Polynomial time algorithms to approximate mixed volumes within a simply exponential factor

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Abstract

Let $K = (K_1,...,K_n)$ be a $n$-tuple of convex compact subsets in the Euclidean space $\mathbb{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbb{R}^n$. The Minkowski polynomial $V_K$ is defined as $V_K(x_1,...,x_n) = V(\lambda_1 K_1 + ... + \lambda_n K_n)$ and the mixed volume $V(K_1,...,K_n)$ as

$$V(K_1...K_n) = \frac{\partial^n}{\partial\lambda_1...\partial\lambda_n} V_K(\lambda_1 K_1 + \cdots + \lambda_n K_n).$$

The mixed volume is one of the backbones of convexity theory. After BKH theorem, the mixed volume (and its generalizations) had become crucially important in computational algebraic geometry.

We present in this talk randomized and deterministic algorithms to approximate the mixed volume of well-presented convex compact sets. Our main result is a poly-time randomized algorithm which approximates $V(K_1,...,K_n)$ with multiplicative error $e^n$ and with better rates if the affine dimensions of most of the sets $K_i$ are small.

Because of the famous Barany-Furedi lower bound, even such rate is not achievable by a poly-time deterministic oracle algorithm.

Our approach is based on the particular, geometric programming, convex relaxation of $\log(V(K_1,...,K_n))$. We prove the mixed volume analogues of the Van der Waerden and the Schrijver/Valiant conjectures on the permanent. These results, interesting on their own, allow to “justify” the above mentioned convex relaxation, which is solved using the ellipsoid method and a randomized poly-time time algorithm for the approximation of the volume of a convex set.

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