Intersection Theory II

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Conics through $p = [1 : 0 : 0]$ revisited

- $\phi([x : y : z]) = [xy : xz : y^2 : yz : z^2]$.
- $\phi([x : 0 : 0]) = [0 : 0 : 0 : 0 : 0]$ (not allowed!) $\Rightarrow \phi$ is not defined at p.

Question

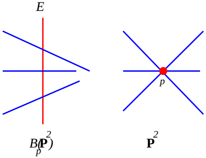
What do we add to $\phi(\mathbf{P}^2 \setminus \{p\})$ when we take the closure?

- If $y, z \neq 0$ then $\frac{z}{y} = t \neq 0 \Rightarrow z = ty$.
- $\phi([x : y : ty]) = [xy : txy : y^2 : ty^2 : t^2y^2]$
- $\lim_{y \to 0} [x : tx : y : ty : t^2y] = [x : tx : 0 : 0 : 0]$.

We add in the line $[s : t : 0 : 0 : 0]$.

Blowing up $p = [1:0:0] \in \mathbf{P}^2$

- Homogenizing the equation $z = ty$ gives $sz = ty$.
- $B_p(\mathbf{P}^2) = \{([x:y:z],[s:t]) \mid sz = ty\} \subset \mathbf{P}^2 \times \mathbf{P}^1$.
- $f : B_p(\mathbf{P}^2) \to \mathbf{P}^2$, where $f(q,[s:t]) = q$.
- $f^{-1}(q) = \begin{cases} q & q \neq p \\ E & q = p \end{cases}$



- $E \cong \mathbf{P}^1$ parametrizes slopes $t = \frac{z}{y}$ of lines through p.

Question

What is intersection theory like on $B_p(\mathbf{P}^2)$?

Warmup: Intersection theory on $\mathbf{P}^2$

- Bézout's Theorem in $\mathbf{P}^n \Rightarrow H_1 \cdots H_n$ only depends on $\deg H_i$.
- Homogeneous form F with $\deg F = d \leftrightarrow$ a hypersurface of degree d.

Example

- $\mathbf{V}(y) =$ line L
- $\mathbf{V}(zy - x^2) =$ conic F
- $\mathbf{V}(y^2) =$ double line $G \equiv 2L \equiv F$

In general:

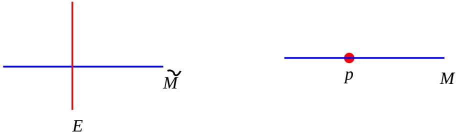
- If $\deg h = d$, then $\mathbf{V}(h) \equiv dL$ where L is a line.
- $aL \cdot bL = ab$

Pulling back to the blowup

- Line $L = \mathbf{V}(x)$. $f^{-1}([0:y:z]) = ([0:y:z],[y:z])$ since $sz = ty$

$\Rightarrow f^{-1}(L) = \tilde{L} \cong L$

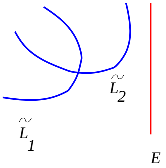
- Line $M = \mathbf{V}(y)$. At $[1:0:0]$, every $[s:t]$ satisfies $sz = ty$.

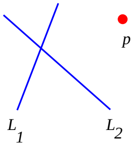


$\Rightarrow f^{-1}(M) \equiv \tilde{M} + E$, $\tilde{M} \cong M$.

Intersection theory on $B_p(\mathbf{P}^2)$: $f^{-1}(C) \cdot f^{-1}(D) = C \cdot D$

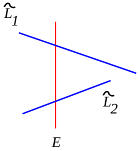
$\tilde{L_1} \cdot \tilde{L_2} = 1$

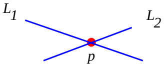




If $L_1 \cap L_2 = \{p\}$, then

$$1 = f^{-1}(L_1) \cdot f^{-1}(L_2) = (\tilde{L_1} + E) \cdot (\tilde{L_2} + E)$$





$\Rightarrow E^2 = -1$

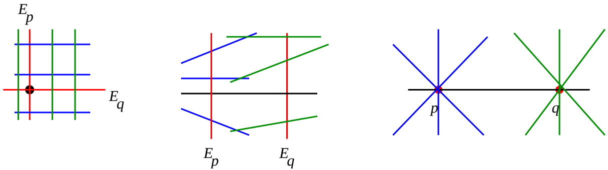
The degree of $\phi(\mathbf{P}^2\backslash\{p\}) \subset \mathbf{P}^4$

- $\phi([x : y : z]) = [xy : xz : y^2 : yz : z^2]$
- Linear form $w_0 - w_4$ corresponds to curve $\mathbf{V}(xy - z^2) \subset \mathbf{P}^2$.
- Hyperplanes $H_1, H_2 \subset \mathbf{P}^4$ correspond to conics $C_i \subset \mathbf{P}^2$ through p.
- $4 = C_1 \cdot C_2 = f^{-1}(C_1) \cdot f^{-1}(C_2) = (\tilde{C}_1 + E) \cdot (\tilde{C}_2 + E)$
- $$\begin{cases} \tilde{C}_i \cdot E = 1 \\ E^2 = -1 \end{cases}$$

$\Rightarrow \tilde{C}_1 \cdot \tilde{C}_2 = \deg \overline{\text{Image}\,\phi} = 3.$

What is $\psi(\mathbf{P}^2 \setminus \{p, q\}) \subset \mathbf{P}^3$?

- $\psi([x : y : z]) = [xy : xz : yz : z^2]$
- $[w_0 : w_1 : w_2 : w_3]$ homogeneous coordinates on $\mathbf{P}^3$
- $w_0 w_3 - w_1 w_2 = (xy)(z^2) - (xz)(yz) = 0$
- $\deg \overline{\text{Image}\,\psi} = 2$
- Replace $[1 : 0 : 0], [0 : 1 : 0]$ with $E_1, E_2 \cong \mathbf{P}^1$.
- Notice that $[x : y : 0] \mapsto [xy : 0 : 0 : 0] =$ point.
- $\text{Image}\,\psi \cong \mathbf{P}^2$ blown up at p, q with $\overline{pq}$ blown down.



The degree of γ

- Replace $p_1, \ldots, p_6$ with $E_1, \ldots, E_6$.
- $E_i^2 = -1$.
- Let H_1, H_2 hyperplanes in $\mathbf{P}^3$.
- The hyperplane correspond to cubic curves in $C_i \subset \mathbf{P}^2$ through the 6 points.
- $C_1 \cdot C_2 = 9$ by Bézout's theorem, $\Rightarrow \tilde{C}_1 \cdot \tilde{C}_2 = 3$.
- $\Rightarrow$ We have a cubic surface.

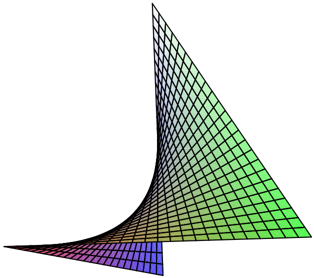
Lines on surfaces

- $\mathbf{P}^2$ contains infinitely many lines = curves of degree 1.
- $L_1 \cdot L_2 = 1$

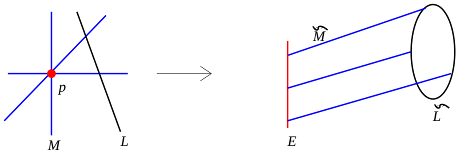
Question

Do our three new surfaces contain lines?

$\psi(\mathbf{P}^2 \backslash \{p, q\})$ contains two infinite families!



Lines in the image of ϕ : What does ϕ do to L and M?

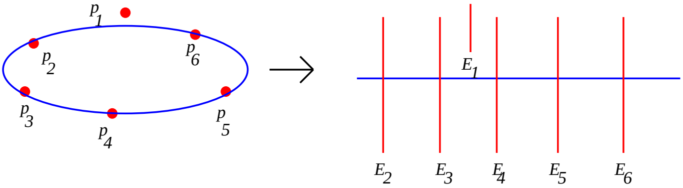


- Let's compute the degrees of $\tilde{L}$ and $\tilde{M}$.
- A hyperplane section corresponds to a conic $C \subset \mathbf{P}^2$ through p and $f^{-1}(C) = \tilde{C} + E$.
- $2 = C \cdot L = (\tilde{C} + E) \cdot \tilde{L} = \tilde{C} \cdot \tilde{L} \Rightarrow \tilde{L}$ is not a line.
- $0 = (\tilde{C} + E) \cdot E \Rightarrow \tilde{C} \cdot E = 1 \Rightarrow E$ is a line.
- $2 = C \cdot M = (\tilde{C} + E) \cdot (\tilde{M} + E) = (\tilde{C} + E) \cdot \tilde{M} = \tilde{C} \cdot \tilde{M} + 1 \Rightarrow \tilde{M}$ is a line.

We have the line E together with an infinite family of lines L_t with $L_{t_1} \cdot L_{t_2} = 0$ and $L_t \cdot E = 1$.

Lines in the image of γ : conics mapping to lines

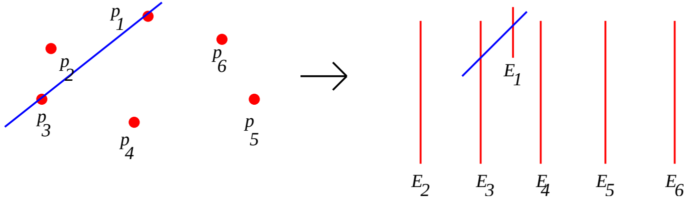
- There is a unique conic Q_i through $\{p_1, \ldots, p_6\} \backslash \{p_i\}$.
- $6 = (\tilde{C} + E_1 + \cdots + E_6) \cdot (\tilde{Q}_1 + E_2 + E_3 + E_4 + E_5 + E_6)$



$\Rightarrow \tilde{C} \cdot \tilde{Q}_1 = \deg \tilde{Q}_1 = 1$

Lines in the image of γ : lines mapping to lines

- $L_{ij} = \overline{p_i p_j}$



- $3 = (\tilde{C} + E_1 + \cdots + E_6) \cdot (\tilde{L_{13}} + E_1 + E_3) = \tilde{C} \cdot \tilde{L_{13}} + 2$

$\Rightarrow \tilde{C} \cdot \tilde{L_{13}} = \deg L_{13} = 1$

How many lines are there on a cubic surface?

- 6 points blow up to lines E_i
- 6 conics Q_i map to lines
- $\binom{6}{2} = 15$ lines L_{ij} map to lines
- It can be shown (using intersection theory) that a smooth cubic surface contains exactly 27 lines!

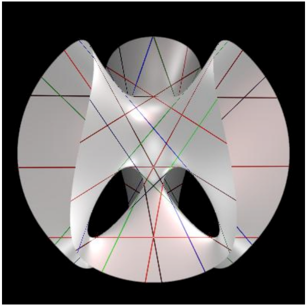


Figure: O. Labs. *The Algebraic Surface Homepage*. University of Mainz (2004) `www.AlgebraicSurface.net`

Summing up and looking forward . . .

- A theory of intersection numbers can help us enumerate solutions to systems of polynomial equations.
- We can use intersection numbers to study geometric questions.
- We can also define the intersection of surfaces in a 3-dimensional space to be a sum of curves (with multiplicity), etc.
- When the algebra becomes unwieldy, we can often translate questions about intersection theory into combinatorial settings.

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