NP, coNP and the Nullstellensatz: Lower Bounds for Stable Set and Graph Coloring Nullstellensätze
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1 Key Points

Systems of polynomial equations over the complex numbers can be used to characterize NP-Complete graph-theoretic decision problems. From the point of view of computer algebra and symbolic computation, these are interesting polynomial systems because they are provably hard: solving them is as hard as solving the underlying NP-Complete problem. Furthermore, unless NP = coNP, there must exist infinite instances of these infeasible systems whose Hilbert Nullstellensatz certificates grow with respect to the underlying graphs.

2 Stable Set as a Zero-Dimensional System of Polynomial Equations (L. Lovász)

Given a graph \( G \) and an integer \( k \), we construct the following equations:

- For every vertex \( i = 1, \ldots, n \), let \( x_i - x_i = 0 \)
- For every edge \( (i, j) \in E(G) \), let \( x_i x_j = 0 \)
- Finally, let \( -k + \sum_{i=1}^{n} x_i = 0 \).

Theorem: Let \( G \) be a graph, \( k \) an integer, encoded as the above \( (n + m + 1) \)-zero-dimensional system of equations. Then this system has a solution if and only if \( G \) has an independent set of size \( k \).

3 Stable Set Nullstellensätze

Theorem 3.1 Given a graph \( G \), there exists a Nullstellensatz certificate of degree \( \alpha(G) \) that certifies the non-existence of a stable set of size \( \alpha(G) + 1 \). Moreover, there exist families of graphs for which the minimum degree possible is at least \( \alpha(G)/2 \).

Corollary 3.2 Given a graph \( G \), there exists a Nullstellensatz certificate of degree \( \alpha(G) \) certifying the non-existence of a stable set of size \( \alpha(G) + 1 \), where all of the terms in every equation is a monomial corresponding to an independent set, and the coefficient for the stable set polynomial contains every independent set.

4 Turán Graph \( T(5, 3) \) Nullstellensatz

Theorem 4.1 The minimum degree Nullstellensatz for odd wheels is four.

5 Experimental Results for Minimum-Degree Stable Set Nullstellensätze

| Graph | \( |V(G)| \) | \( |E(G)| \) | \( \alpha(G) \) | \( |\text{V}(\alpha(G))| \) |
|-------|---------|---------|---------|---------|
| \( T(5, 2) \) | 5 | 4 | 3 | 4 |
| \( T(5, 3) \) | 5 | 9 | 5 | 5 |
| \( T(5, 4) \) | 5 | 12 | 6 | 5 |
| \( T(5, 5) \) | 5 | 16 | 6 | 4 |
| \( T(5, 6) \) | 5 | 20 | 6 | 3 |
| \( T(5, 7) \) | 5 | 24 | 6 | 2 |
| \( T(5, 8) \) | 5 | 28 | 6 | 1 |

6 NP, coNP and the Nullstellensatz

Theorem 6.1 If NP \( \neq \) coNP, then there must exist an infinite family of graphs such that the minimum-degree Nullstellensatz for not-k-colorability or non-existence of a stable set of size \( k \) grows with respect to the size of the graph.

7 Graph Coloring as a Zero-Dimensional System of Polynomial Equations (D. Bayer)

Given a graph \( G \) and an integer \( k \), we construct the following equations:

- vertex polynomials: For every vertex \( i = 1, \ldots, n \), \( x_i^k - 1 = 0 \)
- edge polynomials: For every edge \( (i, j) \in E(G) \), \( x_i x_j + x_i x_j + \cdots + x_i x_j = 0 \).

Theorem: Let \( G \) be a graph, \( k \) an integer, encoded as vertex and edge polynomials. Then this system of equations has a solution if and only if \( G \) is \( k \)-colorable.

8 Odd Wheels, Cat Ears, Cliques and not-3-colorable Nullstellensätze

Theorem 8.1 The minimum degree Nullstellensatz for odd-wheels is four.

Proof Sketch:

We derive a certificate for the \( (n + 1) \)-odd wheel from the \( n \)-th odd wheel, by taking a very particular szczyg on some of the terms from the \( n \)-th odd wheel not 3-colorable certificate.

9 Experimental Results for Minimum-Degree Graph Coloring Nullstellensätze

Kneser graphs, Mycielski graphs, Queen graphs, uniquely-colorable graphs, Mycielski graphs of Mycielski graphs, flowers, assorted triangle-free graphs, Vega graphs, all graphs in six vertices are less.

10 Other Encodings: Hamiltonian Cycle as a Zero-Dimensional System of Polynomial Equations

- For every vertex \( i = 1, \ldots, n \), we have two equations:
  \[ \prod_{s=1}^{n} (x_i - x_s) = 0 \quad \text{and} \quad \prod_{(i,j) \in E(G)} (x_i - x_j + (x_i - x_j - (n - 1))) = 0 \]

- Theorem: A graph \( G \) has a hamiltonian cycle if and only if the above zero-dimensional system of 2n equations has a solution.

- Other Encodings: Longest cycle, graph planarity, max cut and more!

11 Searching for Hilbert Nullstellensatz degrees...

Nullstellensatz: If \( V(\{f_1, \ldots, f_m\}) = \emptyset \), then 1 = \( \sum \alpha_i f_i \)

\[ 1 = (c_1 x + c_2 y + c_3 z + c_4 w + c_5 y^2 - 1) + (c_6 x + c_7 y + c_8 z + c_9 w + c_{10} y^3 - 1) + \cdots \]

Try a degree for the \( \alpha \) polynomials, and construct a large-scale sparse linear system of equations. If infeasible, try a larger degree for \( \alpha \). Note: \( deg \alpha \) cannot exceed known upper bounds for Hilbert Nullstellensatz.

Theorem 8.2 The minimum degree Nullstellensatz for any cat ears graph is four.

Theorem 8.3 The minimum degree Nullstellensatz for \( K_n \), \( n \geq 4 \) is four.

Theorem 8.4 The minimum degree Nullstellensatz for any graph is four.