Crofton Measure in Minkowski Geometry and Singularities

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Sphere bundle over Minkowski $p-$Space

For a Minkowski $p-$Space $\left(\mathbb{R}^2, ||\cdot||_p\right)$, $1 < p < \infty$, where $||(x,y)||_p = (|x|^p + |y|^p)^{1/p}$, the dual norm is $||\cdot||_{\frac{p}{p-1}}$. We know the unit ball B in this space is

$$\left\{(x,y) \in \mathbb{R}^2 : (|x|^p + |y|^p)^{1/p} \leq 1\right\}, \tag{1}$$

and its support function is $h_B((x,y)) = sup\left\{|\varphi((x,y))| : ||\varphi||_{\frac{p}{p-1}} \leq 1\right\}$ for $(x,y) \in B$, whose gradient ∇h_B at (x,y) is a bijection from $S_{(x,y)}\mathbb{R}^2$ to $S^*_{(x,y)}\mathbb{R}^2$,

$$\nabla h_B((\alpha,\beta)) = (\alpha^{p-1}, \beta^{p-1}). \tag{2}$$

Symplectic Form on The Space of Geodesics

Consider a diagram

$$\overline{Gr_1(\mathbb{R}^2)} \overset{projection}{\longleftarrow} S\mathbb{R}^2 \overset{(\nabla h_B)^{-1}}{\underset{\longleftarrow}{\simeq}} S^*\mathbb{R}^2 \overset{inclusion}{\hookrightarrow} T^*\mathbb{R}^2 \tag{3}$$

in which $\overline{Gr_1(\mathbb{R}^2)}$ is the space of geodesics in $\left(\mathbb{R}^2, ||\cdot||_p\right)$.

Theorem

There exists a symplectic form ω on the space of geodesics $\overline{Gr_1(\mathbb{R}^2)}$ in Minkowski space $\left(\mathbb{R}^2, ||\cdot||_p\right)$, which inherits from the canonical symplectic structure on $T^*\mathbb{R}^2$.

Gelfand Transform of Density on the Space of Geodesics

Consider the double fibration on $(\mathbb{R}^2, ||\cdot||_p)$ and $\overline{Gr_1(\mathbb{R}^2)}$:

$$\mathbb{R}^2 \overset{\pi_1}{\leftarrow} \mathcal{I} \overset{\pi_2}{\rightarrow} \overline{Gr_1(\mathbb{R}^2)},$$

where

$$\mathcal{I} = \left\{((x,y),l) : (x,y) \in l, (x,y) \in \mathbb{R}^2, l \in \overline{Gr_1(\mathbb{R}^2)}\right\} \tag{4}$$

is a set of incidence relations, π_1 and π_2 are the natural projections. For a density ϕ on $\overline{Gr_1(\mathbb{R}^2)}$, the Gelfand transform of ϕ is $GT(\phi) := \pi_{1*}\pi_2^*\phi$.

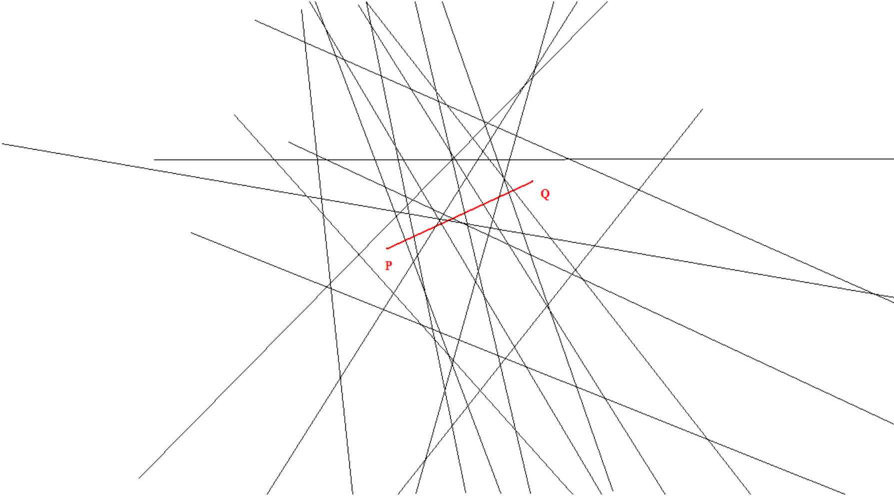
Theorem

Considering the Gelfand transform of ϕ, we have

$$GT(\phi)(v_1, v_2)) = 4||(v_1, v_2)||_p, \tag{5}$$

for any vector $(v_1, v_2) \in T_{(x,y)}\mathbb{R}^2$*,where*
$\phi = |(p-1)\alpha^{p-2} dx \wedge d\alpha + (p-1)\beta^{p-2} dy \wedge d\beta|$ *and*
$||(\alpha, \beta)||_p = 1$.

Measuring Length in Minkowski Space



The Crofton Measure for Length in Minkowski p-Space

Deriving from the previous theorem, we have

Theorem

Let c be a rectifiable curve in Minkowski $p-$space $(\mathbb{R}^2, ||\cdot||_p)$, then the length of c

$$\begin{aligned} L(c) &= \tfrac{1}{4}\int_{l((x,y);\alpha)\in\overline{Gr_1(\mathbb{R}^2)}} \chi(c\cap l((x,y);\alpha)) \\ &\qquad (p-1)(\alpha^{p-2}dxd\alpha - \tfrac{\alpha^{p-1}}{\beta}dyd\alpha), \end{aligned} \tag{6}$$

where $l((x,y);\alpha))$ is the line passing through (x,y) with direction (α,β). In other words, the Crofton measure for Minkowski length is $(p-1)(\alpha^{p-2}dxd\alpha - \frac{\alpha^{p-1}}{\beta}dyd\alpha)$.

Theorem

There exists a Crofton measure

$$\varphi(\Theta) dr d\Theta = \frac{(p-1)^2 |\Theta|^{p(p-2)} |\Omega|^{p^2-3p+1}}{||(\Theta,\Omega)||_{p(p-1)}^{(p-1)(2p-1)}} dr d\Theta, \tag{7}$$

where $|\Omega|^p + |\Theta|^p = 1$, *such that*

$$L(c) = \int_{l(r,\Theta)\in \overline{Gr_1(\mathbb{R}^2)}} \chi(c \cap l(r,\Theta)) \varphi(\Theta) dr d\Theta, \tag{8}$$

for any curve rectifiable curve c in Minkowski $p-$*space* $\left(\mathbb{R}^2, ||\cdot||_p\right)$.

Shortest Curve in Minkowski Spaces and Singularities

As $p = 1$ or $p = \infty$, the map ∇h_B is not a bijection between the sphere-bundle and cosphere-bundle of $\left(\mathbb{R}^2, || \cdot ||_p\right)$ in (2), as the unit ball in the space is not strictly convex.
Let us now consider the shortest path between two points in Minkowski space. By Euler-Lagrange Equation, we can obtain

Theorem

The straight lines joining two points in Minkowski space with strictly convex unit ball is the only shortest curve joining them.

- When $1 < p < \infty$, the unit ball in $(\mathbb{R}^2, ||\cdot||_p)$ is strictly convex, thus the straight line segment joining two point is the shortest path between them.
- When $p = 1$, we don't get a Crofton measure from taking p to infinity in Formula (7) for $1 < p < \infty$. On the other hand, the geodesic in the space is not unique, in other words, the theorem above fails.
- When $p = \infty$, it is similar to $p = 1$.

Shortest Paths in $(\mathbb{R}^2, ||\cdot||_1)$

Choose two points P and Q in $(\mathbb{R}^2, ||\cdot||_1)$, which satisfies that $\overrightarrow{PQ}$ is neither colinear to x axis, nor y axis, since the metric is convex, then the straight line segment is one of the shortest curves connecting P and Q.
Moreover, curves like "staircase" have the equal length with the straight line segment as $||(x, y)||_1 = ||(x, 0)||_1 + ||(0, y)||_1$.

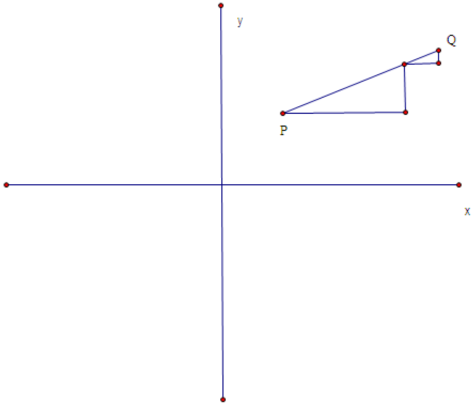


Figure 4.1 Paths of Equal Length

In Figure 4.1, the shortest curve joining P and Q can be the straight line or any "staircase" curves connecting P and Q.