# Intersection Theory I

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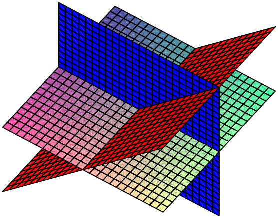
# Systems of polynomial equations: varieties

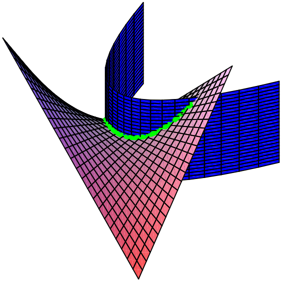
A homogeneous system of linear equations:

$$\begin{aligned} x \quad\quad\quad\quad &= 0 \\ y \quad\quad -z &= 0 \\ z &= 0 \end{aligned}$$



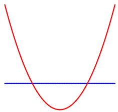
A system of polynomial equations of higher degree:

$$z - xy = 0$$
$$y - x^2 = 0$$

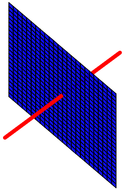


# Our most basic goals: What is intersection theory supposed to do?

- $H_1, \ldots, H_n$ hypersurfaces in an $n$-dimensional space
  $H_1 \cdots H_n$ = number of points in common to all $H_i$



- $X$ subvariety of dimension p
  $Y$ subvariety of dimension n-p
  $X \cdot Y =$ number of points in common



# Applications of intersection theory

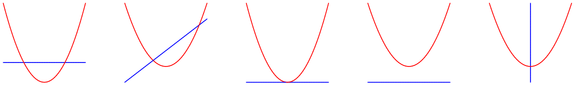
- Schubert conditions (How many lines in $\mathbf{P}^3$ meet 4 given lines in general position?)
- Given 5 conics in $\mathbf{P}^2$, how many conics are tangent to all 5?(3264)
- Bézout's theorem
- Understand the *geometry* of $X \subset \mathbf{P}^n$ : Does $X$ contain lines? Can we enumerate them? How are the lines arranged?

We'll discuss the last two.

# How should $H_1 \cdots H_n$ behave?

Example ($H_1, \ldots, H_n \subset \mathbf{A}^n$, $H_i = \mathbf{V}(f_i)$, $\deg f_i = d_i$ )

If $H_1 = \mathbf{V}(y - x^2)$, $H_2 = \mathbf{V}(f_2)$, $\deg f_2 = 1$, then $H_1 \cdot H_2$ should be 2.



(a) y-1 (b) y-x-1 (c) y (d) y+1 (e) x

We need to consider points of intersection

1. with multiplicity
2. in $\mathbf{C}$
3. and if we also consider points at “infinity” …

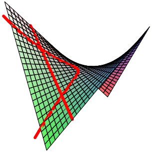
# Towards a definition of multiplicity

**Theorem (Bézout)**

*If* $H_1, \ldots, H_n \subset \mathbf{P}^n_{\mathbf{C}}$, $H_i = V(f_i)$, $\deg(f_i) = d_i$, *then*

$$H_1 \cdots H_n = d_1 \cdots d_n.$$

$F, G$ curves on a smooth surface $X$



The multiplicity of $F \cdot G$ at $p$, denoted $\mathrm{mult}_p(F \cdot G)$, should be defined so that

- if curves meet transversely at $p$, then $\mathrm{mult}_p(F \cdot G) = 1$.
- intersection numbers are constant on "families."

# Looking locally at $p$

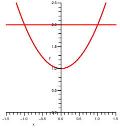
## Definition

The local ring of a point $p$ in a variety $X \subset \mathbf{C}^n$, denoted $\mathbf{C}[X]_p$, consists of all fractions $\frac{f}{g}$ where $f, g \in \mathbf{C}[X]$ and $\frac{f}{g}$ is defined in an open set containing $p$.

## Example (The local ring of the origin in $\mathbf{C}^2$)

$\mathbf{C}[\mathbf{C}^2] = \mathbf{C}[x, y]$, $p = (0, 0)$

$\frac{x}{(y-x^2-1)(y+2)} \in \mathbf{C}[x, y]_p$ is defined on the set with red curves deleted



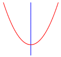
# Intersection multiplicity: smooth surfaces

**Definition (Multiplicity of an isolated point $p \in F \cap G$)**

$F, G$ curves on a smooth surface $X \subset \mathbf{C}^n$, local equations $f, g$
Define $\mathrm{mult}_p(F \cdot G) = \dim \mathbf{C}[X]_p / \langle f, g \rangle$, and

$F \cdot G = \sum_{p \in F \cap G} \mathrm{mult}_p(F \cdot G)$.

**Example (A transverse intersection in $\mathbf{C}^2$)**



$x = 0, y - x^2 = 0, p = (0, 0)$
If $\frac{r}{s} \in \mathbf{C}[x, y]_p$, then there are $r_{ij}, s_{ij} \in \mathbf{C}$, $s_{00} \neq 0$ so that

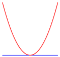
$$r(x, y) = r_{00} + r_{10}x + r_{01}y + r_{20}x^2 + r_{11}xy + \cdots ,$$
$$s(x, y) = s_{00} + s_{10}x + s_{01}y + s_{20}x^2 + s_{11}xy + \cdots .$$

$\mathbf{C}[x, y]_p / \langle x, y - x^2 \rangle \cong \mathbf{C}$ since $x = 0$ and $y = x^2 = 0 \Rightarrow \frac{r}{s} \mapsto \frac{r_{00}}{s_{00}}$.

# Intersections with multiplicity $> 1$.

## Example

$y = 0, y - x^2 = 0, p = (0, 0)$



$\mathbf{C}[x, y]_p / \langle y, y - x^2 \rangle$ is spanned by $\{1, x\}$. Since $y = 0$ and $x^2 = y = 0$, $\frac{r}{s} \in \mathbf{C}[x, y]_p$, maps to

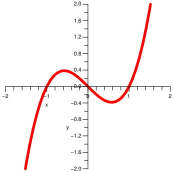
$$\frac{r_{00} + r_{10}x}{s_{00} + s_{10}x} = \frac{r_{00} + r_{10}x}{s_{00} + s_{10}x} \cdot \frac{s_{00} - s_{10}x}{s_{00} - s_{10}x} = \frac{r_{00}s_{00} + (r_{10}s_{00} - r_{00}s_{10})x}{s_{00}^2}$$

Therefore,

$$\text{mult}_p(F \cdot G) = 2.$$

# The degree of a curve in $\mathbf{P}^2$

- $y = x^3 - x$ crosses the $x$-axis at 3 points




- $C = \mathbf{V}(yz^2 - x^3 - xz^2) \subset \mathbf{P}^2$
- For any line $L \subset \mathbf{P}^2$, $3 = C \cdot L$.
- $\deg C = 3$

# The degree of a variety

- $X \subset \mathbf{P}^n$ variety, $\dim X = p$
- $H$ general plane, $\dim H = n - p$
  The *degree* of $X$ is $X \cdot H$.

Example (The degree of a hypersurface I)

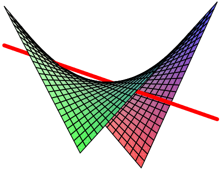
- $X = \mathbf{V}(f) \subset \mathbf{P}^3$, $f = w_0 w_3 - w_1 w_2$
- $L = \{[s:s:s:t]\} \Rightarrow$

  $$f([s:s:s:t]) = st - s^2 = s(t-s).$$

- $X \cap L$ :

  $$\{[0:0:0:1], [1:1:1:1]\}$$

- $\deg X = X \cdot L = 2$



# Projective embeddings: Veronese varieties

**Definition**

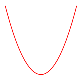
The $d$-uple Veronese embedding $\nu_d : \mathbf{P}^n \to \mathbf{P}^N$ is given by

$$\mathbf{x} = [x_0 : \cdots : x_n] \mapsto [F_0(\mathbf{x}) : \cdots : F_N(\mathbf{x})],$$

where $N = \binom{d+n}{n} - 1$ and $F_0, \ldots, F_N$ is a basis for the homogeneous forms of degree $d$ on $\mathbf{P}^n$.

**Example (Veronese embeddings of $\mathbf{P}^1$ : rational normal curves)**

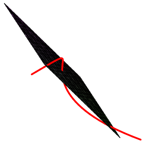
- $\nu_2([x : y]) = [x^2 : xy : y^2]$
- $f = w_0 w_2 - w_1^2$
- $f([x^2 : xy : y^2]) = (x^2)(y^2) - (xy)^2 = 0$



# The degree of $\nu_d(\mathbf{P}^n) \subset \mathbf{P}^N$

## Example ($\deg \nu_3(\mathbf{P}^1) \subset \mathbf{P}^3$)

- $\nu_3([x : y]) = [x^3 : x^2y : xy^2 : y^3]$
- $L = w_0 - w_2$
- $L(x^3, x^2y, xy^2, y^3) = x^3 - xy^2$ has three solutions



To compute $\deg \nu_d(\mathbf{P}^n)$ :

- $H = H_1 \cdots H_n$ where $H_i$ is defined by a linear form $L_i$.
- $\nu_d(\mathbf{P}^n) \cdot H = \nu_d(\mathbf{P}^n) \cdot H_1 \cdots H_n$
- $L_i(x_0^d, x_0^{d-1}x_1, \ldots, x_n^d)$ is a homogeneous form of degree $d$
- Bézout's theorem $\Rightarrow$ hypersurfaces in $\mathbf{P}^n$ defined by "pulling back" equations $L_i$ intersect in $d^n$ points.

Therefore, $\deg \nu_d(\mathbf{P}^n) = d^n$.

# Maps with basepoints I

## Example (Conics through a point)

Let $p = [1 : 0 : 0]$ in $\mathbf{P}^2$ and let $W_p$ be the vector space of homogeneous forms

$$\{F \in \mathbf{C}[x, y, z] | \deg F = 2, F(p) = 0\}.$$

- $W_p = \text{span}\{xy, xz, y^2, yz, z^2\}$.
- Define $\phi : \mathbf{P}^2 \backslash \{p\} \to \mathbf{P}^4$ by $\phi([x : y : z]) = [xy : xz : y^2 : yz : z^2]$.
- We say that $\phi$ has a basepoint at $p$ since it is not defined at $p$.

What is $\deg \overline{\text{Image}\,\phi}$?

# Maps with basepoints II

## Example (Conics through 2 points)

Fix $p = [1 : 0 : 0], q = [0 : 1 : 0]$. Define $\psi : \mathbf{P}^2 \setminus \{p, q\} \to \mathbf{P}^3$ by $\psi([x : y : z]) = [xy : xz : yz : z^2]$.

## Example (Cubics through 6 points)

Fix $p_1, \ldots, p_6 \in \mathbf{P}^2$. Define $\gamma : \mathbf{P}^2 \setminus \{p_1, \ldots, p_6\} \to \mathbf{P}^3$ using a basis of cubics passing through the 6 points.

## Question

What can we say about $\overline{\text{Image}\,\phi}, \overline{\text{Image}\,\psi}, \overline{\text{Image}\,\gamma}$? What are their degrees? How do their geometries differ?