The Voronoi Circle of Smooth Closed Curves

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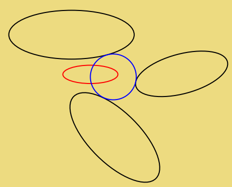
Motivation

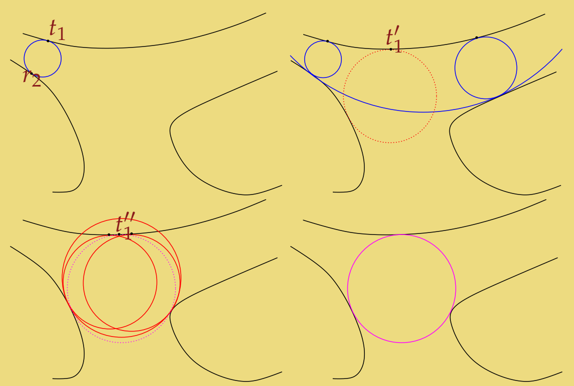


- Voronoi diagram of ellipses in the plane: incremental algorithm, bottleneck is predicate InCircle.
- High algebraic complexity: 184 complex tritangent circles. How many are real?
- Bisector of implicit ellipses: total degree = degree per variable = 28.

Parametric approach

- Bisector $B(t,r)$ of parametric ellipses: total degree = 12, degree per variable = 6.
- Geometric subdivision numerically [2], switch to exact real solving near degeneracy [1].
- [3] similar, more complicated, slower subdivision.

Geometric Subdivision



- Initialize: tangencies inside convex hulls.
- 3 smooth curves $C_t(t), C_r(r), C_s(s)$
- From $t_1 \in C_t$, find ext. bitangent circle of C_t, C_r, i.e. compute tangency point $r_2 \in C_r$.
  $r_2 \to$ ext. bitangent circle of $C_r, C_s \to s_1 \in C_s$.
  $s_1 \to$ ext. bitangent circle of $C_s, C_t \to t_2 \in C_t$.
  Now (t_1, t_2) contains tangency $\hat{t}$ of Voronoi circle.
- Subdivide $t'_1 = \frac{t_1+t_2}{2}$ and iterate, find t'_2.
- Refined $(t'_1, t'_2) \subset (t_1, t_2)$; subdivide $t''_1 = \frac{t'_1+t'_2}{2}$.
- Converge to ext. tritangent (Voronoi) circle.
- Decide InCircle in most cases.

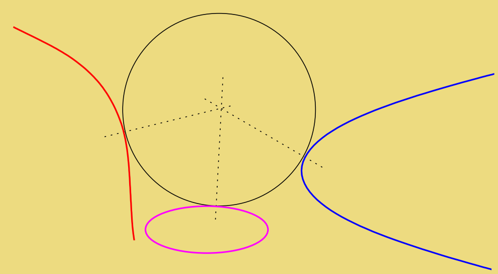
Quadratic Convergence

- $B_1(t_1, r_2) = B_2(r_2, s_1) = B_3(s_1, t_2) = 0 \quad (1)$
  $B_1(t_2, r_1) = B_2(r_1, s_2) = B_3(s_2, t_1) = 0 \quad (2)$
- Eliminate r_2, s_1 from (1) $\to$ resultant $R(t_1, t_2)$.
  Eliminate r_1, s_2 from (2) $\to$ resultant $R(t_2, t_1)$.
- $R(t_1, t_2) \longrightarrow t_2 = f(t_1)$ implicit function.
  Implicit function thm and $R(t_1, t_2) = R(t_2, t_1) \Rightarrow$
  $f'(\hat{t}) = -\left.\frac{\partial R/\partial t_1}{\partial R/\partial t_2}\right|_{t_1=t_2=\hat{t}} = -1.$
- error at one iteration $\epsilon = |f(t_1) - t_1|$;
  next iteration: $\epsilon' = |f(t_1 + \frac{f(t_1)-t_1}{2}) - t_1 - \frac{f(t_1)-t_1}{2}|$
- Taylor expansion around t_1: $\epsilon' = |f(t_1) + \frac{f(t_1)-t_1}{2} f'(t_1) + \frac{(f(t_1)-t_1)^2}{8} f''(\xi) - t_1 - \frac{f(t_1)-t_1}{2}|$
- $\epsilon' \leq \frac{5C}{8}\epsilon^2$

Implementation

Curves of total degree 2, 3, on a P4-2.6GHz:

ALIAS [4] C++, interval-arithmetic	Maple
⋆ 10^{-15} digits, 0.08 sec	⋆ 10^{-30} digits, 0.36 sec
⋆ typically 8 iterations	⋆ time ∼linear in #digits



References

[1] I.Z. Emiris, E.P. Tsigaridas, and G.M. Tzoumas. The predicates for the voronoi diagram of ellipses. In *Proc. 2006 ACM Symp. Computat. Geometry*.

[2] I.Z. Emiris and G.M. Tzoumas. A real-time and exact implementation of the predicates for the Voronoi diagram of parametric ellipses. In *Proc. 2007 ACM Symp. Solid Phys. Modeling*.

[3] I. Hanniel, R. Muthuganapathy, G. Elber, and M.-S. Kim. Precise Voronoi cell extraction of free-form rational planar closed curves. In *Proc. 2005 ACM Symp. Solid Phys. Modeling*. Best paper award.

[4] J-P. Merlet. ALIAS: an interval analysis based library for solving and analyzing systems of equations. In *Systèmes d'Équations Algébriques*, 2000.