Algebra & Algorithms for Differential Elimination & Completion.

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Abstract

Differential algebra provides an algebraic viewpoint on nonlinear differential systems. The motivating questions for this talk are:

- How do we define the general solution of a nonlinear equations
- What are the conditions for a differential system to have a solution
- How do we measure the *degrees of freedom* for the solution set of a differential system

Theory and algorithms for those are extensions of commutative algebra (prime ideal decomposition, Hilbert polynomials) and Groebner bases techniques.

The library diffalg in Maple supports this introduction to constructive differential algebra. It has been developed by F. Boulier (1996) and the speaker afterwards. A recent extension of differential algebra to non-commutative derivations, and its implementation in diffalg, allow to treat systems bearing on differential invariants.

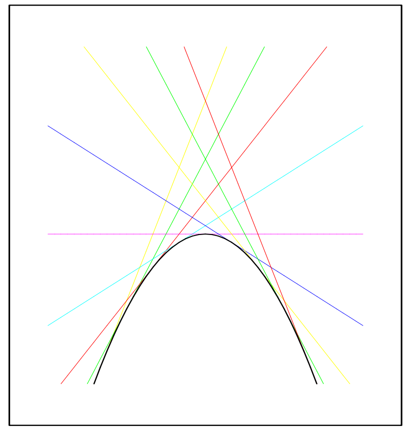
Contents

1 Sample problems

Envelope

Consider a family of curves

$$y = c\,x + c^2$$



What is their envelope?

The differential equation satisfied by this family is:

$$\frac{\mathrm{d}y}{\mathrm{d}x}^2 + x\,\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

General solution: $y = c\,x + c^2$

Singular solution: $y = -1/4x^2$

Kepler $\Rightarrow$ Newton

Kepler's observational laws:

K_1 the planets move along ellipses with the sun as focus
$r = p + e\,x$
where $r^2 = x^2 + y^2$
and $\dot{p}, \dot{e} = 0$

K_2 the vector from the sun to the planet sweeps equal area in equal times
$x\,\dot{y} - \dot{x}\,y = s$
with $\dot{s} = 0$

Newton's gravitational laws:

N_1 the acceleration is inversely proportional to the square of the distance to the sun.
$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\,a\right) = 0$
where $a^2 = (\ddot{x}^2 + \ddot{y}^2)$

N_2 the acceleration vector is directed towards the sun
$x\,\ddot{y} - \ddot{x}\,y = 0$

$K_2 \;\Rightarrow\; N_2, \qquad K_1, K_2 \;\Rightarrow\; N_1$?

Membership [Wu 91]

Orthogonal waves

[G. Metivier]

$$\begin{aligned} s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\ s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\ \psi_x\,\phi_x + \psi_y\,\phi_y &= 0 \end{aligned}$$

- Is there a solution? consistency
- What are the *degrees of freedom* completion

$$
\begin{array}{rcl}
s & = & f_1(y) + f_2(y)\,x + s_{20}\,\frac{x^2}{2} + s_{30}\,\frac{x^3}{6} \dots \\
\psi & = & f_3(y) + c_1\,x + \psi_{20}\,\frac{x^2}{2} + \psi_{11}\,x\,y + \psi_{20}\,\frac{x^2}{2} \dots \\
\phi & = & f_4(y) + \phi_{10}\,x + \phi_{01} y + \phi_{20}\,\frac{x^2}{2} + \phi_{11}\,x\,y + \dots
\end{array}
$$

4 functions of 1 variable, 1 constant.

- Conditions on s elimination

Equivalence

[Neut 03]

$$y'' = f(x,y,y') \quad \xrightarrow[X=\xi(x,y),\,Y=\eta(x,y)]{\exists ?\xi,\eta} \quad Y''=0$$

$$y_2 = f(x,y_0,y_1) \quad \xrightarrow[X=\xi(x,y_0),\,Y_0=\eta(x,y_0)]{\exists ?\xi,\eta} \quad Y_2=0$$

$$
\left\{
\begin{array}{l}
\xi_{y_1} \;=\; 0, \quad \eta_{y_1} \;=\; 0, \\
(\eta_{y_0}\xi_x - \xi_{y_0}\eta_x)\,f + (\eta_{y_0 y_0}\xi_{y_0} - \eta_{y_0}\xi_{y_0 y_0})\,{y_1}^3 \\
+\left(2\,\eta_{x y_0}\xi_{y_0} - 2\,\eta_{y_0 \xi_{x y_0} - \eta_x}\xi_{y_0 y_0} + \eta_{y_0 y_0}\xi_x\right){y_1}^2 \\
+\,(2\,\eta_{x y_0}\xi_x - 2\,\eta_x\xi_{x y_0} - \eta_{y_0}\xi_{xx} + \eta_{x,x}\xi_{y_0})\,y_1 - \eta_x\xi_{xx} + \eta_{xx}\xi_x \;=\; 0
\end{array}
\right.
$$

Differential indeterminates : ξ, η, f, functions of x, y_0, y_1

Derivations : $\partial_x = \frac{\partial}{\partial x}$, $\partial_{y_0} = \frac{\partial}{\partial y_0}$, $\partial_{y_1} = \frac{\partial}{\partial y_1}$

Differential indeterminates : ξ, η, f

Consider ∂_{y_0}, ∂_{y_1}, and $D_x = \partial_x + y_1\,\partial_{y_0} + f\,\partial_{y_1}$

$$\partial_{y_0} D_x - D_x\,\partial_{y_0} = f_{y_0}\,\partial_{y_1}, \quad \partial_{y_1} D_x - D_x\,\partial_{y_1} = \partial_{y_0} + f_{y_1}\,\partial_{y_1}$$

$$
\left\{
\begin{array}{l}
\xi_{y_1} \;=\; 0, \quad \eta_{y_1} \;=\; 0, \\
\xi_x\eta_{xx} - \xi_{xx}\eta_x \;=\; 0
\end{array}
\right.
$$

General case :

$$f_{y_1 y_1 y_1 y_1} = 0, f_{x x y_1 y_1} = 4 f_{x y_0 y_1} + f_{y_1} f_{x y_1 y_1} - 4 f_{y_1} f_{y_0 y_1} - 6 f_{y_0 y_0} + 3 f_{y_1 y_1} f_{y_0}$$

Fiber preserving case ($\xi_{y_0} = 0$)

$$f_{y_1 y_1 y_1} \;=\; 0, \quad f_{x y_1 y_1} \;=\; f_{y_0 y_1}, \quad f_{x y_0 y_1} \;=\; 2\,f_{y_0 y_0} - f_{y_1 y_1} f_{y_0} + f_{y_1} f_{y_0 y_1}$$

2 Software

Software for nonlinear differential systems

diffalg: Differential Algebra — MapleV

1996 Rosenfeld-Gröbner algorithm [BLOP 1997]
by F. Boulier (1996) then at SCG, U. of Waterloo. — Maple V.5

1998 Singular solutions [H 99]; Efficiency improvement [H 00]
by E. Hubert then at SCG, UWaterloo. — Maple 6

1999 Redesign of help pages. — Maple 7

2004 Non-commuting derivations [H 2005]
`http://www.inria.fr/cafe/Evelyne.Hubert/diffalg`

BLAD: Bibliothèques Lilloises d'Algèbre Différentielle — GLGPL

C libraries distributed under Gnu Lesser General Public License

`http://www2.lifl.fr/~boulier/BLAD` by F. Boulier

RegularChains for polynomial systems — Maple10

by F. Lemaire & M. Moreno Maza *et al.* U.WO — Maple 10

RIF: Reduced Involutive Forms — Maple6

A. Wittkopf, G. Reid — Maple 6, DEtools

CRACK: PDE solver — Reduce

T. Wolf (`http://lie.math.brocku.ca/crack`)

Software for linear functional systems

kan/sm1 by N. Takayama (`http://www.math.kobe-u.ac.jp/KAN`)

D-Macaulay by A. Leykin & H. Tsai (`http://www.ima.umn.edu/~leykin/Dmodules`)

Cocoa, `http://cocoa.dima.unige.it/`

Plural:Singular by V. Levandovskyy & H. Schönemann (`http://www.singular.uni-kl.de/plural`)

In Maple: Groebner by F. Chyzak; OreModule with A. Quadrat & D. Robertz. (`http://wwwb.math.rwth-aachen.de/OreModules`)

3 Ring of differential polynomials

Classical construction [Ritt 51, Kolchin 73]

$\mathbb{F} = \mathbb{Q}$ or $\mathbb{Q}(x,y)$

$$\begin{cases} s\,(\phi_{xx}+\phi_{yy})+s_x\,\phi_x+s_y\,\phi_y+\phi=0 \\ s\,(\psi_{xx}+\psi_{yy})+s_x\,\psi_x+s_y\,\psi_y+\psi=0 \\ \psi_x\,\phi_x+\psi_y\,\phi_y=0 \end{cases}$$

$\mathbb{F}$ a field

$\delta_1 = \frac{\partial}{\partial x},\ \delta_2 = \frac{\partial}{\partial y}$

$\Delta = \{\delta_1, \ldots, \delta_m\}$ derivations on $\mathbb{F}$

$\delta_i(a+b) = \delta_i(a) + \delta_i(b) \quad \delta_i(a\,b) = a\delta_i(b) + \delta_i(a)\,b$

$\mathcal{Y} = \{s, \phi, \psi\}$

$\mathcal{Y} = \{y_1, \ldots, y_n\}$

$\mathbb{F}[s, s_x, s_y, s_{xx}\ldots] = \mathbb{F}\,[\![s,\phi,\psi]\!]$

$s_{xxy} \rightsquigarrow s_{x^2y} \rightsquigarrow s_{(2,1)}$

$\mathbb{F}\,[\![\mathcal{Y}]\!] = \mathbb{F}[\,y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,]$

$\frac{\partial}{\partial x}\,(s_{xy}) = s_{xxy}$

$\rightsquigarrow \delta_1\left(s_{(1,1)}\right) = s_{(2,1)}$

$\delta_i\,(y_\alpha) = y_{\alpha+\epsilon_i}$

$\epsilon_i = (0,\ldots,\ \underset{i^{\text{th}}}{1}\ ,\ldots,0)$

$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$

$\delta_i\delta_j = \delta_j\delta_i$

Declare it in Maple

```
> with(diffalg);
```

[Rosenfeld_Groebner, characters, delta_polynomial, denote, derivatives, differential_ring, differentiate, equations, essential_components, field_extension, greater, inequations, initial, initial_conditions, is_prime, leader, power_series_solution, preparation_polynomial, print_ranking, rank, reduce, reduced_form, rewrite_rules, separant]

```
> R := differential_ring( ranking=[[s,phi,psi]], derivations=[x,y], notation=jet );
```

$R := differential\_polynomial\_ring$

```
> differentiate( s[x,y], x, R );
```

$s_{x,x,y}$

```
> denote( s[x,y], vjet, R );
```

$s_{1,1}$

```
> denote( s[x,y], diff, R );
```

$\frac{\partial^2}{\partial x \partial y} s\,(x,y)$

Rankings $\prec$ on $\mathbb{F}[\![\mathcal{Y}]\!]$

A total order on $\mathcal{DY} = \{y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\}$ s.t.

- $y_\alpha \prec y_{\alpha+\gamma}$
- $y_\alpha \prec z_\beta \;\Rightarrow\; y_{\alpha+\gamma} < z_{\beta+\gamma}$

$|\alpha| = \alpha_1 + \ldots + \alpha_m$

Orderly ranking: $|\alpha| < |\beta| \;\Rightarrow\; y_\alpha \prec z_\beta,$ $\quad \forall y,z$

Semi-orderly ranking: $|\alpha| < |\beta| \;\Rightarrow\; y_\alpha \prec y_\beta$

Elimination ranking: $y_\alpha \prec z_\beta,$ $\quad \forall \alpha,\beta$

Rankings with *diffalg*

```
> DY := [seq(seq(seq(u[x$i,y$(j-i)],i=0..j),j=0..2),u=[s,phi,psi])];
```

$$DY := [s, s_y, s_x, s_{y,y}, s_{x,y}, s_{x,x}, \phi, \phi_y, \phi_x, \phi_{y,y}, \phi_{x,y}, \phi_{x,x}, \psi, \psi_y, \psi_x, \psi_{y,y}, \psi_{x,y}, \psi_{x,x}]$$

```
>R2:=differential_ring(ranking=[grlex[s],grlex[phi,psi]],derivations=[x,y]):
> sort(DY, (a,b) -> greater(b,a,R2));
```

$$[\psi, \phi, \psi_y, \psi_x, \phi_y, \phi_x, \psi_{y,y}, \psi_{x,y}, \psi_{x,x}, \phi_{y,y}, \phi_{x,y}, \phi_{x,x}, s, s_y, s_x, s_{y,y}, s_{x,y}, s_{x,x}]$$

```
> R1:=differential_ring(ranking=[[s,phi,psi]],derivations=[x,y]):

> sort(DY, (a,b) -> greater(b,a,R1));
```

$$[\psi, \phi, s, \psi_y, \phi_y, s_y, \psi_x, \phi_x, s_x, \psi_{y,y}, \phi_{y,y}, s_{y,y}, \psi_{x,y}, \phi_{x,y}, s_{x,y}, \psi_{x,x}, \phi_{x,x}, s_{x,x}]$$

Non-commuting derivations

$$y_2 = f(x, y_0, y_1) \quad \underset{Y_0 = \eta(x, y_0)}{\overset{X = \xi(x, y_0)}{\longrightarrow}} \quad Y_2 = 0$$

$$\begin{cases} \xi_{y_1} = 0, \quad \eta_{y_1} = 0, \\ \xi_x \eta_{xx} - \xi_{xx}\eta_x = 0 \end{cases}$$

Differential indeterminates : ξ, η, f

Derivations ∂_{y_0}, ∂_{y_1}, and D_x

$$\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0} \partial_{y_1},$$

$$\partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1} \partial_{y_1}$$

$$\partial_{y_1}\partial_{y_0} - \partial_{y_0}\partial_{y_1} = 0$$

$\mathcal{Y} = \{y_1, \dots, y_n\}$

$\mathcal{D} = \{\delta_1, \dots, \delta_m\}$ with

$$\delta_i\delta_j - \delta_j\delta_i = \sum_{l=1}^{m} c_{ijl}\, \delta_l$$

Differential polynomial ring $\mathbb{K}[\![\mathcal{Y}]\!]$ with non commuting derivations **[H05]**

$\mathcal{Y} = \{y_1, \dots, y_n\}$

$\mathcal{D} = \{\delta_1, \dots, \delta_m\}$

$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$

$$\delta_i(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \text{ if } \alpha_1 = \dots = \alpha_{i-1} = 0 \\ \delta_j \delta_i \left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\, \delta_l \left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \dots = \alpha_{j-1} = 0 \end{cases}$$

Example (m=2): $\delta_1 y_{(1,1)} = y_{(2,1)}$

$$\begin{aligned} \delta_2 y_{(1,1)} = \delta_2\delta_1 y_{(0,1)} = \ & \delta_1\delta_2 y_{(0,1)} + c_{121}\, \delta_1 y_{(0,1)} + c_{122}\, \delta_2 y_{(0,1)} \\ & \delta_1 y_{(0,2)} + c_{121}\, y_{(1,1)} + c_{122}\, y_{(0,2)} \\ & y_{(1,2)} + c_{121}\, y_{(1,1)} + c_{122}\, y_{(0,2)} \end{aligned}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$,
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma}$,
- $\sum_{l \in \mathbb{N}_m} c_{ijl}\, \delta_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then $\delta_i \delta_j(p) = \delta_j \delta_i(p) + \sum_{l=1}^{m} c_{ijl}\, \delta_l(p) \quad \forall p \in \mathbb{K}[\, y_\alpha \,|\, \alpha \in \mathbb{N}^m \,] = \mathbb{K}\,[\![\mathcal{Y}]\!]$

With *diffalg*

$$\partial_{y_0}\, D_x - D_x\, \partial_{y_0} = f_{y_0}\, \partial_{y_1}, \quad \partial_{y_1}\, D_x - D_x\, \partial_{y_1} = \partial_{y_0} + f_{y_1}\, \partial_{y_1}$$

> R := differential_ring(ranking=[[xi,eta],f], derivations=[x,y0,y1], commutations=[[y0,x]=[0,0,f[y0]], [y1,x]=[0,1,f[y1]]]):

> differentiate(differentiate(eta, y0, R), x, R);

$$\eta_{x,y0}$$

> differentiate(eta, x, y0, R);

$$\eta_{x,y0}$$

> differentiate(eta, y0, x, R);

$$\eta_{x,y0} + f_{y0}\, \eta_{y1}$$

Leader, initial, separant

- $\mathcal{Y} = \{y_1, \ldots, y_n\}$, $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$, $\mathbb{F}\,[\![\mathcal{Y}]\!]$, $\prec$

- Nota^o: $\delta^\alpha = \delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m}$

$$\delta^\alpha \delta^\beta \neq \delta^{\alpha+\beta} \text{ but } \delta^\alpha \delta^\beta = \delta^{\alpha+\beta} + \sum_{|\gamma|<|\alpha+\beta|} a_\gamma\, \delta^\gamma, \quad a_\gamma \in \mathbb{K}\,[\![\mathcal{Y}]\!]$$

- $p \in \mathbb{K}\,[\![\mathcal{Y}]\!] \setminus \mathbb{K}$

$$p = i_p\, {y_\alpha}^d + c_1\, {y_\alpha}^{d-1} \ldots + c_d \qquad c_i \prec y_\alpha$$

y_α leader	i_p initial	s_p separant
lead(p)	init(p)	sep(p)

- $s_p = \dfrac{\partial p}{\partial y_\alpha} = d\, i_p\, y_\alpha^{d-1} + (d-1)\, c_1\, y_\alpha^{d-2} + \ldots + c_{d-1}$

Prop:

$$\text{lead}(p) = y_\alpha \quad \Rightarrow \quad \delta^\beta(p) = \text{sep}(p)\; y_{\alpha+\beta} + \underbrace{\ldots\ldots}_{\prec\, y_{\alpha+\beta}}$$

4 Differential ideals

Differential Ideals

$\mathcal{Y} = \{y_1, \ldots, y_n\}$, $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$, $\mathbb{F}[\![\mathcal{Y}]\!]$

$\{p_1, \ldots, p_k\} \subset \mathbb{F}[\![\mathcal{Y}]\!]$

I, a differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a \in I \Rightarrow \delta a \in I, \forall \delta \in \Delta$
- I is an ideal

$[p_1, \ldots, p_k]$

$[p] = (\, \delta^\alpha p \,|\, \alpha \in \mathbb{N}^m)$

J, radical differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a^k \in J \;\Rightarrow\; a \in J$
- J is a differential ideal

$[\![p_1, \ldots, p_k]\!]$

$[\![p]\!] = \{q | \, q^r \in [p]\}$

P, prime differential ideal of $\mathbb{F}[\![\mathcal{Y}]\!]$:

- $a\, b \in P \Rightarrow a \in P$ or $b \in P$
- is a differential ideal

P radical and irreducible.

Differential Nullstellensatz

$p_1, \ldots, p_r \in \mathbb{F}[\![\mathcal{Y}]\!]$

NOTE:
- if $q \in [\![p_1, \ldots, p_r]\!]$ then $q^e = \sum c_{i\alpha} \delta^\alpha p_i$ and q vanishes on all the common zeros of $p_1, \ldots, p_r$.
- if $1 \in [\![p_1, \ldots, p_r]\!]$ then $1 = \sum c_{i\alpha} \delta^\alpha p_i$ so that $p_1, \ldots, p_r$ have no common zero.

THEO:
- $p_1, \ldots, p_k$ admit a common zero iff $1 \notin [\![p_1, \ldots, p_r]\!]$
- $q \in \mathbb{F}[\![\mathcal{Y}]\!]$ vanishes on all the common zeros of $p_1, \ldots, p_k$ iff $q \in [\![p_1, \ldots, p_k]\!]$.

Ritt-Raudenbush Theorem

THEO: A radical differential ideal J of $\mathbb{F}[\![\mathcal{Y}]\!]$ is

- finitely generated: $J = [\![p_1, \ldots, p_r]\!]$ for some $p_1, \ldots, p_r \in \mathbb{F}[\![\mathcal{Y}]\!]$.

  Differential ideals need not be finitely generated.
- the intersection of finitely many prime differential ideals.

  The irredundant decomposition $J = \bigcap_{i=1}^{r} P_i$ is unique.

EX: $[\![y_x^2 + x\, y_x - y]\!] = [\![y_x^2 + x\, y_x - y, y_{xx}]\!] \cap [\![4\, y + x^2]\!]$.

Saturation ideals

- $h \in \mathbb{F}[\![\mathcal{Y}]\!]$ $\quad I : h^\infty = \left\{ q \mid \exists k \text{ s.t. } h^k\, q \in I \right\}$

- $H = \{h_1, \ldots, h_s\}$ $\quad I : H^\infty = I : h^\infty \quad \text{where } h = h_1 \ldots h_s$

NOTE: The zeros of $[\![p]\!] : h^\infty$ are the zeros of p that don't vanish on h, except for some *adherent* piece.

EX: $[\![y_x^2 + x\, y_x - y]\!] = [\![y_x^2 + x\, y_x - y]\!] : (2y_x + x)^\infty \cap [\![4\, y + x^2]\!]$.

5 Representation of radical differential ideals

Purpose

Given $p_1, \ldots, p_r$ in $\mathbb{F}[\![\mathcal{Y}]\!]$ we want to compute a representation of $[\![p_1, \ldots, p_r]\!]$ that allows to

- test membership to $[\![p_1, \ldots, p_r]\!]$
- *measure* the zero set (completion)
- compute $[\![p_1, \ldots, p_r]\!] \cap \mathbb{F}[\![\mathcal{Z}]\!]$ (elimination)

In a factorisation free way.

There is no strict analogue of Gröbner bases.

Differential ideals do not admit in general

- a finite set
- that is generating for the differential ideal
- and *reduces* to zero the elements of the differential ideal

Characterisable differential ideals are defined by

- a finite set
- that *reduces* to zero the elements of the differential ideal
- generate the ideal *outside of some hypersurface*

Prime differential ideals are characterizable

Radical diff. ideal are $\cap^o$ of characterisable diff. ideals

Characteristic Decomposition

ALGO: (Rosenfeld-Groebner in *diffalg*)

In: $\{p_1, \ldots, p_k\} \subset \mathbb{F}[\![\mathcal{Y}]\!]$, $\prec$

Out: $C_1, \ldots C_r$ s.t.

$$[\![p_1, \ldots, p_k]\!] = [C_1] : S_1^\infty \cap \ldots \cap [C_r] : S_r^\infty$$

Membership: $p \in [C_i] : S_i \;\Leftrightarrow\; p \longrightarrow_{C_i} 0$

Completion: If $\prec$ orderly then $[C_i] : S_i^\infty \cap \mathbb{F}[\mathcal{D}_\kappa \mathcal{Y}] = (\mathcal{D}_\kappa C_i) : S_i^\infty$

Elimination: If $\prec$ eliminates $\mathcal{Y} \setminus \mathcal{Z}$ and $C_i' = C_i \cap \mathbb{F}[\![\mathcal{Z}]\!]$ then

$$[C_i] : S_i^\infty \cap \mathbb{F}[\![\mathcal{Z}]\!] = [C_i'] : {S_i'}^\infty$$

Output characteristic sets C

or pseudo-Gröbner bases

- C is coherent:

  $a, b \in C, \qquad \mathsf{lead}(a) = y_\alpha,\ \mathsf{lead}(b) = y_\beta$

  $\gamma = \alpha + \bar{\alpha} = \beta + \bar{\beta} \quad \Rightarrow \quad s_a\, \delta^{\bar{\beta}} b - s_b\, \delta^{\bar{\alpha}} a \in (\mathcal{D}C_{<y_\gamma}) : S^\infty$

  Recall:

  $$\begin{array}{rcl} \delta^{\bar{\beta}} b &=& s_b\, y_\gamma + \ldots \\ \delta^{\bar{\alpha}} a &=& s_a\, y_\gamma + \ldots \end{array}$$

  A generalisation of

  $$\left.\begin{array}{l} u_x = f(x,y) \\ u_y = g(x,y) \end{array}\right\} \Rightarrow\ f_y - g_x = 0$$

- C is a differential triangular set:

  $a, b \in C, \quad \mathsf{lead}(a) = y_\alpha,\ \mathsf{lead}(b) = y_\beta \quad \Rightarrow \quad \beta \neq \alpha + \gamma$

- $[C] : S^\infty$ is a radical differential ideal.

- $p \longrightarrow_c q$ means:
  - q is free of proper derivatives of $\mathsf{lead}(c)$

    $s_c^e\, p = q_1 \mod (\delta^\beta c, \ldots, \delta^\gamma c)$
  - the degree of q in $\mathsf{lead}(p)$ is lower than that of p

    $i_p^f\, q_1 = q \mod (p)$

Envelope

> R:= differential_ring(ranking=[c,y],derivations=[x]):
> C := Rosenfeld_Groebner($[y - c\,x - c^2,\ c_x]$, R);

$$C := [characterisable]$$

> equations(G);

$$[[-y_x + c,\ y_x^2 + x\,y_x - y]]$$

> p := equations(G[1])[-1];

$$p := y_x^2 + x\,y_x - y$$

> C:= Rosenfeld_Groebner([p],R);

$$C := [characterisable, characterisable]$$

> equations(C), inequations(C)

$$[[y_x^2 + x\,y_x - y],\ [4\,y + x^2]],\quad [[2y_x + x],\ [\,]]$$

That is: $[\![p]\!] = [p] : (2y_x + x)^\infty \cap [4\,y + x^2]$

Kepler $\Rightarrow$ Newton

> R:= differential_ring(ranking=[[x,y],[a,r],[p,e,s]],derivations=[t], parameters=[p,e,s]):
> K := $\{a^2 - x_{t,t}^2 - y_{t,t}^2, r^2 - x^2 - y^2), r - p + e\,x, x\,y_t - y\,x_t - s\}$:
> C := Rosenfeld_Groebner(K, $\{p, e, s\}$, R);

$$C := [characterisable]$$

> N := differentiate($r^2\,a$,t,R):

$$N := r\,(a_t r + 2\,r_t a)$$

> reduce(N, C[1]);

$$0$$

With $[\![K]\!] : (pes)^\infty = [e\,\mathbf{x} + r - p, es\,\mathbf{y} - prr_t, r^2p^2\mathbf{r_t}^2 - s^2(e^2r^2 + r^2 - 2\,rp + p^2), r^4p^2\mathbf{a}^2 - s^4] : (epsarr_t)^\infty$

Orthogonal Waves

$$\begin{aligned} s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\ s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\ \psi_x\,\phi_x + \psi_y\,\phi_y &= 0 \\ s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0 \end{aligned}$$

> with(diffalg)

> R:=differential_ring(ranking=[[s,phi,psi]],derivations=[x,y]):

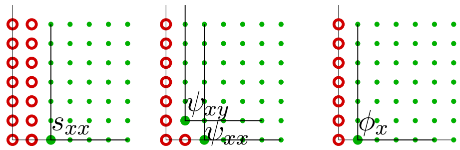
> S:=[s*(phi[x,x]+phi[y,y])+s[x]*phi[x]+...]

> H :={s[x],s[y],phi[x],...}

> C :=Rosenfeld_Groebner(S,H,R);

$$C := [characterisable]$$

$$C = \{s_{xx} = \ldots, \psi_{xx} = \ldots, \psi_{xy} = \ldots, \phi_x = \ldots\}$$



$s \qquad \psi \qquad \phi$

$$\begin{aligned}
s &= s_{00} + s_{10}\,x + s_{01}y + s_{20}\tfrac{x^2}{2} + s_{11}\,x\,y + s_{02}\,\tfrac{y^2}{2} + s_{30}\,\tfrac{x^3}{6}\ldots \\
\psi &= \psi_{00} + \psi_{10}\,x + \psi_{01}y + \psi_{20}\tfrac{y^2}{2} + \psi_{11}\,x\,y + \psi_{02}\,\tfrac{x^2}{2}\ldots \\
\phi &= \phi_{00} + \phi_{10}\,x + \phi_{01}y + \phi_{20}\tfrac{x^2}{2} + \phi_{11}\,x\,y + \phi_{02}\,\tfrac{y^2}{2}\ldots
\end{aligned}$$

$$\begin{aligned}
s &= f_1(y) + f_2(y)\,x + s_{20}\tfrac{x^2}{2} + s_{30}\,\tfrac{x^3}{6}\ldots \\
\psi &= f_3(y) + c_1\,x + \psi_{20}\tfrac{x^2}{2} + \psi_{11}\,x\,y + \psi_{20}\,\tfrac{x^2}{2}\ldots \\
\phi &= f_4(y) + \phi_{10}\,x + \phi_{01}y + \phi_{20}\tfrac{x^2}{2} + \phi_{11}\,x\,y + \ldots
\end{aligned}$$

Challenge Computational Problem

$$\begin{aligned}
s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\
s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\
\psi_x\,\phi_x + \psi_y\,\phi_y &= 0 \\
s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0
\end{aligned}$$

```
> R:=differential_ring(ranking=[[phi,psi],s],derivations=[x,y]):
```

6 Symmetric Systems

$$\mathcal{S}\left\{\begin{aligned}
s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\
s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\
\psi_x\,\phi_x + \psi_y\,\phi_y &= 0
\end{aligned}\right.$$

Idea : factor out the symmetry [Mansfield 01], [Lisle & Reid 06] [15pt]
Tool : moving frame construction [Fels & Olver 99]

[Desolv,Vessiot]

$$\mathcal{S}\left\{\begin{aligned}
s\,(\phi_{xx} + \phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi &= 0 \\
s\,(\psi_{xx} + \psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi &= 0 \\
\psi_x\,\phi_x + \psi_y\,\phi_y &= 0
\end{aligned}\right.$$

$$(x, y, s, \phi, \psi) \overset{g_{\alpha,\beta,\rho,\tau,\mu,\nu,a,b}}{\longrightarrow} (X, Y, S, \Phi, \Psi) \qquad \beta^2 + \alpha^2 = 1$$

$$X = \frac{\alpha}{\rho}\,x - \frac{\beta}{\rho}\,y + \frac{a}{\rho} \quad Y = \frac{\alpha}{\rho}\,x + \frac{\beta}{\rho}\,y + \frac{b}{\rho}$$

$$S \;=\; \frac{s}{\rho^2\tau} \quad \Phi \;=\; \frac{\phi}{\mu}\Psi \;=\; \frac{\psi}{\nu}$$

$$\begin{aligned}
S_X &= \tfrac{\beta}{\rho}\,s_x - \tfrac{\alpha}{\rho}\,s_y & S_Y &= \tfrac{\alpha}{\rho}\,s_y + \tfrac{\beta}{\rho}\,s_x \\
\Phi_X &= \tfrac{\rho\alpha}{\mu}\,\phi_x - \tfrac{\rho\beta}{\mu}\,\phi_y & \Phi_Y &= \tfrac{\rho\alpha}{\mu}\,\phi_y + \tfrac{\rho\beta}{\mu}\,\phi_x \\
& & \ldots
\end{aligned}$$

All differential invariants can be written in terms of the fundamental invariants [Vessiot, Groebner]

$$s_1^2 := \frac{s_x^2 + s_y^2}{4\,s},$$

$$\begin{array}{rclrcl} s_2 &:=& \dfrac{s_{xy}(s_y^2 - s_x^2) + s_x s_y (s_{xx} - s_{yy})}{8\,s\,s_1^3}, & s_3 &:=& \dfrac{s_x^2 s_{yy} + s_y^2 s_{xx} - 2\,s_x s_y s_{xy}}{8\,s\,s_1^3} - s_1, \\ \psi_1 &:=& \dfrac{s_y \psi_x - s_x \psi_y}{2\,s_1\,\psi}, & \psi_2 &:=& \dfrac{s_x \psi_x + s_y \psi_y}{2\,s_1\,\psi}, \\ \phi_1 &:=& \dfrac{s_y \phi_x - s_x \phi_y}{2\,s_1\,\phi}, & \phi_2 &:=& \dfrac{s_x \phi_x + s_y \phi_y}{2\,s_1\,\phi}. \end{array}$$

(s_1, s_2, s_3 depend only on s and derivatives)

and their derivatives with respect to the invariant derivations:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{\pm\sqrt{s\,(s_y^2 + s_x^2)}}{s_x^2 - s_y^2} \begin{pmatrix} -s_y & s_x \\ s_x & -s_y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$\mathcal{P} = (x,\, y,\, s - 1,\, \phi - 1,\, \psi - 1,\, s_x)$

Invariants: $s_1 = \iota(s_y)/2$, $s_2 = \frac{\iota(s_{xy})}{\iota(s_y)}$, $s_3 = \frac{\iota(s_{xx})}{\iota(s_y)} - \iota(s_y)/2$, $\psi_1 = \iota(\psi_x)$, $\psi_2 = \iota(\psi_y)$, $\phi_1 = \iota(\phi_x)$ and $\phi_2 = \iota(\phi_y)$.

We can write $\mathcal{S}$ in terms of $\mathcal{Y} = \{s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2\}$ and $\Delta = \{\delta_1, \delta_2\}$: expressions of $s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2$ not needed

$$\mathcal{S} \left\{ \begin{array}{c} \delta_1(\phi_1) + \delta_2(\phi_2) + \phi_1^2 + \phi_2^2 - s_2\phi_1 + (2\,s_1 + s_3)\phi_2 + 1 = 0, \\ \delta_1(\psi_1) + \delta_2(\psi_2) + \psi_1^2 + \psi_2^2 - s_2\psi_1 + (2\,s_1 + s_3)\psi_2 + 1 = 0, \\ \phi_1\,\psi_1 + \phi_2\,\psi_2 = 0. \end{array} \right.$$

but now

$$\delta_1\delta_2 - \delta_2\delta_1 = s_3\,\delta_1 + s_2\,\delta_2$$

and the fundamental invariants are not differentially independent.
The *syzygies* are

$$\mathcal{Z} \left\{ \begin{array}{rcl} \delta_1(s_1) &=& s_1 s_2 \\ \delta_1(s_2) - \delta_2(s_3) &=& s_3^2 + s_2^2 + s_1\,(s_2 + s_3) \\ \delta_1(\phi_2) - \delta_2(\phi_1) &=& \phi_1\,s_3 + \phi_2\,s_2, \\ \delta_1(\psi_2) - \delta_2(\psi_1) &=& \psi_1\,s_3 + \psi_2\,s_2. \end{array} \right.$$

Project status

Rational Invariants of a Group action. Construction and Rewriting, with I. Kogan, JSC, in press.

Smooth and Algebraic Invariants. Local and Global construction, with I. Kogan, submitted.

Algebraic Algorithms for Computing Differential Invariants, with I. Kogan, in preparation.

Algebra for differential invariants, in preparation.

Differential polynomial algebra with non commuting derivations. Journal of Pure and Applied Algebra, 200:1-2, p 163-190 (2005).

Software: *aida*, *diffalg*
`inria.fr/cafe/Evelyne.Hubert/aida`
`inria.fr/cafe/Evelyne.Hubert/diffalg`

Thanks.