On the Newton polytope of specialized resultants

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Outline

Toric elimination theory

Newton polytopes

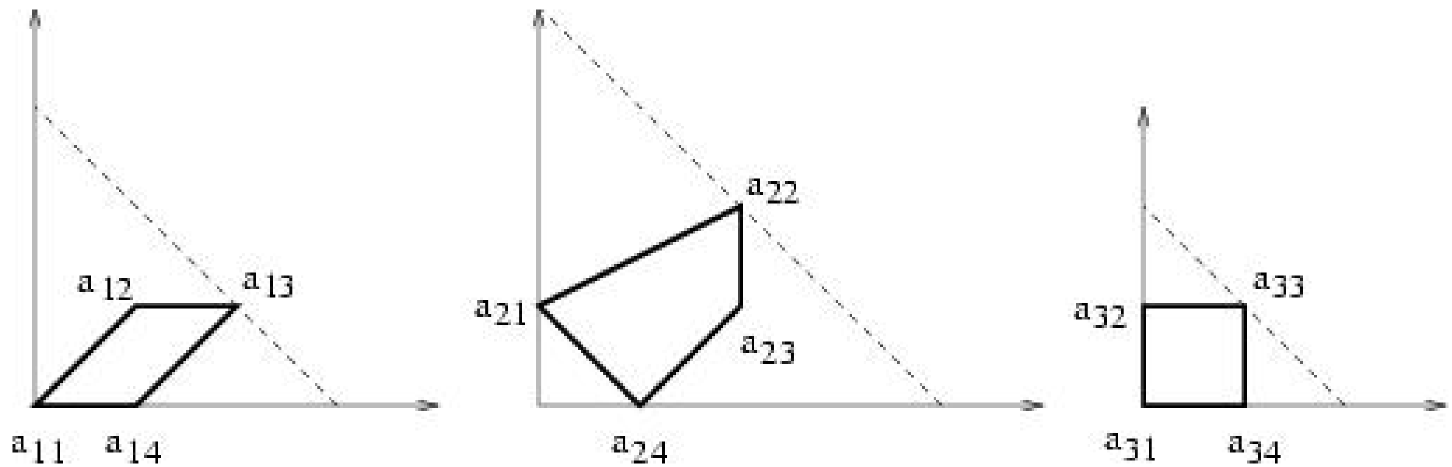
The support A_i of a polynomial $f_i \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, s.t.

$$f_i = \sum_j c_{ij} x^{a_{ij}}, \qquad c_{ij} \neq 0,$$

is defined as the set $A_i := \{a_{ij} \in \mathbb{Z}^n : c_{ij} \neq 0\}$.
The Newton polytope $Q_i \subset \mathbb{R}^n$ of f_i is the Convex Hull of all $a_{ij} \in A_i$.

Example:

$$\begin{aligned} f_1 &= c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x \\ f_2 &= c_{21}y + c_{22}x^2y^2 + c_{23}x^2y + c_{24}x + c_{25}xy \\ f_3 &= c_{31} + c_{32}y + c_{33}xy + c_{34}x \end{aligned}$$



Minkowski addition

- The Minkowski sum of convex polytopes $P_1, P_2 \subset \mathbb{R}^n$ is convex polytope $P_1 + P_2 = \{p_1 + p_2 \mid p_i \in P_i\} \subset \mathbb{R}^n$.
If P_1, P_2 have integral vertices, then so does $P_1 + P_2$.

- Minkowski addition of polytopes $P_i \subset \mathbb{R}^n$, $i \in I$ is a many-to-one map

$$(P_i)_{i\in I} \to P := \sum_{i\in I} P_i \subset \mathbb{R}^n : (p_i \in P_i)_{i\in I} \mapsto \sum_{i\in I} p_i.$$

Mixed volume

1. The mixed volume $\text{MV}(P_1, \ldots, P_n) \in \mathbb{R}$ of convex polytopes $P_i \subset \mathbb{R}^n$

- is multilinear wrt Minkowski addition and scalar multiplication:

$$\text{MV}(P_1, \ldots, \lambda P_i + \mu P_i', \ldots, P_n) =$$
$$= \lambda \text{MV}(P_1, \ldots, P_i, \ldots, P_n) + \mu \text{MV}(P_1, \ldots, P_i', \ldots, P_n), \qquad \lambda, \mu \in \mathbb{R},$$

- st. $\text{MV}(P_1, \ldots, P_1) = n!\ \text{vol}(P_1)$.

2. Equivalently, $\text{vol}(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a polynomial in scalar variables $\lambda_1, \ldots, \lambda_n$, with multilinear term $\text{MV}(P_1, \ldots, P_n)\ \lambda_1 \cdots \lambda_n$.

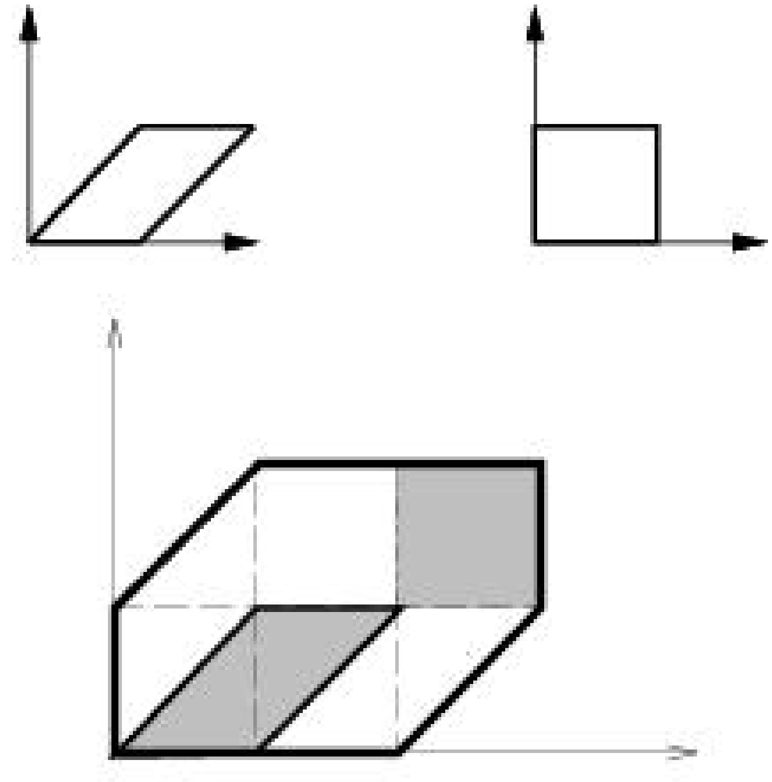
Bernstein (BKK) bound

Theorem [Bernstein'75,Kushnirenko'75,Khovanskii'78] [Danilov'78]:
Given polynomials $f_1, \ldots, f_n \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, for any field K, the number of common isolated zeros in $(\overline{K} - \{0\})^n$, counting multiplicities, is bounded by the mixed volume of the Newton polytopes $\mathrm{MV}(Q_1, \ldots, Q_n)$ (irrespective of the variety's dimension).

[Canny,Rojas'91]: Exact bound if the extremal coefficients are generic.

Dense homogeneous: $\mathrm{MV}(Q_1, \ldots, Q_n) = \prod_{i=1}^n d_i =$ Bézout's bound, where $d_i = \deg(f_i)$ and $Q_i = \mathrm{simplex}\{0, (d_i, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_i)\}$.

Example: mixed subdivision for well-constrained problem



Given $f_1 = c_{11} + c_{12}xy + c_{13}x^2y + c_{14}x$, $f_3 = c_{31} + c_{32}y + c_{33}xy + c_{34}x$,

- construct their Newton polytopes in $\mathbb{R}^2$
- compute a mixed subdivision of the Minkowski Sum (3 mixed cells)
- compute the Mixed Volume using the formula MV$= \sum_\sigma V(\sigma)$, over all mixed cells σ of the mixed subdivision (here MV=3).

Mixed subdivisions

Induced subdivisions

For $Q_i \subset \mathbb{R}^n$, $$(Q_i)_{i\in I} \to Q = \sum_{i\in I} Q_i \; : \; (q_i)_{i\in I} \mapsto \sum_{i\in I} q_i.$$

Pick [affine] liftings $l_i : \mathbb{Z}^n \to \mathbb{R}$, which define

$$\widehat{Q}_i := \{(p_i, l_i(p_i)) : q_i \in Q_i\} \subset \mathbb{R}^{n+1}.$$

Each face in the lower-hull of $\widehat{Q}$ is written uniquely as $\sum_i \widehat{F}_i$, where convex polytope [face] $\widehat{F}_i \subset \widehat{Q}_i$.

Minkowski sum $\widehat{Q} := \sum_i \widehat{Q}_i$ projects onto Q, so the lower-hull faces induce a regular subdivision of Q. The faces (cells) are $\sum_i F_i$.
In particular, facets on the lower-hull project to maximal cells (dim$= n$).

Example lifting for the over-constrained problem

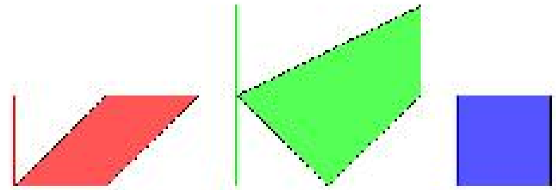


Figure 1: The given polytopes.

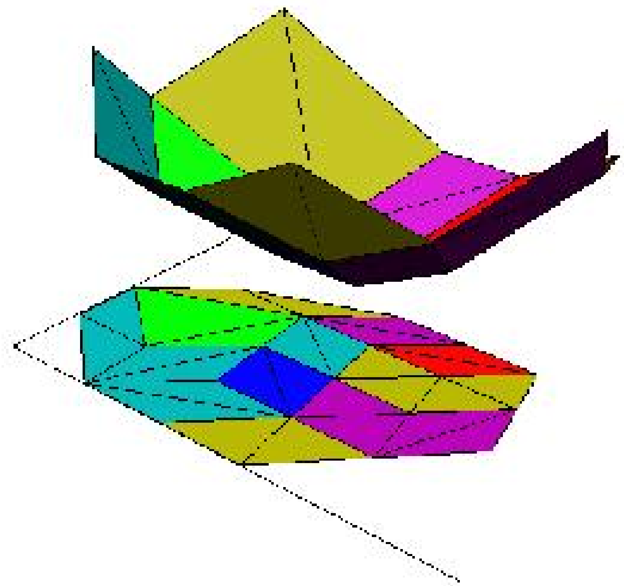


Figure 2: The lower hull of the lifted Minkowski Sum and its planar projection.

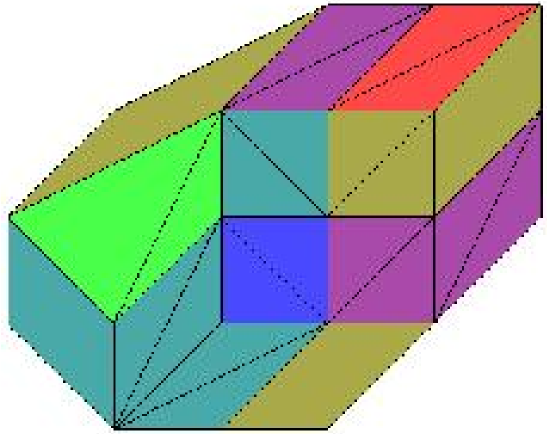
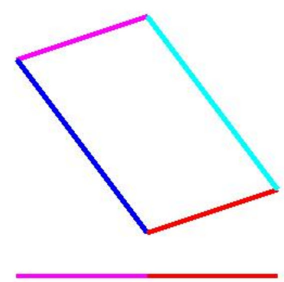


Figure 3: The mixed subdivision.

Coherent subdivisions

A subdivision is coherent i.e. there is a continuous change of the unique (“optimal”) expressions of cells, as we move from a cell to its subcells and its adjacent cells.
We also say that the cells intersect properly as Minkowski sums.



Eg: INcoherent subdivision of $Q_0 + Q_1,\ Q_i = [0, 1]$.
Leftmost cell = projection of $\widehat{0} + \widehat{Q}_1$, so $\widehat{0} + \widehat{1}$ projects to 1.
Rightmost cell = projection of $\widehat{1} + \widehat{Q}_1$ so $\widehat{1} + \widehat{0}$ projects to 1: different expression.

Induced subdivisions are coherent.

Tight coherent mixed subdivisions

In general: $\dim\left(\sum_i F_i\right) \leq \sum_i \dim F_i$.
A generic lifting implies equality, i.e. a tight subdivision.
In particular, for a maximal-dimension cell, $n = \sum_i \dim F_i$.

E.g: NOT tight subdivision: 2 segments lifted in parallel:
$\dim(F_0 + F_1) = 1 < \dim F_0 + \dim F_1 = 1 + 1$.

This leads to a (tight coherent) mixed subdivision, which partitions Q.
A generic lifting implies the lower-hull of $\widehat{Q}$ corresponds bijectively to Q.

Mixed cells

A maximal cell σ, in a mixed decomposition Δ, is mixed iff it has n linear summands, ie. n edges F_i : $\dim F_i = 1$.

- n polytopes: $Q = Q_1 + \cdots + Q_n$, mixed cells are sums of edges.

Thm: $$\mathrm{MV}\ (Q_1, \ldots, Q_n) = \sum_{\sigma} \mathrm{vol}(\sigma),$$ over all mixed cells $\sigma \in \Delta$.

- $n+1$ polytopes: $Q = Q_0 + Q_1 + \cdots + Q_n$, i-mixed cells are sums of edges plus vertex $a_i \in Q_i$.

Thm: $$\mathrm{MV}(Q_0, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_n) = \sum_{\sigma} \mathrm{vol}(\sigma),$$

over all i-mixed cells $\sigma \in \Delta$.

Resultants

Resultant definition

Given $n+1$ Laurent polynomials $f_0, \ldots, f_n \in K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ with indeterminate coefficients $\vec{c}$, their projective, resp. toric / sparse, *resultant* is the unique (up to sign) irreducible polynomial $R(\vec{c}) \in \mathbb{Z}[\vec{c}]$ such that

$$R(\vec{c}) = 0 \Leftrightarrow \exists\, \xi = (\xi_1, \ldots, \xi_n) \in X \,:\, f_0(\xi) = \cdots = f_n(\xi) = 0$$

where the variety X equals:

- the projective space $\mathbb{P}^n$ over the algebraic closure $\overline{K}$,
- resp. the toric variety X, $(\overline{K}^*)^n \subset X \subset \mathbb{P}^N$.

[van der Waerden, Gelfand-Kapranov-Zelevinsky, Cox-Little-O'Shea]

Resultant degree

The projective, resp. toric, resultant polynomial $R \in \mathbb{Z}[\vec{c}]$ is separately homogeneous in the coefficients of each f_i, with *degree* equal to $\prod_{j \neq i} \deg f_j$ (Bézout's number), resp. the n-fold mixed volume:

$$\mathrm{MV}_{-i} := \mathrm{MV}(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n),$$

provided the supports of the f_i generate $\mathbb{Z}^n$.

Generalizations

The toric resultant reduces to:

- the determinant of the coefficient matrix of a *linear* system,
- the Sylvester or Bézout determinant of 2 *univariate* polynomials,
- the projective resultant for $n+1$ *dense* polynomials, where the toric variety equals $\mathbb{P}^n$ and $\mathrm{MV}_{-i} = \prod_{j \neq i} \deg f_j$.

Sylvester matrix

Overconstrained system

$$\begin{aligned} f_0 &= a_{d_0} x^{d_0} + \cdots + a_0, \\ f_1 &= b_{d_1} x^{d_1} + \cdots + b_0. \end{aligned}$$

$$\begin{array}{ccccc} x^{d_0+d_1-1} & & \cdots & x & 1 \end{array}$$

$$R = \det \begin{bmatrix} a_{d_0} & \cdots & a_0 & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_{d_0} & \cdots & a_0 \\ b_{d_1} & \cdots & \cdots & b_0 & 0 \\ 0 & b_{d_1} & \cdots & \cdots & b_0 \end{bmatrix} \quad \begin{matrix} f_0 * & \begin{matrix} x^{d_1-1} \\ \vdots \\ 1 \end{matrix} & \Bigg\} B_0 \\ f_1 * & \begin{matrix} x^{d_0-1} \\ \vdots \\ 1 \end{matrix} & \Bigg\} B_1 \end{matrix}$$

Poisson formula: $R = b_{d_1}^{d_0} \prod_{\alpha : f_1(\alpha)=0} f_0(\alpha).$

The u-resultant

$$\begin{aligned} {\color{red} f_0} &{\color{red} = u_1 x_1 + u_2 x_2 + u_0,} \\ f_1 &= c_{12}x_1^2 + c_{11}x_1 + c_{10} \\ f_2 &= c_{22}x_2^2 + c_{21}x_2 + c_{20} \end{aligned}$$

$$\text{A toric resultant matrix} = \begin{bmatrix} c_{10} & c_{12} & c_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{10} & c_{11} & c_{12} & 0 \\ c_{20} & 0 & 0 & c_{22} & c_{21} & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 & c_{22} & 0 & c_{22} \\ u_0 & 0 & u_1 & 0 & u_2 & 0 & 0 & 0 \\ 0 & u_1 & u_0 & 0 & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & u_2 & u_0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_0 & u_1 & u_2 \end{bmatrix} \quad \begin{matrix} f_1 \\ x_2 f_1 \\ f_2 \\ x_1 f_2 \\ f_0 \\ x_1 f_0 \\ x_2 f_0 \\ x_1 x_2 f_0 \end{matrix}$$

defined by "row" monomials s.t. the matrix be square (dialytic elimination).
There also exist algorithms [Canny-E'92,'93,'00].

The number of rows per polynomial = 4,2,2 = MV$_{-i}$ i.e. optimal, hence:

$$R(u) = \det M(u) = C \cdot \prod_{f_i(\alpha)=0, i=1,2} (u_1\alpha_1 + u_2\alpha_2 + u_0).$$

Matrices of Sylvester-type

Algorithms: subdivision [Canny-E'93,'00], incremental [E-Canny'95] yield a square Newton matrix M such that:

$$\det(M) \neq 0,$$
$$R \mid \det(M),$$
$$\deg_{f_0} \det(M) = \deg_{f_0} R,$$

where R is the toric resultant. Same properties for the Macaulay matrix of the projective resultant [Mac'02].

Complexity [E'96] $O(e^n \deg R\, (\text{vtx} Q_i)^3)$, when n-fold Mixed Volumes > 0, and the Newton polytopes do not differ "too much" (bounded scaling).

Rational form [D'Andrea'01] : $\quad R = \det(M) / \det(M')$,
where M' is a submatrix of M, generalizing Macaulay's construction.
Open: single lifting.

Toric resultant support

Newton polytope of the toric resultant

Given are supports $A_0, \ldots, A_n$ s.t. $k := \sum_i |A_i|$ and $\dim(\sum_i A_i) = n$. Consider the toric resultant $R \in \mathbb{Z}[c]$ and its Newton polytope in $\mathbb{R}^k$.

Let lifting $l \in \mathbb{R}^k$ define a (tight coherent) mixed subdvision of $Q_0 + \cdots + Q_n$. Consider the trailing monomial of R with respect to l, which corresponds to the vertex of $\text{supp}(R) \subset \mathbb{Z}^k$ with inner normal l.

Theorem. This trailing monomial is

$$\text{In}_l(R) = \prod_{i=0}^{n} \prod_{i-\text{mixed } \sigma} \text{coef}(f_i, a_i)^{\text{vol}(\sigma)},$$

where $\text{vol}(\cdot)$ denotes Euclidean volume and the i-mixed cells are $\sigma = F_0 + \cdots + a_i + \cdots + F_n : \dim a_i = 0$.

[Sturmfels'94]

Newton polytope of the toric resultant (cont'd)

Theorem. If $l \in \mathbb{R}^k$ defines a mixed subdvision, then the trailing monomial of R, wrt l, is

$$\mathrm{In}_l(R) = \prod_{i=0}^{n} \prod_{i-\text{mixed } \sigma} \mathrm{coef}(f_i, a_i)^{\mathrm{vol}(\sigma)},$$

where the i-mixed cells are $\sigma = F_0 + \cdots + a_i + \cdots + F_n$.

Proof. Let M be the subdivision-based Newton matrix.

- Specialize $\mathrm{coef}(f_i, a_j) \mapsto t^{l_i(a_j)}$, for new variable t,

then, the product of diagonal entries = trailing term of $\det M(t)$.

- Each $\mathrm{coef}(f_i, a_i)$ appears on the diagonal, at least $\mathrm{vol}(\sigma)$ times.

Lastly, $\deg_{f_i} \mathrm{In}_l(R) = \sum_{i-\text{mxd } \sigma} \mathrm{vol}(\sigma) = \mathrm{MV}_{-i} = \deg_{f_i} R$. □

Corollary. A surjection exists from the set of mixed-cell configurations onto the extreme monomials of R (vertices of its Newton polytope).

Corollary. The coefficient of the trailing term is ± 1.

Enumerating mixed subdivisions

The Cayley trick introduces point set $C \subset \mathbb{Z}^{2n}$:

$$C := \{(0^n, a_{0j}) \ : \ a_{0j} \in A_0\} \ \cup \ \{(e_i, a_{ij}) \ : \ i = 1, \ldots, n, \ a_{ij} \in A_i\},$$

where $e_i = (\ldots, 0, 1, 0, \ldots) \in \mathbb{N}^n$. So $|C| = |A_0| + \cdots + |A_n|$.

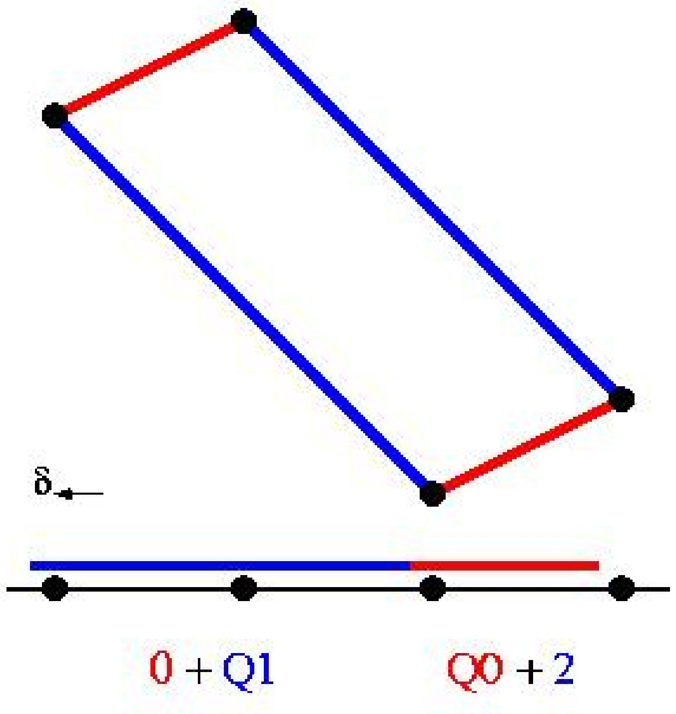
Theorem. The set of all mixed subdivisions of $A_0, \ldots, A_n \subset \mathbb{Z}^n$ corresponds bijectively to the set of all regular triangulations of point-set C.

Caley trick in sparse Sylvester case

$f_0 = c_{00} + c_{01}x,$
$f_1 = c_{10} + c_{12}x^2.$

Cayley point set

$$C := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$



Triangulations: $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ shown,

and also $\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$

Secondary polytope

Consider the graph of regular triangulations of point-set $C \subset \mathbb{Z}^d$, where edges correspond to (bistellar) flips.

Theorem [Gelfand-Kapranov-Zelevinsky, Billera-Sturmfels]
If C affinely spans $\mathbb{R}^d$, then the graph can be embedded in $\mathbb{R}^{|C|-d-1}$ as the secondary polytope $\Sigma(C)$. For triangulation T,

$$(v_T)_i = \sum_{i \in \text{vtx}(\sigma):\sigma \in T} \text{vol}(\sigma), \qquad i = 1, \ldots, |C|,$$

where vtx(σ) are the vertices of simplex σ.

E.g. $C \subset \mathbb{Z}^2$, $|C| = 4$:

Computation

Computation of all mixed-cell configurations by even/odd flips, followed by regularity check [Michiels-Verschelde'97]

Reverse search computes a spanning tree of $\Sigma(C)$ [Imai-Masada-Takeuchi-Imai'02]

All works assume entire secondary polytope is available.
Specialized resultant corresponds to projection: silhouette only.

Circuits

A circuit $Z = \{c_1, \ldots, c_t\}$ is a minimal affinely-dependent subset of C, satisfying $\lambda_1 c_1 + \cdots + \lambda_t c_t = 0$, where $\lambda_i \neq 0$, $\sum_i \lambda_i = 0$.

Z admits triangulations $Z^+ = \{Z \setminus \{c_i\} \mid \lambda_i > 0\}$, $Z^- = \{Z \setminus \{c_i\} \mid \lambda_i < 0\}$. Each flip $T \leftrightarrow T'$ corresponds to precisely one circuit Z s.t.

$$T' \simeq T \setminus Z^+ \cup Z^-$$

E.g. $Z = C$, $$-\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$Z^+ = T =$  $= T' = Z^-$

Projecting $\Sigma(C)$ to $\mathbb{R}$

Project to 1st coordinate, corresponding to $c_1 \in C$.

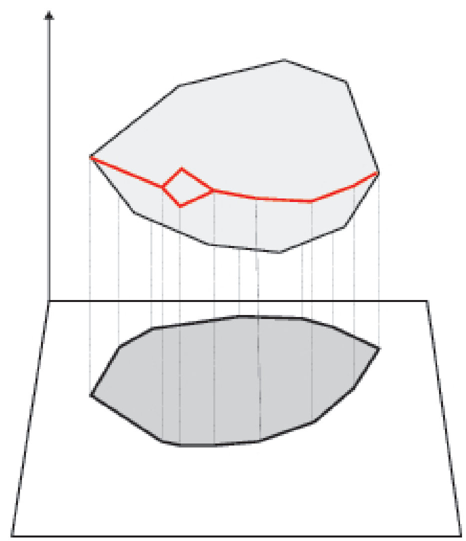
- Let T be a regular triangulation, Z the circuit supporting flip $T \leftrightarrow T'$. Then c_1 is a vertex of every simplex in Z^- iff $(v_T)_1 < (v_{T'})_1$.

- Let Z_j be the circuits that make $(v)_1$ increase, and $\sigma_j \in Z_j^+$ the unique simplex not containing c_1. Then, the triangulation T maximizing $(v_T)_1$ is s.t. the volume of simplices containing σ_j is max.

- Hence $(v)_1 \uparrow$. If strictly$\uparrow$ then min-path.

[E-Konaxis-Palios'07]

Projecting $\Sigma(C)$ to $\mathbb{R}^k, k \geq 2$



- Complexity: Time $= O^*(s^2 m)$LP$(\dim \Sigma, s)$, Space $= O(ns)$,

$s =$ max #any-dim simplices$= O(k^n)$, $m =$ #mixed-cell config's.

- Gift-wrapping, CCW$(u, v, w) = \det \begin{bmatrix} 1 & u_1 & u_2 \\ 1 & v_1 - u_1 & v_2 - u_2 \\ 1 & w_1 - v_1 & w_2 - v_2 \end{bmatrix}$.

Goal $= O^*(sH)$LP$(\cdot)$, $H =$ #silhouette-points.

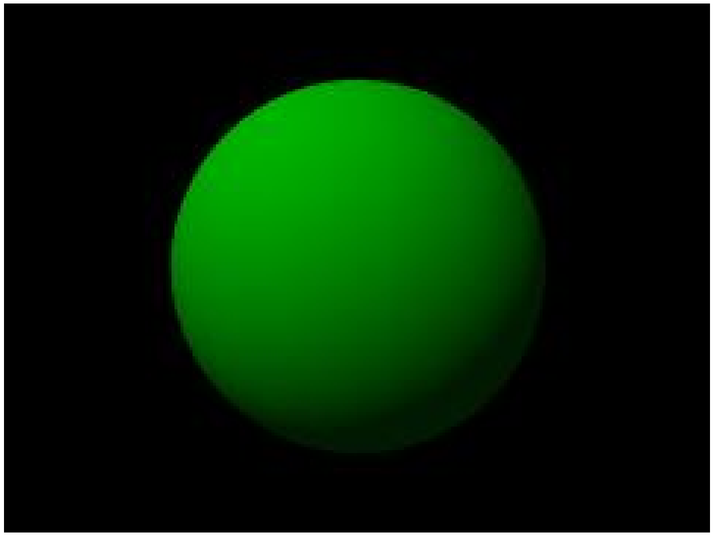
Implicitization

Example: sphere

The sphere in $\mathbb{R}^3$ is the set of values (x, y, z):

$$x = \frac{t_1^2 - t_2^2 - 1}{t_1^2 + t_2^2 + 1}, y = \frac{2t_1}{t_1^2 + t_2^2 + 1}, z = \frac{2t_1 t_2}{t_1^2 + t_2^2 + 1}, \ t_1, t_2 \in [0, 1],$$

as well as the set of roots of $H(x, y, z) := x^2 + y^2 + z^2 - 1 = 0$.



Modeling/CAD use parametric *and* implicit/algebraic representations

$\Rightarrow$ must implicitize a (hyper)surface given a (rational) parameterization

Implicitization by linear algebra

$S =$ vector of monomials forming a superset of the support of the implicit equation $H(x,y,z)$, evaluated at the parameters s,t.
$C =$ unknown coefficients of $H(x,y,z)$ with respect to S.
Solve for C in $(SS^T)C = \vec{0}$ [Corless-Galligo-Kotsireas-Watt'00].

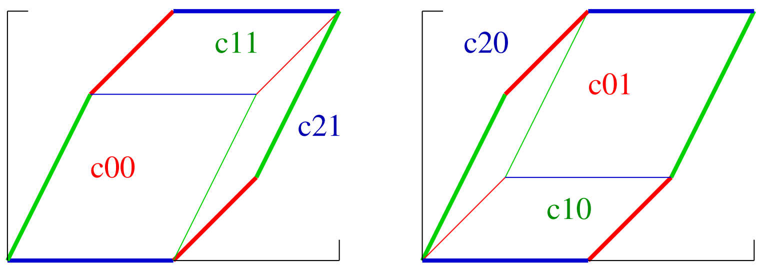
Example: supp$(H) \subset \{x^3y, x^3, x^3y^2, y^2z^3\}$, then

$$SS^T = \begin{bmatrix} x^6y^2 & x^6y & x^6y^3 & x^3y^3z^3 \\ x^6y & x^6 & x^6y^2 & x^3y^2z^3 \\ x^6y^3 & x^6y^2 & x^6y^4 & x^3y^4z^3 \\ x^3y^3z^3 & x^3y^2z^3 & x^3y^4z^3 & y^4z^6 \end{bmatrix} \Rightarrow C = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Mixed subdivisions: example

A sparse example [Buchberger'88]

$$f_0 = c_{00} - c_{01}st,\ f_1 = c_{10} - c_{11}st^2,\ f_2 = c_{20} - c_{21}s^2.$$

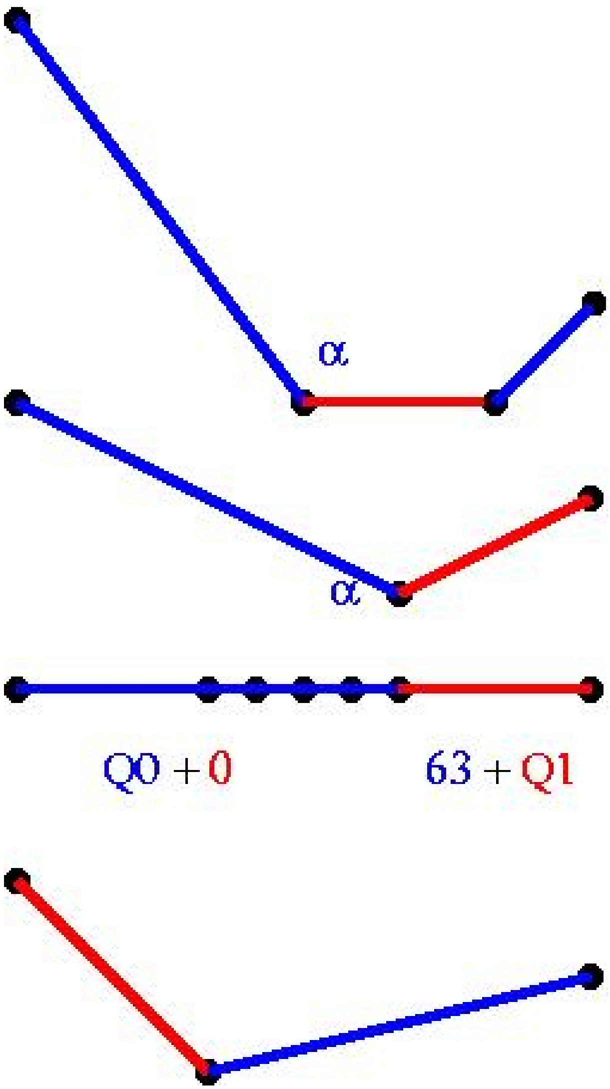


The mixed subdivisions yield extreme monomials $c_{00}^4 c_{11}^2 c_{21},\ c_{01}^4 c_{10}^2 c_{20}$.

The toric resultant turns out to be $R = c_{00}^4 c_{11}^2 c_{21} - c_{01}^4 c_{10}^2 c_{20}$.

The Fröberg-Dickenstein example

$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, \quad y = t^{32}.$$



$Q_0' + 0, a + Q_1, Q_0'' + 32$ $\qquad \pm y^a c_{0a}^{32} c_{1,32}^{63-a}$

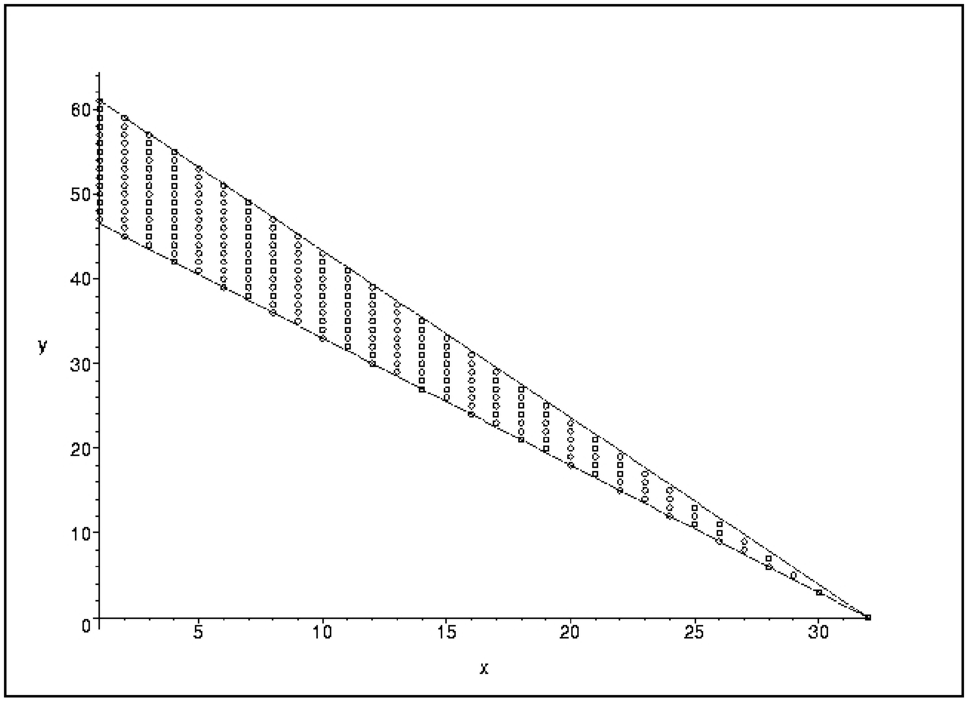
$a = 48, 56, 60, 62, 63$ $\qquad$ yields $\pm y^a$

$a = 63$ $\qquad \pm y^{63} c_{0,63}^{32}$

$0 + Q_1, Q_0 + 32, (a = 0)$ $\qquad \pm x^{32} c_{1,32}^{63}$

The Fröberg-Dickenstein example (cont'd)

The projected support is defined by points $(0, 48), (0, 63), (32, 0)$:



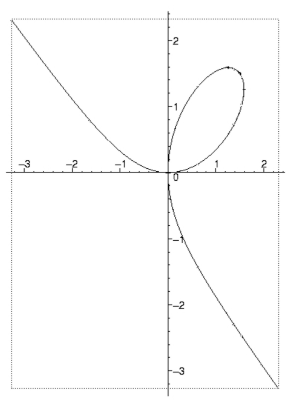
This triangle includes 257 integral points, optimally.

Implicitization examples

[Descartes' folium]

[1596-1650]

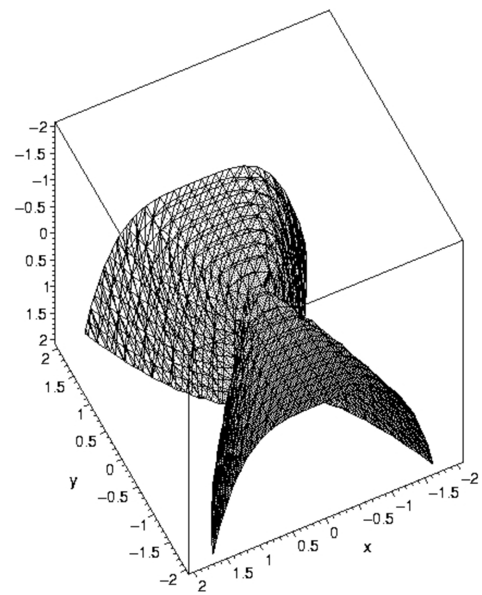
$$(x, y) = \left(\frac{3\ t^2}{t^3 + 1}, \frac{3\ t}{t^3 + 1} \right)$$



$$H = x^3 + y^3 - 3\ x\ y$$

[Buchberger'88]

$$(x, y, z) = (st, st^2, s^2)$$



$$H = x^4 - y^2 z$$

[Busé'01]

$$x = \frac{s^2}{s^3 + t^3},$$

$$y = \frac{s^3}{s^3 + t^3},$$

$$z = \frac{t^2}{s^3 + t^3}$$

$$H = x^3 - 2x^3 y + x^3 y^2 - y^2 z^3$$

Application to implicitization

Curve / Surface	param. (bi)deg.	impl. deg.	general #mon's	[E-Kotsireas'03] #monom's	actual #mon's
Unit circle	2	2	6	3	3
[Descartes'folium]	3	3	10	3	3
[Fröberg-Dickenstein]	63	63	1057	257	257
[Buchberger]	3	4	35	2	2
[Busé]	3	5	56	4	4
Bilinear	1,1	2	10	9	9

Maple code [E-Kotsireas'03]

- regular triangulations via TopCom [Rambau],
- cannot handle bicubic surfaces.

Specify implicit Newton polytope

Optimal generic Newton polytope

Consider parameterizations with fixed supports and generic coefficients.

◦ Toric elimination leads to above algorithm for resultant's Newton polytope, in $O(\exp(k/2-n-1))$.

• Specify vertices of Newton polygon of rationally parameterized curves in $O(k)$ [E-Konaxis-Palios'07].

◦ Specify faces of the Newton polytope of the Sylvester resultant [Gelfand-Kapranov-Zelevinsky'90].

• Tropical geometry leads to an algorithm for the polytope of Laurent-polynomial (hyper)surfaces and varieties: codim>1; for curves, specifies polygon in $O(k)$ [Sturmfels-Tevelev-Yu'06].

Sylvester support

Let polynomials $\in K[x]$ have degree d_0, d_1 and resultant

$$\sum_{p \in \mathbb{N}^k} \text{const}_p \prod_{j=0}^{d_0} c_{0j}^{p_{0j}} \prod_{j=0}^{d_1} c_{1j}^{p_{1j}} = \sum_{p \in \mathbb{N}^k} \text{const}_p \, c^p.$$

Its support is defined by [Gelfand-Kapranov-Zelevinsky]:

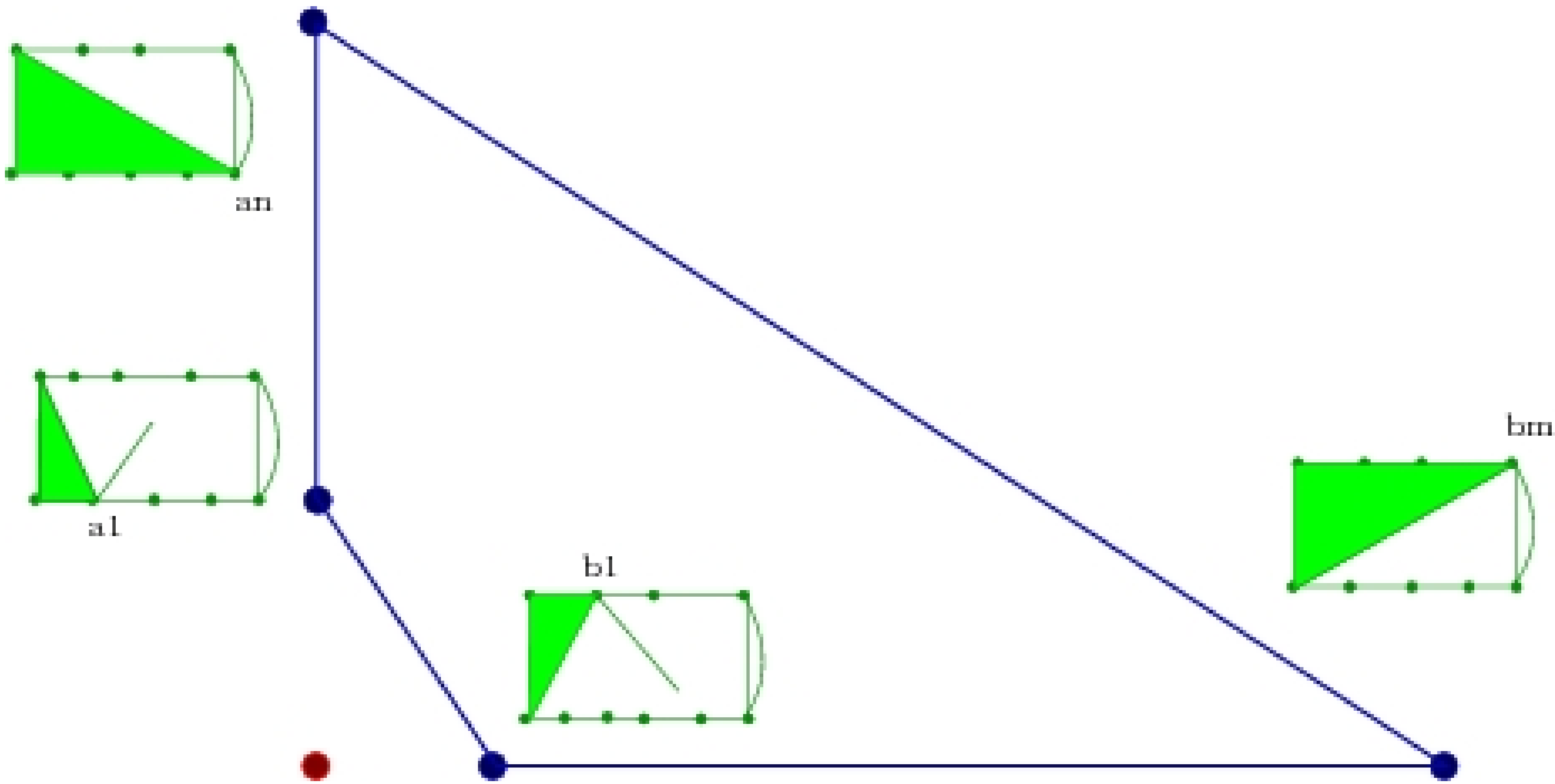
$$0 \le i \le 1, 0 \le j \le d_i \ : \qquad p_{ij} \ \ge \ 0,$$

$$\begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & d_0 & 0 & \cdots & d_1 \end{bmatrix} \begin{bmatrix} p_{00} \\ \vdots \\ p_{0d_0} \\ p_{10} \\ \vdots \\ p_{1d_1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_0 \\ d_0 d_1 \end{bmatrix},$$

$$\left.\begin{array}{l} 0 \le a \le d_0 \\ 0 \le b \le d_1 \end{array}\right\} \quad \sum_{j=a}^{d_0} (j-a) p_{0j} + \sum_{j=b}^{d_1} (j-b) p_{1j} \ \ge \ (d_0 - a)(d_1 - b)$$

Polynomially parameterized curves

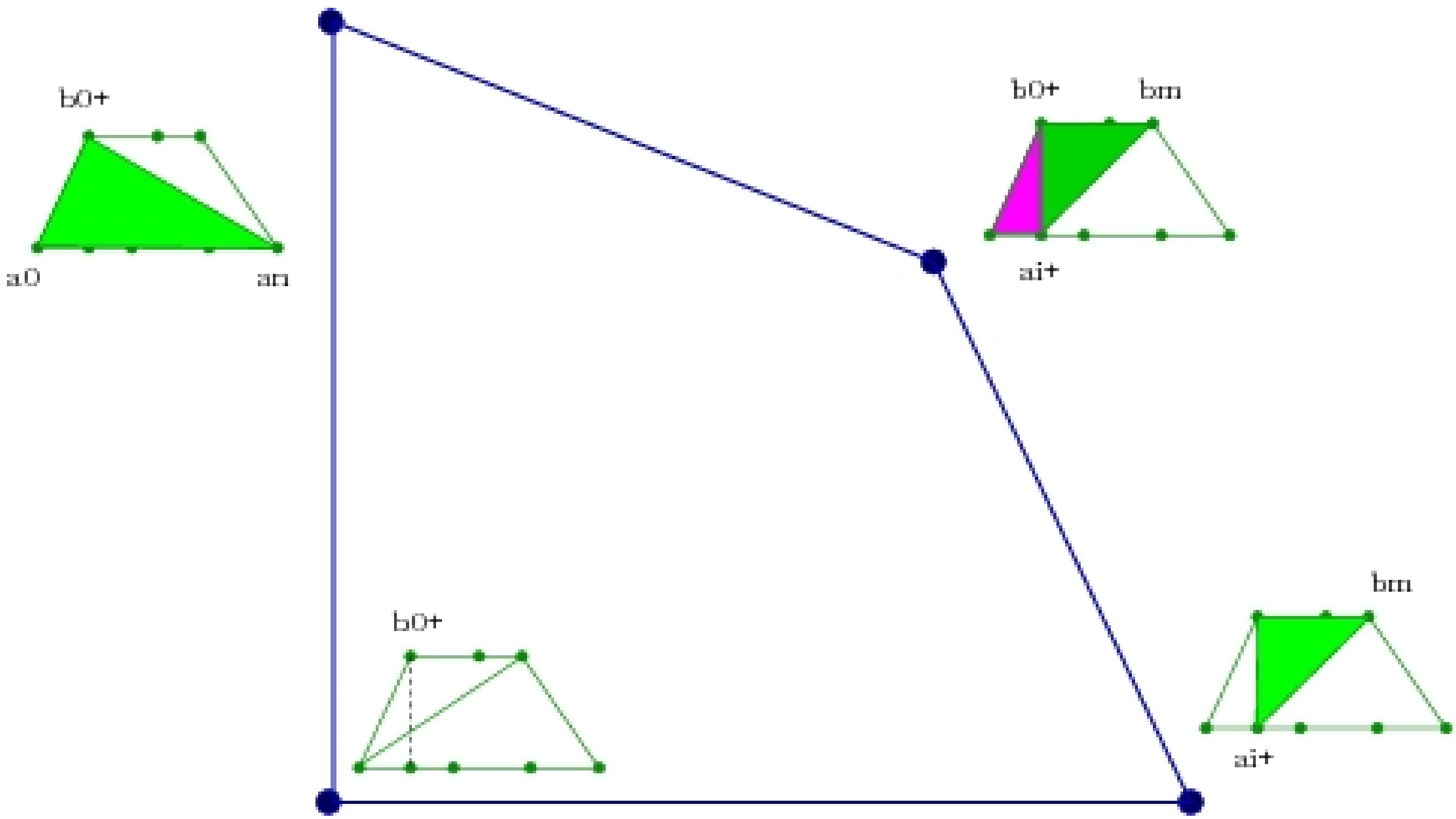
If ∃ const.term in $P_i(t)$, then ∃ implicit vertex $(0,0)$



Cor: $\text{coef}(x^{b_m}) = \pm(-c_{1m})^{a_n}$, $\text{coef}(y^{a_n}) = \pm c(-c_{0n})^{b_m}$

Laurent-polynomial parameterization

$\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_m\} \subset \mathbb{Z}$, unique selected 0^+.



Up-right vertex $= (b_m, |a_0|)$ iff $\det \begin{bmatrix} |a_0| & a_n \\ |b_0| & b_m \end{bmatrix} > 0$, $(|b_0|, a_n)$ iff $\det < 0$.

Rationally parameterized curves

$$x_i = \frac{P_i(t)}{Q(t)} \;\rightarrow\; f_i = x_i Q(t) - P_i(t) \in K[t], \;\; i = 0, 1,$$

with supports $\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_m\} \subset \mathbb{N}$.

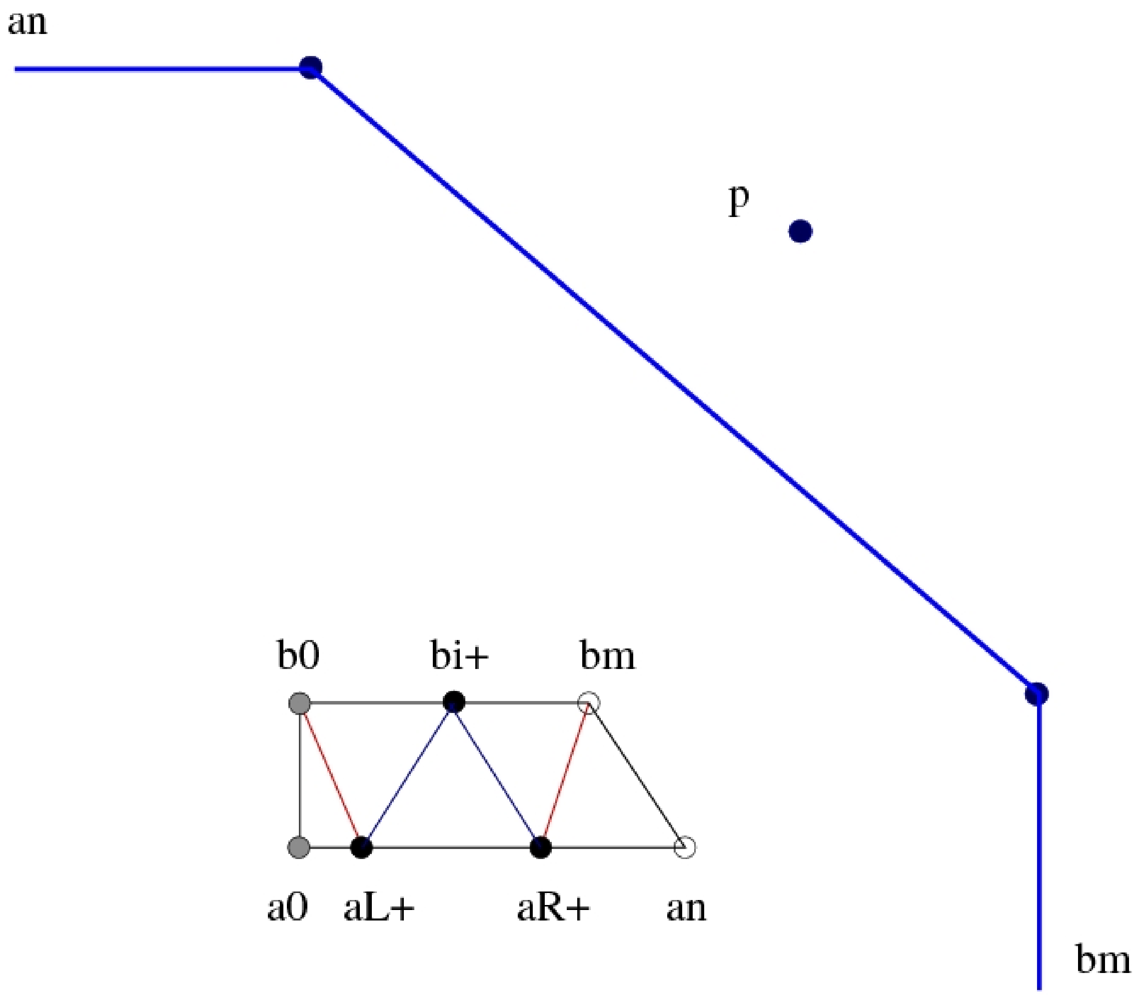
Lemma. Consider direction $(1, 1)$. The upper [resp. lower] hull of the Newton polygon has vertices of the form:

$$\left(\sum_{i,l,r} \text{vol}(a_i^+, b_l, b_r), \sum_{l,r,j} \text{vol}(a_l, a_r, b_j^+) \right) \;\in \mathbb{N}^2,$$

where $a_i^+, b_j^+ \in \text{supp}(Q)$ [resp. $\text{supp}(Q) - \text{supp}(P_i)$].

Pf. Consider monomials or binomials $\in \mathbb{C}[x_i]$, i.e. $c_1 x_i$ or $c_0 + c_1 x_i$, [resp. monomials $\in \mathbb{C}[x_i]$, i.e. $c_1 x_i$].

Upper-right corner



if $a_0^-, b_0^-, a_n^-, b_m^-$ then:

$$\exists\ p = (b_R^+, a_n - a_L^+) \Leftrightarrow$$

$$\det \begin{bmatrix} a_n - a_R^+ & a_L^+ \\ b_L^+ & b_m - b_R^+ \end{bmatrix} > 0,$$

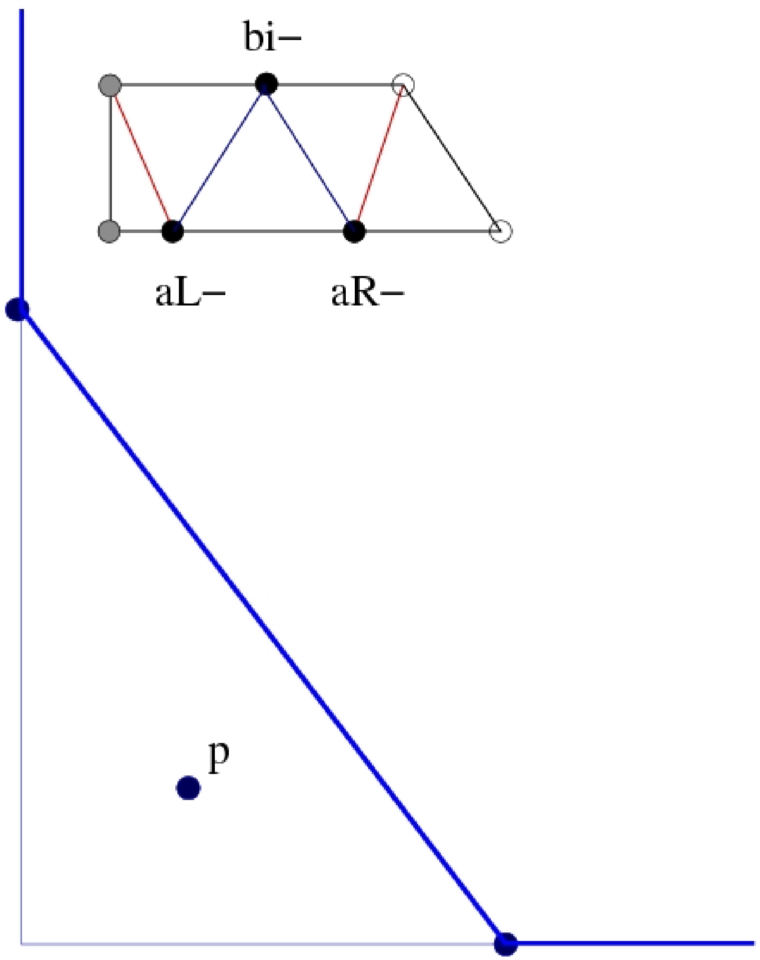
$$p = (b_m - b_L^+, a_R^+) \Leftrightarrow \det < 0$$

$$x_{\max} = b_m,$$

$$y_{\max} = (a_R^+ - a_L^+) + \mathcal{X}(b_m^+) \cdot (a_n - a_R^+)$$

Selected $a_i^+, b_j^+ \in \text{supp}(Q)$, not selected $a_i^-, b_j^- \notin \text{supp}(Q)$.

Lower-left corner



$$x_{\mathrm{min}} = 0,$$
$$y_{\mathrm{min}} = \mathcal{X}(b_0^+)a_L^+ +$$
$$+\mathcal{X}(b_m^+)(a_n - a_R^-)$$

if $a_0^+, b_0^+, a_n^+, b_m^+$ then:

$$\exists\ p = (b_L^-, a_n - a_R^-) \Leftrightarrow$$
$$\det \begin{bmatrix} a_n - a_R^- & a_L^- \\ b_m - b_R^- & b_L^- \end{bmatrix} < 0,$$
$$p = (b_m - b_R^-, a_L^-) \Leftrightarrow \det > 0$$

$a_i^+, b_j^+ \in \mathrm{supp}(Q) - \mathrm{supp}(P_i)$, $a_i^-, b_j^- \notin \mathrm{supp}(Q) - \mathrm{supp}(P_i)$.

Polynomially parameterized surfaces

$$x_i = P_i(s,t) \in K[s,t],\ i = 0,1,2.$$

- $(0,0,0)$ is in Implicit support S iff some P_i contains a constant.

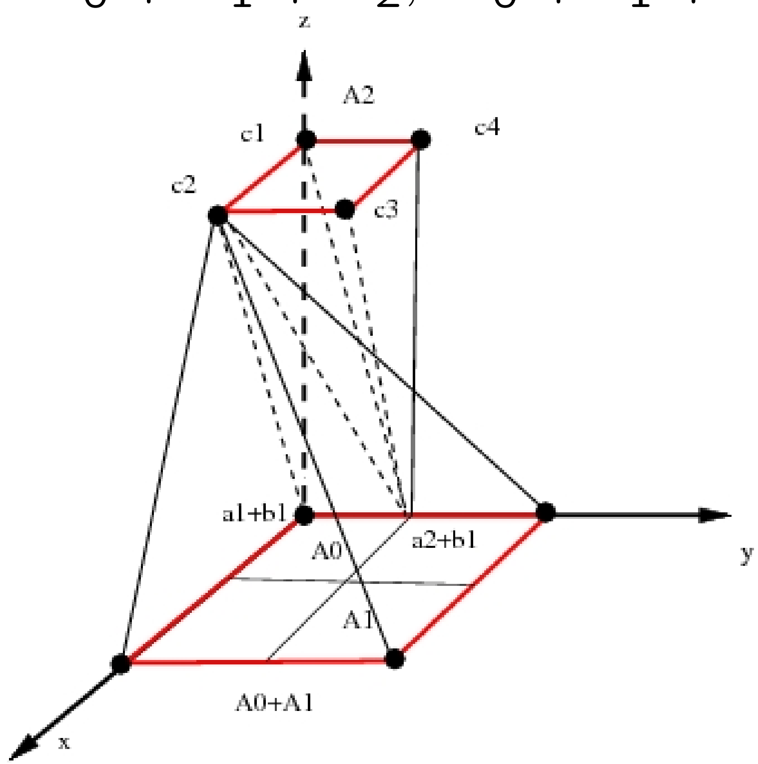
- For (small) unmixed systems, if no P_i contains a constant, then S contains $(k,0,0)$, for $k \in \mathbb{N}^*$, and the symmetric points.

- S contains $(0, \mathrm{MV}(A_0, A_2), k)$, for $k \in \mathbb{N}$, and the symmetric points.

Partial Cayley trick

Apply (partial) Cayley trick to $A_0 + A_1, A_2 \Rightarrow C \subset \mathbb{Z}^3$.

- Define good subdivisions comprised of maximal cells $M = (\sigma, \tau)$, s.t. $\sigma \in$ mxd.subdiv$(A_0 + A_1)$, $\tau \in$ mxd.subdiv(A_2), dim$(M) = 3$.
- The mixed cells are $a_0 + E_1 + E_2$, $E_0 + a_1 + E_2$, or $E_0 + E_1 + a_2$.



Lemma. There is a bijection between the good subdivisions of C and the mixed subdivisions of $A_0 + A_1 + A_2$.