty)) \subseteq X_n \text{ for all } n \geq 0$$


0
1
2
3

Suppose $X_n = B_{1/n}(0) - \{0\} \subseteq \mathbb{R}^2$

$\bigcap X_n = \emptyset$

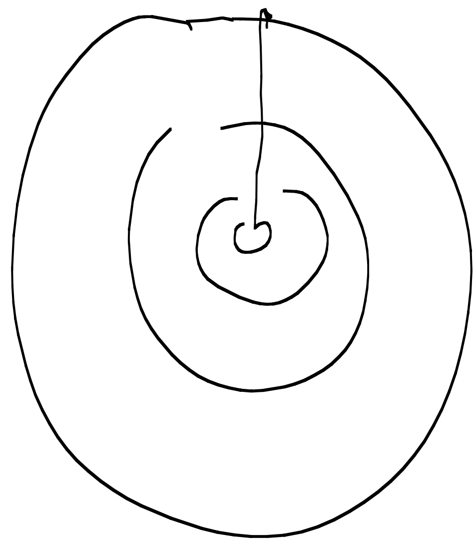
$\operatorname{holim}_{\overleftarrow{n}} X_n$ is non-empty. In fact, it has homotopy type of S^1.

$X_n = B_{1/n}(0) - \{0\} \simeq S^1$. The diagram of X_n's

is equivalent to the diagram

$$\to S^1 \xrightarrow{id} S^1 \xrightarrow{id} S^1$$

Therefore, limit is <u>not homotopy invariant</u>.



Homotopy pullback?

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[r, "q"] & X \end{tikzcd}$$

Mapping space $X^I \times Y \times Z$

$$\cup|$$

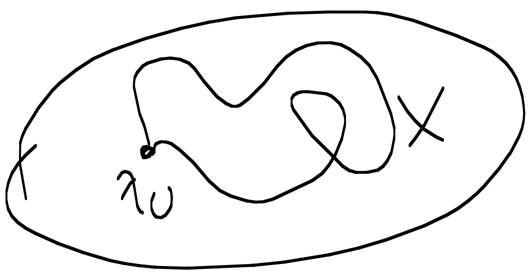
$$P$$

(f, y, z) s.t. $p(y) = f(0)$

& $q(z) = f(1)$

$$\begin{tikzcd} & \bullet \arrow[d, "x_0"] \\ \bullet \arrow[r, "x_0"'] & (X, x_0) \end{tikzcd}$$

Homotopy construction amounts to maps $I \xrightarrow{\alpha} X$ s.t. $\alpha(0) = x_0$ & $\alpha(1) = x_0$



$= \Omega(X, x_0)$ loop space of X based @ x_0

Group Action

X given a G-action. EG = contractible free G-space

$EG \to *$

$$\operatorname{holim}_G X \simeq X^{hG} = F^G(EG, X)$$

"homotopy fixed set"

$$F^G(*, X) = X^G$$

In general, $X^G \to X^{hG}$ is not an equivalence.

Sullivan Conjecture (1980's) Miller, Lannes, CC)

X a finite complex w/ action by a finite p-gp.

$X^G \xrightarrow{\eta} X^{hG}$ Th, η_* is an isomorphism for mod p homology.