1. Consider the integrate-and-fire model

$$\tau dV/dt = I - V, \; V(t^-) = V_{th}(t^-) \; \Rightarrow \; V(t^+) = V_r,$$

with $I > V_{th}$ and with an adapting threshold $V_{th}(t)$ that satisfies

$$\tau dV_{th}/dt = -V_{th}, \; V(t^-) = V_{th}(t^-) \; \Rightarrow \; V_{th}(t^+) = V_{th}(t^-) + \Delta.$$

Derive a formula for the firing rate for a periodic solution to this model.

2. Consider a nondimensionalized integrate-and-fire model of the form

$$dV/dt = E - V + I(t), \; V(t^-) = V_{th} \; \Rightarrow \; V(t^+) = V_r,$$

where $E = -10, V_{th} = 0, V_r = -25$. Suppose that $\int_0^{1000} I(t)dt = K$, a fixed constant. Explore which form of $I(t)$ gives the highest firing rate over T time units. Try various (K, T) combinations, e.g. $K = 200, T = 1000$ (or greater). Examples of $I(t)$ might include a function that is a nonzero constant over some time interval and then 0 for the rest of the alloted time, a sinusoidal function, or a series of short, concentrated pulses.

3. Analytically compute the period of oscillations in the quadratic integrate-and-fire model,

$$dV/dt = I + V^2, \; V(t-) = V_{th} \; \Rightarrow \; V(t^+) = V_r,$$

in the following cases:

   (a) $I > 0$,

   (b) $I < 0$ and $V_r > \sqrt{|I|}$.

4. Find changes of variables that show that the Izhikevich model

$$\begin{aligned} \dot{v} &= I + v^2 - u, \\ \dot{u} &= a(bv - u), \end{aligned}$$

with threshold $v = v_t$ and reset $v = v_r$, is equivalent to each of the following:

   (a) QIF with a leaky dendrite:

$$\begin{aligned} \dot{v} &= k + v^2 + g_1(v_d - v), \\ \dot{v_d} &= g_L(v_L - v_d) + g_2(v - v_d). \end{aligned}$$

(b)

$$\begin{aligned} c\dot{v} &= k(v - v_r)(v - v_t) - u + I, \\ \dot{u} &= a[b(v - v_r) - u]. \end{aligned}$$

5. Plot the ionic current
$$I = PFz\xi \left(\frac{[C]_{out} e^{-\xi} - [C]_{in}}{e^{-\xi} - 1} \right)$$
as a function of $\xi = zFV_m/RT$ (hence effecively as a function of V_m) for various values of $[C]_{out}/[C]_{in}$.

6. Write the voltage equation for the Hodgkin-Huxley (HH) model as $CdV/dt = F(V, m, h, n) + I$. Assume that when $m = m_\infty(V), h = h_\infty(V)$, and $n = n_\infty(V)$, F is a monotone function of V, such that the model has a unique critical point for each I. Write down the (mammilary) Jacobian matrix (4x4) from linearization of the model about the critical point for arbitrary fixed I. Use the Routh-Hurwitz criteria to derive a condition under which an Andronov-Hopf bifurcation occurs.

7. For each $x \in \{m, h, n\}$, define a new function $V_x(t)$ by writing $x(t) = x_\infty(V_x(t))$. Using the HH model, derive an ordinary differential equation for each $V_x(t)$. Solve the resulting system numerically for I sufficiently large that oscillations occur. Check the claim that $V_m \approx V$ and $V_h \approx V_n$. Setting $V_m = V$ and $V_n = V_h$, plot the nullclines for the (V, V_h) system. Determine conditions under which oscillations can or cannot occur.

8. Couple together two identical elliptic bursting cells (see Bard's collection of ode files), each with applied current I, with chemical synapses. Determine whether they synchronize at the level of bursts and/or spikes. Try this under a variety of conditions, such as with excitatory or inhibitory synapses with fast or slow decay, with a variety of different values of applied current I.

9. Consider the HH model written as
$$\begin{aligned} \dot{V} &= F + I, \\ \dot{m} &= M, \\ \dot{n} &= (1/\tau_n)N, \\ \dot{h} &= (1/\tau_h)H, \end{aligned}$$

for constants τ_n, τ_h. Generate the bifurcation diagram, with bifurcation parameter I, for the desingularized system

$$\begin{aligned} \dot{h} &= -F_v H/\tau_h, \\ \dot{V} &= F_n N/\tau_n + F_h H/\tau_h, \end{aligned}$$

for various values of (τ_h, τ_n). Be sure to include singularities $(H = N = 0)$ and folded singularities $(F_v = F_n N/\tau_n + F_h H/\tau_h = 0)$. Determine for which ranges of I there exist folded nodes, and hence the possibility of mixed mode oscillations.

10. Suppose that two identical cells send fast, direct, excitatory synaptic input to a third cell, and that this cell sends fast, direct inhibitory synaptic input to the first two. Represent each cell as a two-dimensional system

$$\begin{aligned} \dot{v}_i &= f(v_i, w_i) + I_{syn}, \\ \dot{w}_i &= \epsilon(w_\infty(v_i) - w_i)/\tau(v_i), \end{aligned} \tag{1}$$

with $0 < \epsilon \ll 1$ and with cubic v nullcline and monotone w nullcline, as discussed in the lecture on small networks. (Here, $I_{syn} = g_{syn} \sum s_j (v_{syn} - v_i)$ where the s_j come from cells that give inputs to cell i and where v_{syn} depends on whether the input is excitatory or inhibitory, and $\dot{s}_i = \alpha H(v_i - \theta_s)(1 - s_i) - \beta s_i$.) Determine what periodic singular solutions exist for this network, assuming either that the excitatory cells are oscillatory and the inhibitory cell is excitable or that all of the cells are excitable. Repeat this exercise with slow synaptic decay, $\beta = \epsilon\hat{\beta}$.

11. Suppose that two cells evolve according to (1) and are coupled to each other with fast, direct excitatory synapses with $s \in I_s := [0, s_{max} = \alpha/(\alpha + \beta)]$. Suppose that one cell has a critical point on the left branch of its cubic v nullcline for all $s \in I_s$ and the other has a critical point on the right branch of its cubic v nullcline for all $s \in I_s$. Determine conditions under which a periodic oscillation exists for the coupled pair of cells (see Rubin, Phys. Rev. E, 2006).