A PROBLEM ABOUT NILPOTENT MATRICES

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We consider n-by-n matrices with real entries. In particular, consider tridiagonal matrices having the following form:

$$M(x) := tridiag(a, d, x)$$

where

- $a := (-1, -1, \ldots, -1)$,
- $d := (-1, 0, 0, \ldots, 0, 0, 1)$, and
- $x := (x_1, x_2, \ldots, x_2, x_1)$.

In other words, M(x) is a tri-diagonal matrix which is all -1 on the subdiagonal, all 0 on the diagonal except for a -1 in the (1,1) entry and a 1 in the (n,n) entry, and a symmetric vector of variables on the superdiagonal. For example, when n is 5 we have:

$$\begin{matrix} -1 & x_1 & 0 & 0 & 0 \\ -1 & 0 & x_2 & 0 & 0 \\ 0 & -1 & 0 & x_2 & 0 \\ 0 & 0 & -1 & 0 & x_1 \\ 0 & 0 & 0 & -1 & 1 \end{matrix}$$

Conjecture: For all n there exist positive values for the variables such that the matrix M(x) is nilpotent.

Irv Hentzel (Iowa State University, hentzel@iastate.edu) has found matrices which satisfy the conjecture for n from 1 to 10. He has also proved that the solution with positive values for the unknown variables is unique.

The conjecture is related to the 2n-conjecture which involves the following question: Can matrices with 2n nonzero signed entries take on all possible eigenvalues? One way to prove that a general signed matrix takes on all possible eigenvalues is to show that for some values the matrix is nilpotent, and there is a small ball with positive radius around this nilpotent matrix where the matrix in not zero. The conjecture concerning nilpotent matrices is related to the first part of this proof program.

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Here is a more detailed description of the 2n-conjecture. Consider the following sign pattern:

$$\begin{matrix} + & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ + & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \\ + & 0 & 0 & 0 & - \end{matrix}.$$

This pattern is spectrally arbitrary in the sense that every monic polynomial with degree 5 is the characteristic polynomial at least one real matrix with this sign pattern. The 2n-conjecture asserts that an n-by-n spectrally arbitrary sign pattern has at least 2n nonzero entries.

Remark: From October 23 to 27, 2006 the American Institute of Mathematics is sponsoring a workshop on "Spectra of families of matrices described by graphs, digraphs and sign patterns". One of the aims of this workshop is to discuss the 2n-conjecture for spectrally arbitrary sign patterns.