On sparse polynomial systems, Mixed Volume and Condition Numbers

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ABSTRACT

A *sparse polynomial* is an expression of the form:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} f_{\mathbf{a}} x_1^{a_1} \dots x_n^{a_n}$$

where $A \subset (\mathbb{Z}^+)^n$ is a finite set, known as the *support* of f. We consider a system of equations of the form

$$f_1(\mathbf{x}) = \dots = f_n(\mathbf{x}) = 0$$

where each f_i is a sparse polynomial with support A_i. The number of solutions in $(\mathbb{C}^*)^n$ for a *generic* choice of the coefficients $f_{\mathbf{a}} \in \mathbb{C}$ in the equation above is a geometric invariant known as the *scaled mixed volume* of $\mathcal{A}_1, \dots, \mathcal{A}_n$, where $\mathcal{A}_i$ is the smallest convex set containing A_i. We will see that the scaled mixed volume is the integral over $(\mathbb{C}^*)^n$ of a certain *mixed volume form*, corresponding to a certain *toric compactification* of $(\mathbb{C}^*)^n$.

A non-degenrate solution $\mathbf{x}_0 \in (\mathbb{C}^*)^n$ of a polynomial system $\mathbf{f}_0$ may be extended to an implicit function $\mathbf{x}(\mathbf{f})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{f})) \equiv 0$ and $\mathbf{x}(\mathbf{f}_0) = \mathbf{x}_0$. The *condition number* $\mu(\mathbf{f}, \mathbf{x})$ is the norm of the derivative of the implicit function $\mathbf{x}(\mathbf{f})$ at $\mathbf{f}_0$.

In general, condition numbers are an important 'invariants' of a numerical problem. They measure the accuracy of the solution in terms of the acuracy of the coefficients. However, a key fact usually overlooked is that the norm of a linear operator such as $\partial \mathbf{x}/\partial \mathbf{f}$ depends on the choice of norms in problem-space and in solution-space.

In the particular case of sparse polynomial systems, the solution-space may be endowed with a metric associated to the mixed volume form. This allows to significatively improve some bounds related to numerical polynomial solving.

Main reference for this talk: Malajovich and Rojas, High probability analysis of the condition number of sparse polynomial systems, *Theoretical Computer Science* **315** (2004) 525-555.