Linkage Problems and Real Algebraic Geometry

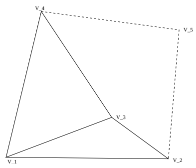
Definitions

Definition (Framework). A *framework* is a graph $G = (V, E)$ together with a set $L = \{L_{ij} : ij \in E\}$ of non-negative real numbers L_{ij} interpreted as edge lengths.

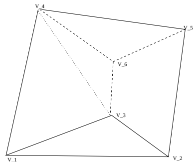
Definition (Laman Graph). Let G be a graph with n vertices and $m = 2n-3$ edges. If each subset of k vertices spans at most $2k-3$ edges, we say that G has the *Laman property* and call it a *Laman graph.*

- Laman graphs can be constructed via a sequence of Henneberg steps starting with a simple bar, and all graphs constructed in this way have the Laman property.
- Laman graphs are minimally rigid.

Henneberg I step:

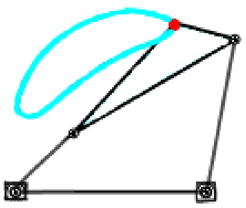


Henneberg II step:



Definition (Rigidity). A graph is *generically rigid* if it is rigid for all embeddings on generic sets of points. An embedding is *rigid* if it cannot be deformed continuosly into another (non-congruent) embedding of the same framework.

Definition (1DOF Mechanism). A one-degree-of-freedom (1DOF) mechanism is a framework whose configuration space is a one-dimensional curve.

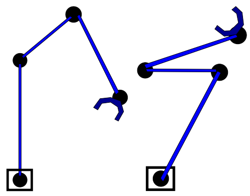


Applications

These concepts appear naturally in the following scenarios.

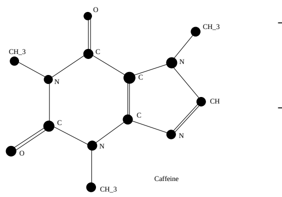
- Robot Kinematics

→ Planar non-colliding robot arm motion planning



- Molecular biology

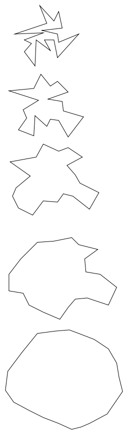
→ Efficient computer simulation of molecule conformation.



- The **Carpenter's Rule Problem**: Is it possible to move a closed linkage continuosly from an arbitrary initial configuration to any final configuration avoiding collisions between the bars?

→ Always possible (*Connelly, Demain and Rote 2000*)

→ Algebraic-combinatorial algorithm by *Streinu 2000*



Combinatorial and algebraic questions

- How many embeddings has a given framework of a minimally rigid graph?
  - → Best bound so far, *Borcea and Streinu '02*: For a generic choice of edge lengths, a Laman Graph with n vertices has at most $\binom{2n-4}{n-2}$ distinct embeddings in $\mathbb{R}^2$, up to rigid motions.
  - → This is a bound for the complex solutions.
  - → Even over $\mathbb{C}$ it is in general not sharp.
- Given a 1-degree-of-freedom framework, how can one characterize and compute the trajectory of the vertices?

Our viewpoint

A framework (G, L) can be formulated by the following system of 4 linear and $|E|-1$ quadratic equations in $2|V|$ unknowns.

$$\begin{aligned} x_1 - c_1 &= 0 \\ y_1 - c_2 &= 0 \\ x_2 - (L_{1\,2} + c_1) &= 0 \\ y_2 - c_2 &= 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - L_{i\,j}^2 &= 0 \quad \forall \{i,j\} \neq \{1,2\} \in E \end{aligned}$$

The real roots of this system define a real algebraic variety. Questions concerning the framework (G, L) can be answered by studying the components of this variety.

Our Tools

Definition (Mixed Volume). Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$. Then the *mixed volume* of $P_1, \ldots, P_n$ is defined as

$$\mathrm{MV}_n(P_1, \ldots, P_n) = \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } \sum P_j}} \mathrm{vol}_n(Q)$$

Definition (Newton Polytope). For $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ the *Newton polytope* of f is defined as

$$\mathrm{conv}\left(\{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}\right) .$$

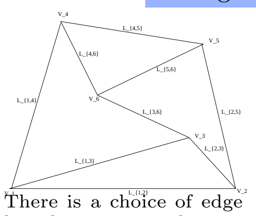
Theorem (Bernstein '75). *Let $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials with finitely many common zeros in $(\mathbb{C}^*)^n$, and let $P_1, \ldots, P_n$ be the Newton Polytopes of the f_i's. Then the number of common zeros in $(\mathbb{C}^*)^n$ is less than or equal the mixed volume $MV_n(P_1, \ldots, P_n)$. Equality holds if the coefficients of the f_i are generic.*

Lemma (Seperation Lemma). *Let $\Gamma_1, \ldots, \Gamma_k$ be polytopes in $\mathbb{R}^{m+k}$ and $\Delta_1, \ldots, \Delta_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(\Delta_1, \ldots, \Delta_m, \Gamma_1, \ldots, \Gamma_k) = \mathrm{MV}_m(\Delta_1, \ldots, \Delta_m) * \mathrm{MV}_k(\pi(\Gamma_1), \ldots, \pi(\Gamma_k)) ,$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Desargues Graph



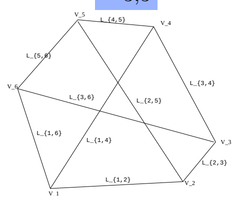
There is a choice of edge lengths giving **24** distinct real embeddings of this framework. Borcea and Streinu's theorem tells us, that there are at most $\binom{8}{4} = 70$. With our approach we get a better bound.
The truncated system of equations correponding to this framework is:

$$\begin{aligned} & x_1 - 1 = 0 \\ & y_1 - 1 = 0 \\ & x_2 - (L_{1,2} + 1) = 0 \\ & y_2 - 1 = 0 \\ & (x_1 - x_3)^2 + (y_1 - y_3)^2 - L_{1,3}^2 = 0 \\ & x_3^2 - x_1^2 - 2x_2x_3 + 2x_1x_3 + y_2^2 - y_1^2 - 2y_2y_3 + 2y_1y_3 - L_{2,3}^2 + L_{1,3}^2 = 0 \\ & (x_1 - x_4)^2 + (y_1 - y_4)^2 - L_{1,4}^2 = 0 \\ & 2x_4 + 2x_5 - 2x_4x_5 + 2y_4 + 2y_5 - 2y_4y_5 - L_{4,5}^2 + L_{1,4}^2 + L_{2,5}^2 = 0 \\ & (x_2 - x_5)^2 + (y_2 - y_5)^2 - L_{2,5}^2 = 0 \\ & (x_3 - x_6)^2 + (y_3 - y_6)^2 - L_{3,6}^2 = 0 \\ & 2x_4 + 2x_6 - 2x_4x_6 + 2y_4 + 2y_6 - 2y_4y_6 - L_{4,6}^2 + L_{1,4}^2 + L_{3,6}^2 = 0 \\ & 2x_5 + 2x_6 - 2x_5x_6 + 2y_5 + 2y_6 - 2y_5y_6 - L_{5,6}^2 + L_{2,5}^2 + L_{3,6}^2 = 0 \end{aligned}$$

.

The Mixed Volume of this system is **32**.

$K_{3,3}$



Again, the above theorem gives 70 as a bound on the number of possible embedings in $\mathbb{R}^2$, and as above we can improve that bound using Berstein's theorem.
The Mixed Volume of the truncated system of equations correponding to this framework is **32**.

Perspective

- Frameworks coming from Laman graphs that involve only Henneberg I steps in their construction are well understood. The (sharp) upper bound for the number of embeddings is $2^{|V|-2}$, which we can obtain with our method or even much simpler considerations. But graphs which involve Henneberg II steps still pose some difficulties.
- On the left we see the only examples of Non-Henneberg I-constructable graphs with 6 vertices. Since the Separation Lemma allows to 'split' larger graphs and compute the mixed volume for each component separately we can already treat a large class of graphs with our method.
- We would like to otain a bound that respects the structure of the graph. (i.e. That takes into account the number of Henneberg I and Henneberg II steps needed to construct it.)

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