

# Grassmannians and Other Schubert Varieties

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What is Algebraic Geometry?

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# Famous Quotations

**Arnold Ross (and PROMYS).**

”Think deeply of simple things.”

**Angela Gibney.** Why do algebraic geometers love moduli spaces?

“It is just like with people, if you want to get to know someone, go to their family reunion.”

**Goal.** Focus our attention on a particular family of varieties which are indexed by combinatorial data where lots is known about their structure and yet lots is still open.

# Schubert Varieties

A *Schubert variety* is a member of a family of projective varieties whose points are indexed by matrices and whose defining equations are determinantal minors.

Typical properties:

- This family of varieties is indexed by combinatorial objects; e.g. partitions, permutations, or Weyl group elements.
- Some are smooth and some are singular.
- Their cohomology rings include familiar subsets and quotients of polynomial rings including the symmetric functions.

# Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in  $\mathbb{R}^2$ ?
- How many lines intersect two given lines and a given point in  $\mathbb{R}^3$ ?
- How many lines intersect four given lines in  $\mathbb{R}^3$  ?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

# Why Study Schubert Varieties?

1. It can be useful to see points, lines, planes etc as families with certain properties.
2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry.
3. Applications in discrete geometry, computer graphics, and computer vision.

# Outline

1. Review of vector spaces and projective spaces
2. Introduction to Grassmannians, Flag Manifolds and Schubert Varieties
3. Five Fun Facts on Schubert Varieties
4. Permutation Arrays = higher dimensional analog of permutation matrices
5. Solving Schubert problems with Permutation Arrays

# Vector Spaces

- $V$  is a *vector space* over a field  $\mathbb{F}$  if it is closed under addition and multiplication by scalars in  $\mathbb{F}$ .
- $B = \{b_1, \dots, b_k\}$  is a *basis* for  $V$  if for every  $a \in V$  there exist unique scalars  $c_1, \dots, c_k \in \mathbb{F}$  such that

$$a = c_1 b_1 + c_2 b_2 + \dots + c_k b_k = (c_1, c_2, \dots, c_k) \in \mathbb{F}^k.$$

- The *dimension* of  $V$  equals the size of a basis.
- A *subspace*  $U$  of a vector space  $V$  is any subset of the vectors in  $V$  that is closed under addition and scalar multiplication.

**Fact.** Any basis for  $U$  can be extended to a basis for  $V$ .

# Projective Spaces

## Defn.

- $\mathbb{P}(V) = \{1\text{-dim subspaces of } V\} = \frac{V}{\langle d \cdot a = a \rangle}$  .

1-dim subspaces = lines in  $V$  through 0  $\longleftrightarrow$  points in  $\mathbb{P}(V)$   
2-dim subspaces = planes in  $V$  through 0  $\longleftrightarrow$  lines in  $\mathbb{P}(V)$

- Given a basis  $B = \{b_1, b_2, \dots, b_k\}$  for  $V$ , the line spanned by  $a = c_1 b_1 + c_2 b_2 + \dots + c_k b_k \in \mathbb{P}(V)$  has *homogeneous coordinates*

$$[c_1 : c_2 : \dots : c_k] = [dc_1 : dc_2 : \dots : dc_k] \quad \text{for any } d \in \mathbb{F}.$$

# The Grassmannian Varieties

**Definition.** Fix a vector space  $V$  over  $\mathbb{C}$  (or  $\mathbb{R}, \mathbb{Z}_2, \dots$ ) with basis  $B = \{e_1, \dots, e_n\}$ . The *Grassmannian variety*

$$G(k, n) = \{k\text{-dimensional subspaces of } V\}.$$

**Question.**

How can we impose the structure of a variety or a manifold on this set?

# The Grassmannian Varieties

**Answer.** Relate  $G(k, n)$  to the vector space of  $k \times n$  matrices.

$$U = \text{span}\langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4 \rangle \in G(3, 4)$$

$$M_U = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix}$$

- $U \in G(k, n) \iff$  rows of  $M_U$  are independent vectors in  $V \iff$  some  $k \times k$  minor of  $M_U$  is NOT zero.

# Plücker Coordinates

- Define  $f_{j_1, j_2, \dots, j_k}$  to be the polynomial given by the determinant of the matrix

$$\begin{bmatrix} x_{1, j_1} & x_{1, j_2} & \cdots & x_{1, j_k} \\ x_{2, j_1} & x_{2, j_2} & \cdots & x_{2, j_k} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k, j_1} & x_{k, j_2} & \cdots & x_{k, j_k} \end{bmatrix}$$

- $G(k, n)$  is an open set in the Zariski topology on  $k \times n$  matrices defined as the complement of the union over all  $k$ -subsets of  $\{1, 2, \dots, n\}$  of the varieties  $V(f_{j_1, j_2, \dots, j_k})$ .
- All the determinants  $f_{j_1, j_2, \dots, j_k}$  are homogeneous polynomials of degree  $k$  so  $G(k, n)$  can be thought of as an open set in projective space.

# The Grassmannian Varieties

**Canonical Form.** Every subspace in  $G(k, n)$  can be represented by a unique matrix in row echelon form.

**Example.**

$$U = \text{span}\langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4 \rangle \in G(3, 4)$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, 2e_1 + e_3, 7e_1 + e_4 \rangle$$

# Subspaces and Subsets

## Example.

$$U = \text{RowSpan} \begin{bmatrix} 5 & 9 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 9 & 7 & 9 & \textcircled{1} & 0 & 0 & 0 \\ 4 & 6 & 0 & 2 & 6 & 4 & 0 & 3 & \textcircled{1} & 0 \end{bmatrix} \in G(3, 10).$$

$$\text{position}(U) = \{3, 7, 9\}$$

## Definition.

If  $U \in G(k, n)$  and  $M_U$  is the corresponding matrix in canonical form then the columns of the leading 1's of the rows of  $M_U$  determine a subset of size  $k$  in  $\{1, 2, \dots, n\} := [n]$ . There are 0's to the right of each leading 1 and 0's above and below each leading 1. This  $k$ -subset determines the *position* of  $U$  with respect to the fixed basis.

# The Schubert Cell $C_j$ in $G(k, n)$

**Defn.** Let  $\mathbf{j} = \{j_1 < j_2 < \dots < j_k\} \in [n]$ . A *Schubert cell* is

$$C_j = \{U \in G(k, n) \mid \text{position}(U) = \{j_1, \dots, j_k\}\}$$

**Fact.**  $G(k, n) = \bigcup C_j$  over all  $k$ -subsets of  $[n]$

**Example.** In  $G(3, 10)$ ,

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & \textcircled{1} & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & \textcircled{1} & 0 \end{bmatrix} \right\}$$

- Observe  $\dim(C_{\{3,7,9\}}) = 13$ .
- In general,  $\dim(C_j) = \sum j_i - i$ .

# Schubert Varieties in $G(k, n)$

**Defn.** Let  $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$ . A *Schubert variety* is

$X_{\mathbf{j}} = \text{Closure of } C_{\mathbf{j}} \text{ under Zariski topology.}$

**Question.** In  $G(3, 10)$ , what polynomials vanish on  $C_{\{3,7,9\}}$ ?

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & \textcircled{1} & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & \textcircled{1} & 0 \end{bmatrix} \right\}$$

**Answer.** All minors  $f_{j_1, j_2, j_3}$  with  $\left\{ \begin{array}{l} 4 \leq j_1 \leq 8 \\ \text{or } j_1 = 3 \text{ and } 8 \leq j_2 \leq 9 \\ \text{or } j_1 = 3, j_2 = 7 \text{ and } j_3 = 10 \end{array} \right\}$

In other words, the canonical form for any subspace in  $\overline{C_{\mathbf{j}}}$  has 0's to the right of column  $j_i$  in each row  $i$ .

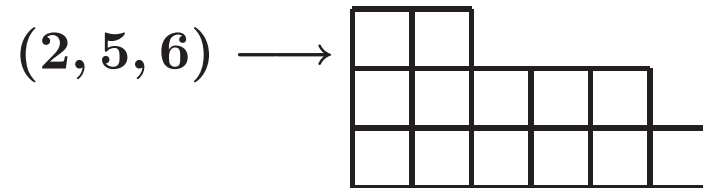
# $k$ -Subsets and Partitions

**Defn.** A *partition* of a number  $n$  is a weakly increasing sequence of non-negative integers

$$\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$$

such that  $n = \sum \lambda_i = |\lambda|$ .

Partitions can be visualized by their *Ferrers diagram*



**Fact.** There is a bijection between  $k$ -subsets of  $\{1, 2, \dots, n\}$  and partitions whose Ferrers diagram is contained in the  $k \times (n - k)$  rectangle given by

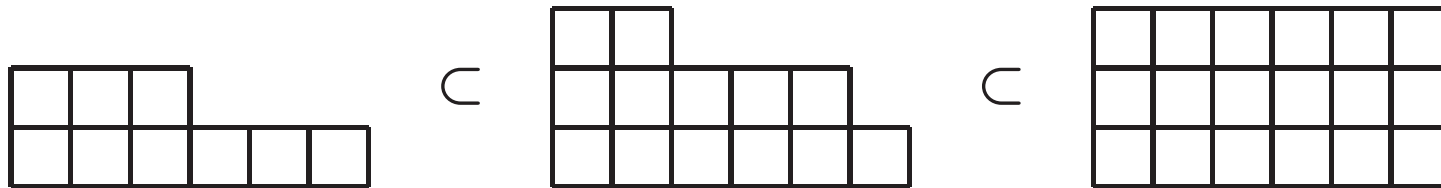
$$\text{shape} : \{j_1 < \dots < j_k\} \mapsto (j_1 - 1, j_2 - 2, \dots, j_k - k).$$

# A Poset on Partitions

**Defn.** A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

**Defn.** *Young's Lattice*

If  $\lambda = (\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k)$  and  $\mu = (\mu_1 \leq \mu_2 \leq \dots \leq \mu_k)$  then  $\lambda \subset \mu$  if the Ferrers diagram for  $\lambda$  fits inside the Ferrers diagram for  $\mu$ .



**Facts.**

1.  $X_j = \bigcup_{\text{shape}(i) \subset \text{shape}(j)} C_i.$
2. The dimension of  $X_j$  is  $|\text{shape}(j)|.$
3. The Grassmannian  $G(k, n) = X_{\{n-k+1, \dots, n-1, n\}}$  is a Schubert variety!

# Enumerative Geometry Revisited

**Question.** How many lines intersect four given lines in  $\mathbb{R}^3$  ?

**Translation.** Given a line in  $\mathbb{R}^3$ , the family of lines intersecting it can be interpreted in  $G(2, 4)$  as the Schubert variety

$$X_{\{2,4\}} = \overline{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}}$$

with respect to a suitably chosen basis determined by the line.

**Reformulated Question.** How many subspaces  $U \in G(2, 4)$  are in the intersection of 4 copies of the Schubert variety  $X_{\{2,4\}}$  each with respect to a different basis?

**Modern Solution.** Use Intersection Theory!

# Intersection Theory

- Schubert varieties induce canonical elements of the cohomology ring for  $G(k, n)$  called *Schubert classes*:  $[X_j]$ .
- Multiplication of Schubert classes corresponds with intersecting Schubert varieties with respect to different bases.

$$[X_i][X_j] = [X_i(B^1) \cap X_j(B^2)]$$

- Schubert classes add and multiply just like *Schur functions*. Schur functions are a fascinating family of symmetric functions indexed by partitions which appear in many areas of math, physics, theoretical computer science.
- Expanding the product of two Schur functions into the basis of Schur functions can be done via linear algebra:

$$S_\lambda S_\mu = \sum c_{\lambda, \mu}^\nu S_\nu.$$

- The coefficients  $c_{\lambda, \mu}^\nu$  are non-negative integers called the Littlewood-Richardson coefficients. In a 0-dimensional intersection, the coefficient of  $[X_{\{1, 2, \dots, k\}}]$  is the number of subspaces in  $X_i(B^1) \cap X_j(B^2)$ .

# Enumerative Solution

**Reformulated Question.** How many subspaces  $U \in G(2, 4)$  are in the intersection of 4 copies of the Schubert variety  $X_{\{2,4\}}$  each with respect to a different basis?

# Enumerative Solution

**Reformulated Question.** How many subspaces  $U \in G(2, 4)$  are in the intersection of 4 copies of the Schubert variety  $X_{\{2,4\}}$  each with respect to a different basis?

**Solution.**

$$[X_{\{2,4\}}] = S_{(1)} = x_1 + x_2 + \dots$$

By the recipe, compute

$$\begin{aligned} & [X_{\{2,4\}}(B^1) \cap X_{\{2,4\}}(B^2) \cap X_{\{2,4\}}(B^3) \cap X_{\{2,4\}}(B^4)] \\ &= S_{(1)}^4 = 2S_{(2,2)} + S_{(3,1)} + S_{(2,1,1)}. \end{aligned}$$

**Answer.** The coefficient of  $S_{2,2} = [X_{1,2}]$  is 2 representing the two lines meeting 4 given lines in general position.

# Recap

1.  $G(k, n)$  is the Grassmannian variety of  $k$ -dim subspaces in  $\mathbb{R}^n$ .
2. The Schubert varieties in  $G(k, n)$  are nice projective varieties indexed by  $k$ -subsets of  $[n]$  or equivalently by partitions in the  $k \times (n - k)$  rectangle.
3. Geometrical information about a Schubert variety can be determined by the combinatorics of partitions.
4. Intersection theory applied to Schubert varieties can be used to solve problems in enumerative geometry.

# Generalizations

"Nothing is more disagreeable to the hacker than duplication of effort. The first and most important mental habit that people develop when they learn how to write computer programs is to generalize, generalize, generalize.: –Neil Stephenson "In the Beginning was the Command Line"

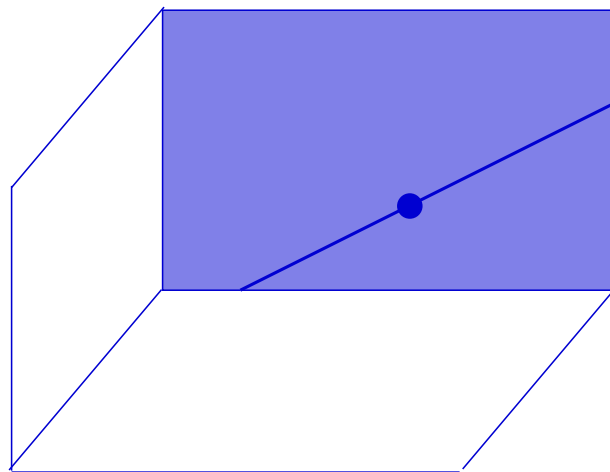
Same goes for mathematicians!

# The Flag Manifold

**Defn.** A *complete flag*  $F_\bullet = (F_1, \dots, F_n)$  in  $\mathbb{C}^n$  is a nested sequence of vector spaces such that  $\dim(F_i) = i$  for  $1 \leq i \leq n$ .  $F_\bullet$  is determined by an ordered basis  $\langle f_1, f_2, \dots, f_n \rangle$  where  $F_i = \text{span}\langle f_1, \dots, f_i \rangle$ .

**Example.**

$$F_\bullet = \langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4, e_2 \rangle$$



# The Flag Manifold

## Canonical Form.

$$F_{\bullet} = \langle 6e_1 + 3e_2, 4e_1 + 2e_3, 9e_1 + e_3 + e_4, e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, 2e_1 + e_3, 7e_1 + e_4, e_1 \rangle$$

$\mathcal{Fl}_n(\mathbb{C}) :=$  *flag manifold* over  $\mathbb{C}^n \subset \prod_{k=1}^n G(n, k)$

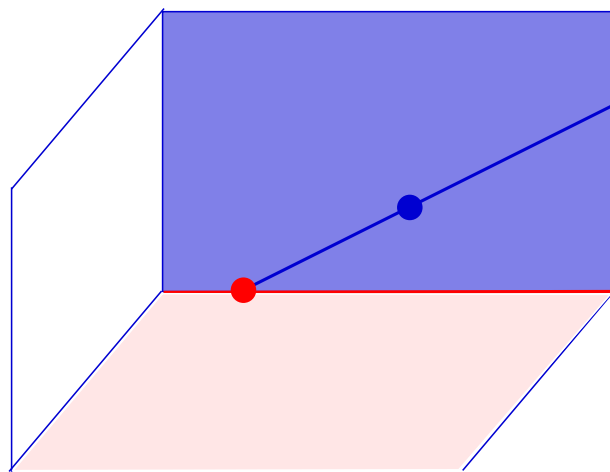
$=$  {complete flags  $F_{\bullet}$ }

$= B \setminus GL_n(\mathbb{C})$ ,  $B =$  lower triangular mats.

# Flags and Permutations

**Example.**  $F_{\bullet} = \langle 2e_1 + e_2, 2e_1 + e_3, 7e_1 + e_4, e_1 \rangle \approx \begin{bmatrix} 2 & \textcircled{1} & 0 & 0 \\ 2 & 0 & \textcircled{1} & 0 \\ 7 & 0 & 0 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \end{bmatrix}$

**Note.** If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag  $F_{\bullet}$  with respect to the reference flag  $E_{\bullet} = \langle e_1, e_2, e_3, e_4 \rangle$ .



# Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = 2341 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

matrix  
notation

two-line  
notation

one-line  
notation

rank  
table

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} = \begin{array}{c} 1234 \\ \text{YH} \\ 2341 \end{array} = (1, 2, 3)$$

diagram of a  
permutation

string diagram

reduced  
word

# The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

**Defn.**  $C_w(E_\bullet) =$  All flags  $F_\bullet$  with  $\text{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \text{rk}(w[i, j])\}$$

**Example.**  $F_\bullet = \begin{bmatrix} 2 & \textcircled{1} & 0 & 0 \\ 2 & 0 & \textcircled{1} & 0 \\ 7 & 0 & 0 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

## Easy Observations.

- $\dim_{\mathbb{C}}(C_w) = l(w) = \#$  inversions of  $w$ .
- $C_w = w \cdot B$  is a  $B$ -orbit using the right  $B$  action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

# The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

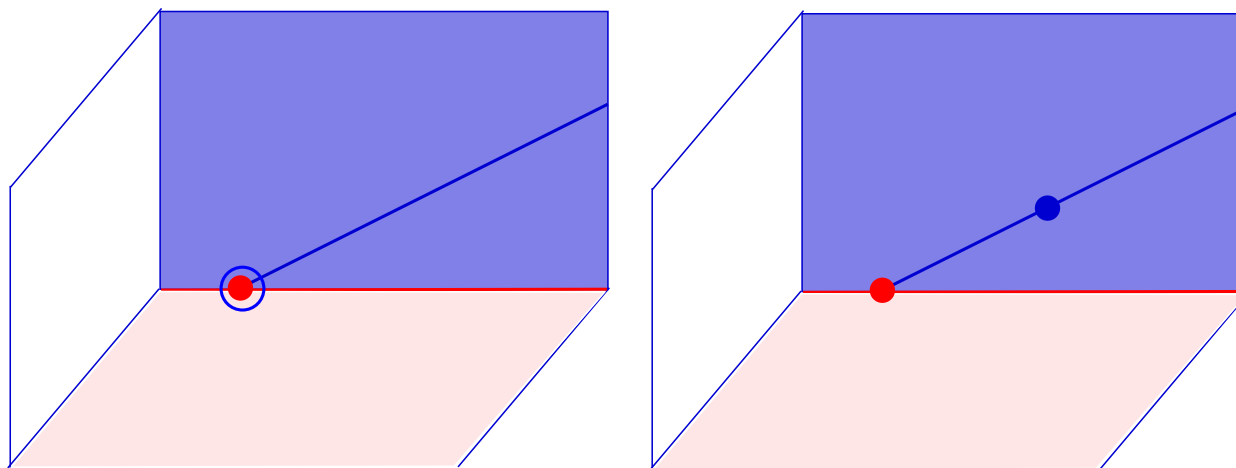
**Defn.**  $X_w(E_\bullet) = \text{Closure of } C_w(E_\bullet) \text{ under the Zariski topology}$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \text{rk}(w[i, j])\}$$

where  $E_\bullet = \langle e_1, e_2, e_3, e_4 \rangle$ .

**Example.** 
$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & * & \textcircled{1} & 0 \\ 0 & * & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$$

**Why?.**



# Five Fun Facts

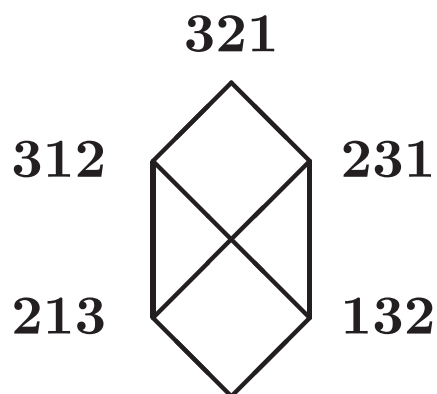
**Fact 1.** The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \leq w} C_v = \bigcup_{v \leq w} X_v$$

**Bruhat order** (Ehresmann 1934, Chevalley 1958) is the transitive closure of

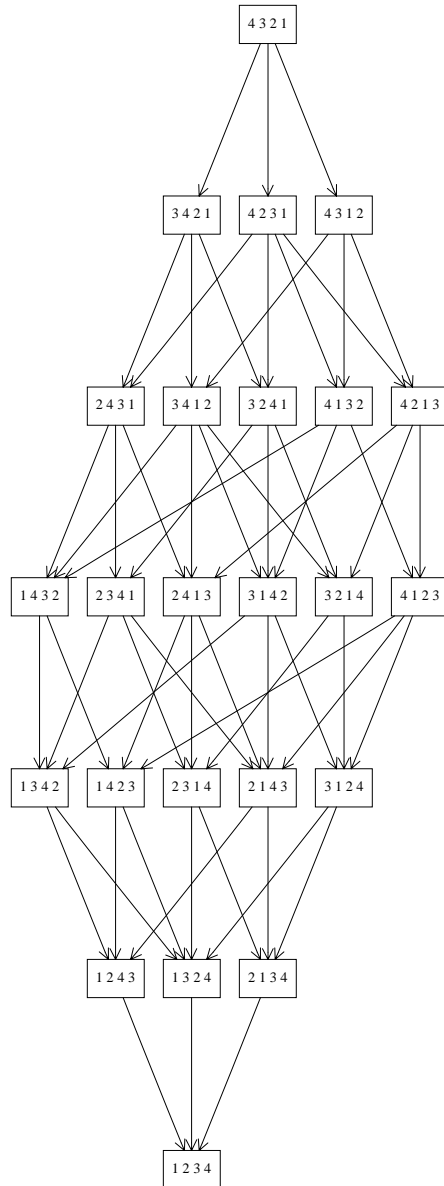
$$w < wt_{ij} \iff w(i) < w(j).$$

**Example.** Bruhat order on permutations in  $S_3$ .

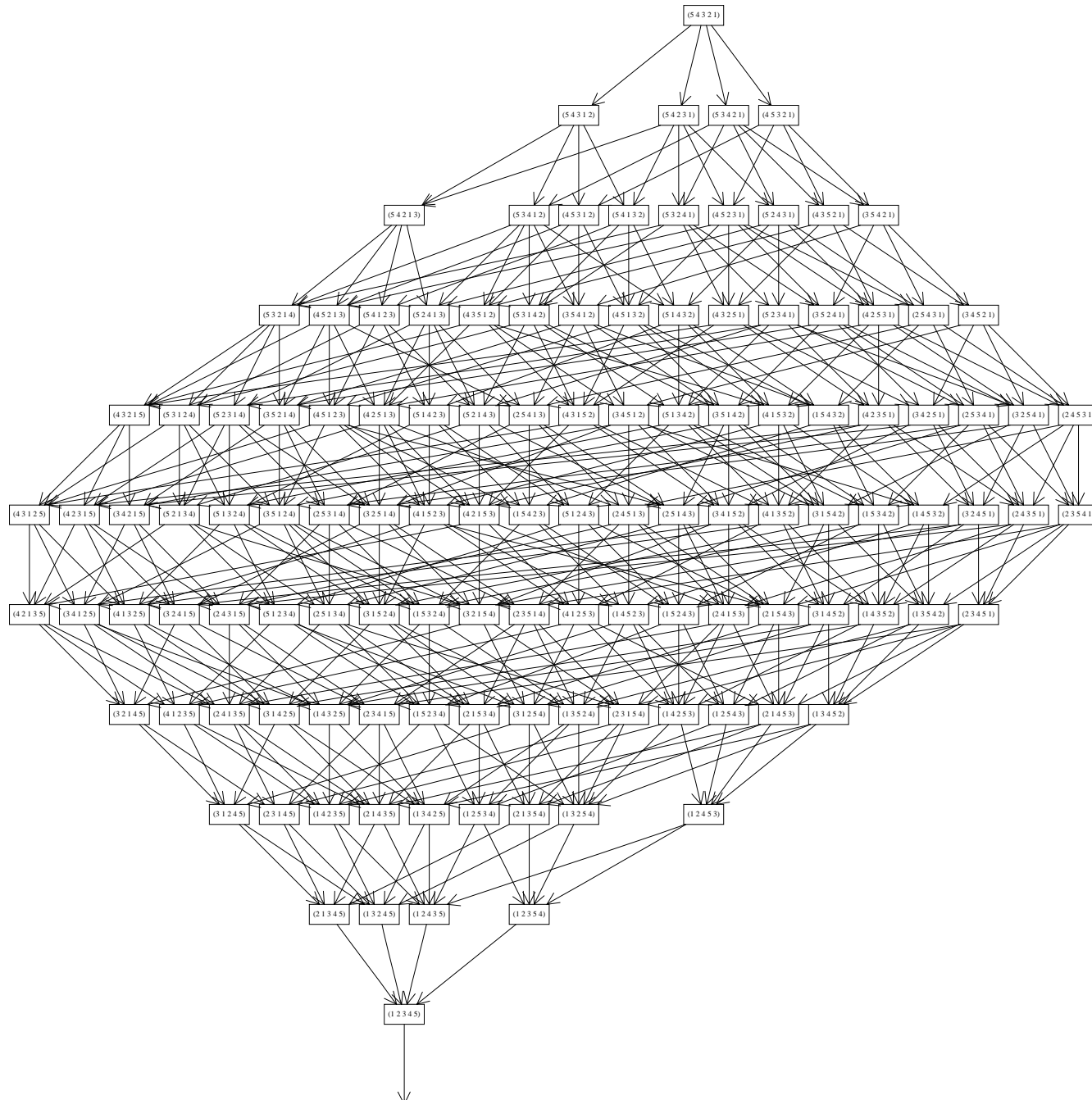


**Observations.** Self dual, rank symmetric, rank unimodal.

# Bruhat order on $S_4$



# Bruhat order on $S_5$



# 10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in  $\mathcal{S}_\infty$
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5.  $[x, y]$  Determines the Composition Series for Verma Modules
6. Symmetric Interval  $[\hat{0}, w] \iff X(w)$  rationally smooth
7. Order Complex of  $(u, v)$  is shellable
8. Rank Symmetric, Rank Unimodal and  $k$ -Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

# Singularities in Schubert Varieties

**Defn.**  $X_w$  is *singular* at a point  $p \iff$   
 $\dim X_w = l(w) <$  dimension of the tangent space to  $X_w$  at  $p$ .

**Observation 1.** Every point on a Schubert cell  $C_v$  in  $X_w$  looks locally the same. Therefore,  $p \in C_v$  is a singular point  $\iff$  the permutation matrix  $v$  is a singular point of  $X_w$ .

**Observation 2.** The singular set of a varieties is a closed set in the Zariski topology. Therefore, if  $v$  is a singular point in  $X_w$  then every point in  $X_v$  is singular. The irreducible components of the singular locus of  $X_w$  is a union of Schubert varieties:

$$\text{Sing}(X_w) = \bigcup_{v \in \text{maxsing}(w)} X_v.$$

# Singularities in Schubert Varieties

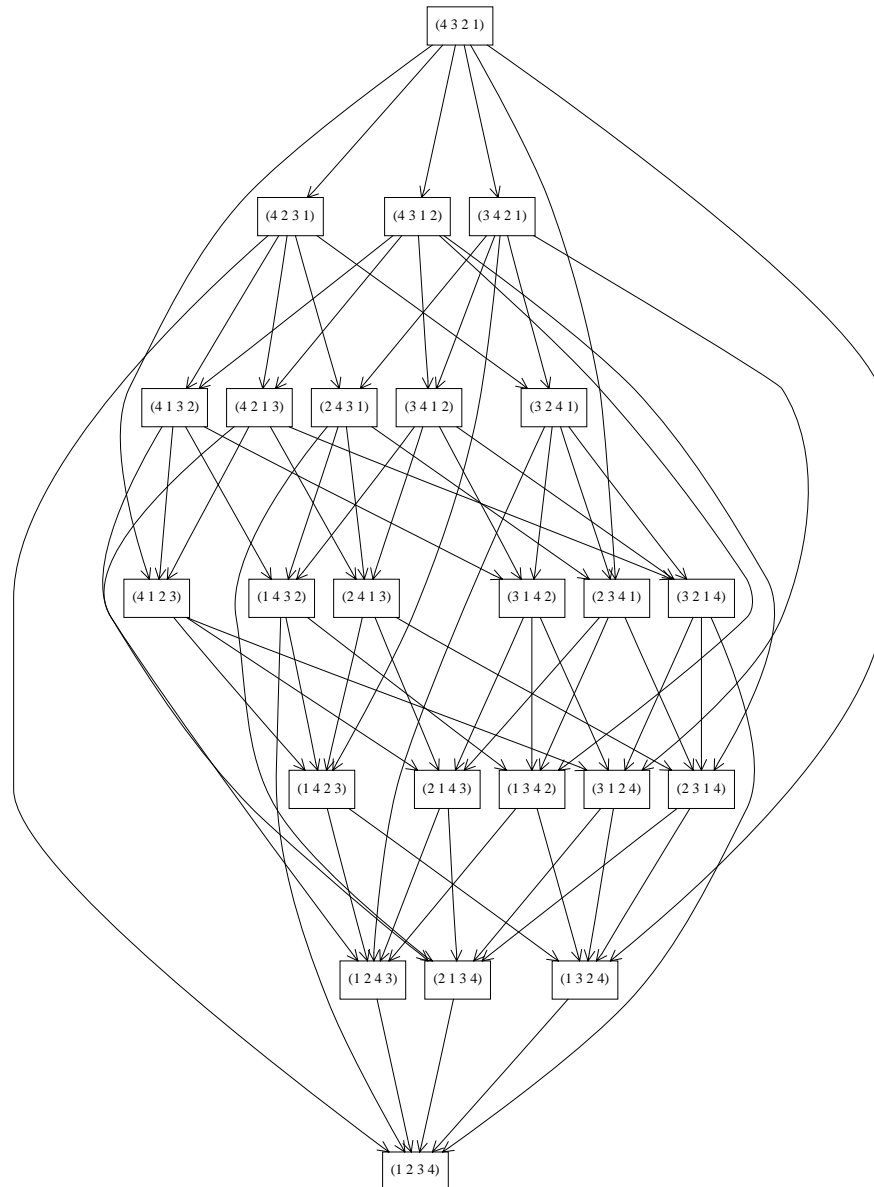
**Fact 2.** (Lakshmibai-Seshadri) A basis for the tangent space to  $X_w$  at  $v$  is indexed by the transpositions  $t_{ij}$  such that

$$vt_{ij} \leq w.$$

## Definitions.

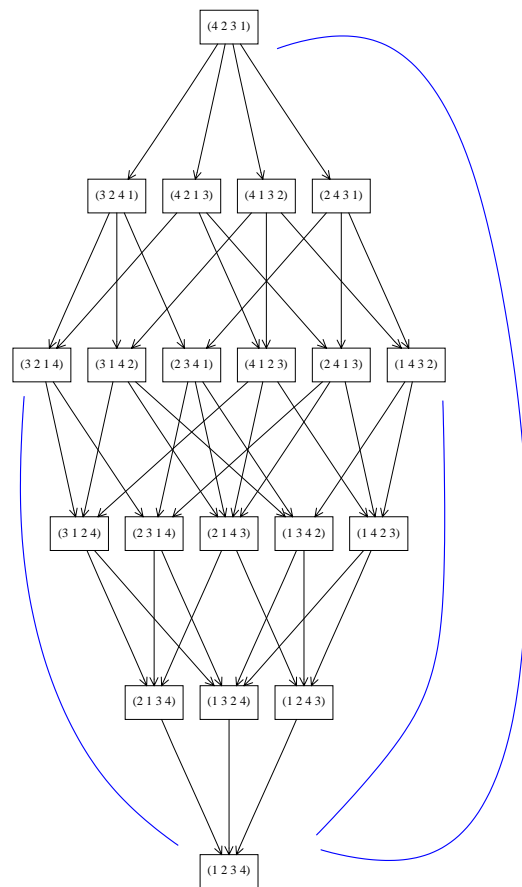
- Let  $T =$  invertible diagonal matrices. The  $T$ -fixed points in  $X_w$  are the permutation matrices indexed by  $v \leq w$ .
- If  $v, vt_{ij}$  are permutations in  $X_w$  they are connected by a  $T$ -stable curve. The set of all  $T$ -stable curves in  $X_w$  are represented by the *Bruhat graph* on  $[id, w]$ .

# Bruhat Graph in $S_4$



# Tangent space of a Schubert Variety

**Example.**  $T_{1234}(X_{4231}) = \text{span}\{x_{i,j} \mid t_{ij} \leq w\}$ .



$$\dim X(4231) = 5$$

$$\dim T_{id}(4231) = 6 \implies X(4231) \text{ is singular!}$$

# Five Fun Facts

**Fact 3.** There exists a simple criterion for characterizing singular Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)

$X_w$  is non-singular  $\iff w$  has no subsequence with the same relative order as **3412** and **4231**.

$w = 625431$	contains	<b>6241</b> $\sim$ <b>4231</b>	$\implies X_{625431}$ is singular
<i>Example:</i> $w = 612543$	avoids	<b>4231</b> & <b>3412</b>	$\implies X_{612543}$ is non-singular

# Five Fun Facts

## Consequences of Fact 3.

- (Billey-Warrington, Kassel-Lascoux-Reutenauer, Manivel 2000) The bad patterns in  $w$  can also be used to efficiently find the singular locus of  $X_w$ .
- (Bona 1998, Haiman) Let  $v_n$  be the number of  $w \in S_n$  for which  $X(w)$  is non-singular. Then the generating function  $V(t) = \sum_n v_n t^n$  is given by

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1 - 4t}}{1 - 6t + 8t^2 - 4t^3}.$$

- (Billey-Postnikov 2001) Generalized pattern avoidance to all semisimple simply-connected Lie groups  $G$  and characterized smooth Schubert varieties  $X_w$  by avoiding these generalized patterns. Only requires checking patterns of types  $A_3, B_2, B_3, C_2, C_3, D_4, G_2$ .

# Five Fun Facts

**Fact 4.** There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Theorem: Woo-Yong (Sept. 2004)

$X_w$  is Gorenstein  $\iff$

- $w$  avoids **31542** and **24153** with Bruhat restrictions  $\{t_{15}, t_{23}\}$  and  $\{t_{15}, t_{34}\}$
- for each descent  $d$  in  $w$ , the associated partition  $\lambda_d(w)$  has all of its inner corners on the same antidiagonal.

# Five Fun Facts

**Fact 5.** Schubert varieties are useful for studying the cohomology ring of the flag manifold.

*Theorem (Borel):*  $H^*(\mathcal{Fl}_n) \cong \frac{\mathbb{Z}[x_1, \dots, x_n]}{\langle e_1, \dots, e_n \rangle}.$

- The symmetric function  $e_i = \sum_{1 \leq k_1 < \dots < k_i \leq n} x_{k_1} x_{k_2} \dots x_{k_i}.$
- $\{[X_w] \mid w \in S_n\}$  form a basis for  $H^*(\mathcal{Fl}_n)$  over  $\mathbb{Z}.$

**Question.** What is the product of two basis elements?

$$[X_u] \cdot [X_v] = \sum [X_w] c_{uv}^w.$$

# Cup Product in $H^*(\mathcal{Fl}_n)$

**Answer.** Use Schubert polynomials! Due to Lascoux-Schützenberger, Bernstein-Gelfand-Gelfand, Demazure.

- BGG: If  $\mathfrak{S}_w \equiv [X_w] \bmod \langle e_1, \dots, e_n \rangle$  then

$$\frac{\mathfrak{S}_w - s_i \mathfrak{S}_w}{x_i - x_{i+1}} \equiv [X_{ws_i}] \text{ if } l(w) < l(ws_i)$$

$$[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \equiv \prod_{i>j} (x_i - x_j) \equiv \cdots$$

Here  $\deg[X_w] = \text{codim}(X_w)$ .

- LS: Choosing  $[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$  works best because product expansion can be done without regard to the ideal!

# Schubert polynomials for $S_4$

$$\begin{aligned}\mathfrak{S}_{w_0(1234)} &= 1 \\ \mathfrak{S}_{w_0(2134)} &= x_1 \\ \mathfrak{S}_{w_0(1324)} &= x_2 + x_1 \\ \mathfrak{S}_{w_0(3124)} &= x_1^2 \\ \mathfrak{S}_{w_0(2314)} &= x_1x_2 \\ \mathfrak{S}_{w_0(3214)} &= x_1^2x_2 \\ \mathfrak{S}_{w_0(1243)} &= x_3 + x_2 + x_1 \\ \mathfrak{S}_{w_0(2143)} &= x_1x_3 + x_1x_2 + x_1^2 \\ \mathfrak{S}_{w_0(1423)} &= x_2^2 + x_1x_2 + x_1^2 \\ \mathfrak{S}_{w_0(4123)} &= x_1^3 \\ \mathfrak{S}_{w_0(2413)} &= x_1x_2^2 + x_1^2x_2 \\ \mathfrak{S}_{w_0(4213)} &= x_1^3x_2 \\ \mathfrak{S}_{w_0(1342)} &= x_2x_3 + x_1x_3 + x_1x_2 \\ \mathfrak{S}_{w_0(3142)} &= x_1^2x_3 + x_1^2x_2 \\ \mathfrak{S}_{w_0(1432)} &= x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2 \\ \mathfrak{S}_{w_0(4132)} &= x_1^3x_3 + x_1^3x_2 \\ \mathfrak{S}_{w_0(3412)} &= x_1^2x_2^2 \\ \mathfrak{S}_{w_0(4312)} &= x_1^3x_2^2 \\ \mathfrak{S}_{w_0(2341)} &= x_1x_2x_3 \\ \mathfrak{S}_{w_0(3241)} &= x_1^2x_2x_3 \\ \mathfrak{S}_{w_0(4321)} &= x_1^2x_2x_3 + x_1^2x_2x_2\end{aligned}$$

# Cup Product in $H^*(\mathcal{F}l_n)$

**Key Feature.** Schubert polynomials have distinct leading terms, therefore expanding any polynomial in the basis of Schubert polynomials can be done by linear algebra just like Schur functions.

Buch: Fastest approach to multiplying Schubert polynomials uses Lascoux and Schützenberger's transition equations. Works up to about  $n = 15$ .

**Draw Back.** Schubert polynomials don't prove  $c_{uv}^w$ 's are nonnegative (except in special cases).

# Cup Product in $H^*(\mathcal{Fl}_n)$

## Another Answer.

- By intersection theory:  $[X_u] \cdot [X_v] = [X_u(E_\bullet) \cap X_v(F_\bullet)]$
- Perfect pairing:  $[X_u(E_\bullet)] \cdot [X_v(F_\bullet)] \cdot [X_{w_0 w}(G_\bullet)] = c_{uv}^w [X_{id}]$

||

$$[X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)]$$

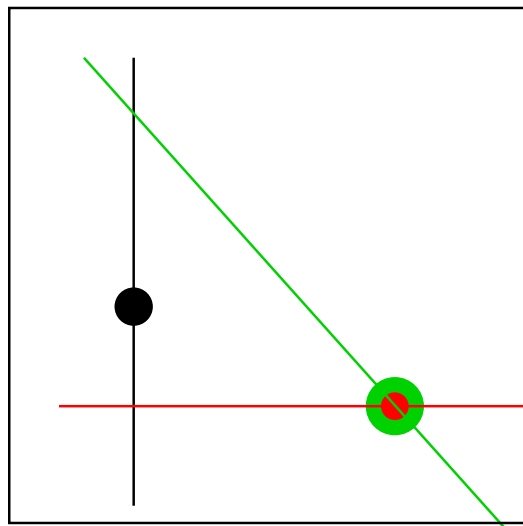
- The Schubert variety  $X_{id}$  is a single point in  $\mathcal{Fl}_n$ .

**Intersection Numbers:**  $c_{uv}^w = \#X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)$

Assuming all flags  $E_\bullet, F_\bullet, G_\bullet$  are in sufficiently general position.

# Intersecting Schubert Varieties

**Example.** Fix three flags  $R_\bullet$ ,  $G_\bullet$ , and  $B_\bullet$ :

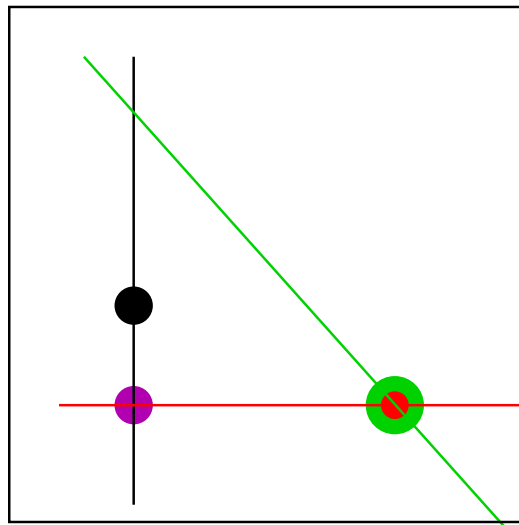


Find  $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$  where  $u, v, w$  are the following permutations:

	$R_1$ $R_2$ $R_3$	$G_1$ $G_2$ $G_3$	$B_1$ $B_2$ $B_3$																											
$P_1$	<table style="border-collapse: collapse; width: 100px; height: 100px;"> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px; color: red; text-align: center;">1</td><td style="width: 33px; height: 33px;"></td></tr> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td></tr> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td></tr> </table>		1								<table style="border-collapse: collapse; width: 100px; height: 100px;"> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px; color: green; text-align: center;">1</td></tr> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px; color: green; text-align: center;">1</td><td style="width: 33px; height: 33px;"></td></tr> <tr><td style="width: 33px; height: 33px; color: green; text-align: center;">1</td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td></tr> </table>			1		1		1			<table style="border-collapse: collapse; width: 100px; height: 100px;"> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px; text-align: center;">1</td><td style="width: 33px; height: 33px;"></td></tr> <tr><td style="width: 33px; height: 33px; text-align: center;">1</td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td></tr> <tr><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px;"></td><td style="width: 33px; height: 33px; text-align: center;">1</td></tr> </table>		1		1					1
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# Intersecting Schubert Varieties

**Example.** Fix three flags  $R_\bullet$ ,  $G_\bullet$ , and  $B_\bullet$ :

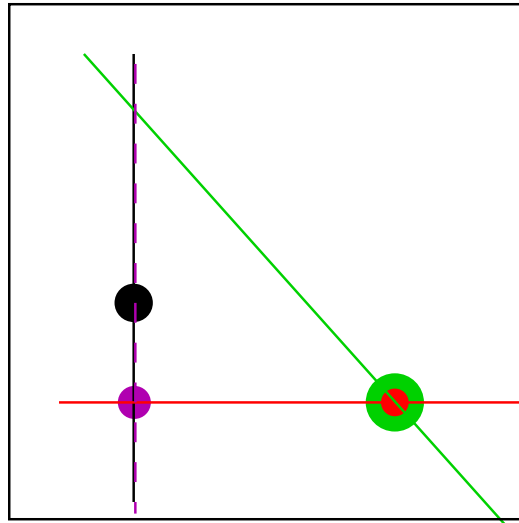


Find  $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$  where  $u, v, w$  are the following permutations:

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# Intersecting Schubert Varieties

**Example.** Fix three flags  $R_\bullet$ ,  $G_\bullet$ , and  $B_\bullet$ :



Find  $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$  where  $u, v, w$  are the following permutations:

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# Intersecting Schubert Varieties

**Schubert's Problem.** How many points are there usually in the intersection of  $d$  Schubert varieties if the intersection is 0-dimensional?

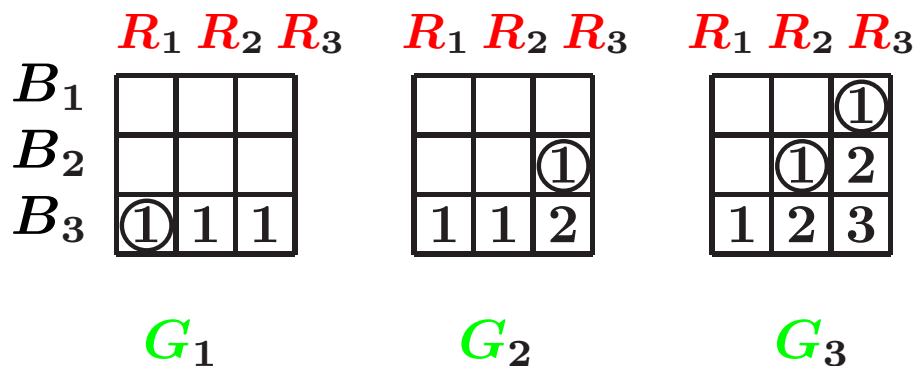
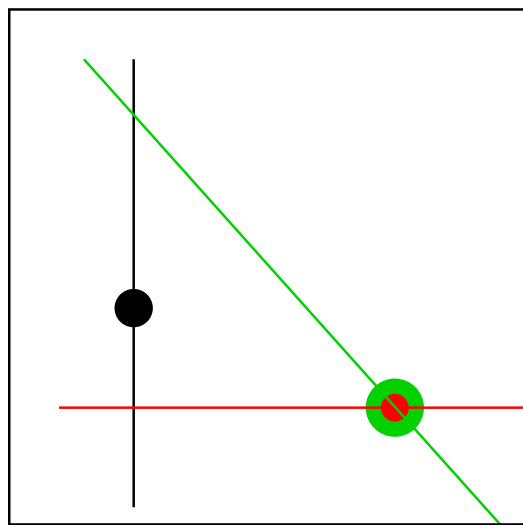
- Solving approx.  $n^d$  equations with  $\binom{n}{2}$  variables is challenging!

**Observation.** We need more information on spans and intersections of flag components, e.g.  $\dim(E_{x_1}^1 \cap E_{x_2}^2 \cap \dots \cap E_{x_d}^d)$ .

# Permutation Arrays

**Theorem.** (Eriksson-Linusson, 2000) For every set of  $d$  flags  $E_{\bullet}^1, E_{\bullet}^2, \dots, E_{\bullet}^d$ , there exists a unique permutation array  $P \subset [n]^d$  such that

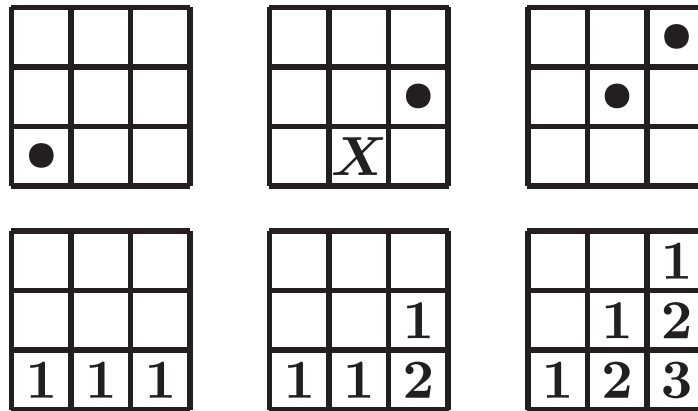
$$\dim(E_{x_1}^1 \cap E_{x_2}^2 \cap \dots \cap E_{x_d}^d) = \text{rk}P[x].$$



# Totally Rankable Arrays

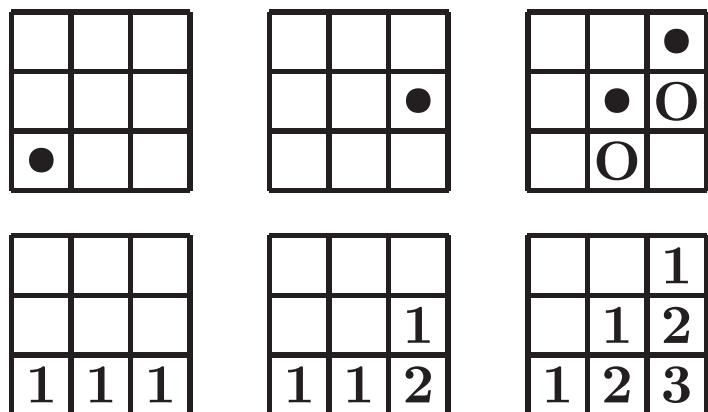
**Defn.** For  $P \subset [n]^d$ ,

- $\text{rk}_j P = \#\{k \mid \exists x \in P \text{ s.t. } x_j = k\}$ .
- $P$  is *rankable* of rank  $r$  if  $\text{rk}_j(P) = r$  for all  $1 \leq j \leq d$ .
- $y = (y_1, \dots, y_d) \preceq x = (x_1, \dots, x_d)$  if  $y_i \leq x_i$  for each  $i$ .
- $P[x] = \{y \in P \mid y \preceq x\}$
- $P$  is *totally rankable* if  $P[x]$  is rankable for all  $x \in [n]^d$ .



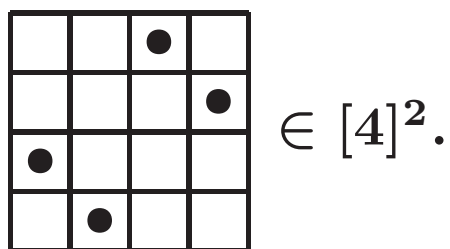
- Union of dots is totally rankable. Including  $X$  it is not.

# Permutation Arrays



- Points labeled  $O$  are redundant, i.e. including them gives another totally rankable array with same rank table.

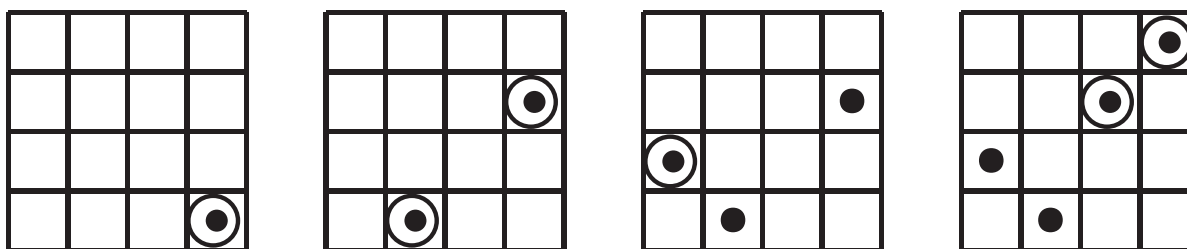
**Defn.**  $P \subset [n]^d$  is a *permutation array* if it is totally rankable and has no redundant dots.



**Open.** Count the number of permutation arrays in  $[n]^k$ .

# Permutation Arrays

**Theorem.** (Eriksson-Linusson) Every permutation array in  $[n]^{d+1}$  can be obtained from a unique permutation array in  $[n]^d$  by identifying a sequence of antichains.



This produces the 3-dimensional array

$$P = \{(4, 4, 1), (2, 4, 2), (4, 2, 2), (3, 1, 3), (1, 4, 4), (2, 3, 4)\}.$$

			4
		4	2
3			
	2		1

# Unique Permutation Array Theorem

**Theorem.** (Billey-Vakil 2005) If

$$X = X_{w^1}(E_{\bullet}^1) \cap \cdots \cap X_{w^d}(E_{\bullet}^d)$$

is nonempty 0-dimensional intersection of  $d$  Schubert varieties with respect to flags  $E_{\bullet}^1, E_{\bullet}^2, \dots, E_{\bullet}^d$  in general position, then there exists a **unique** permutation array  $P \in [n]^{d+1}$  such that

$$X = \{F_{\bullet} \mid \dim(E_{x_1}^1 \cap E_{x_2}^2 \cap \cdots \cap E_{x_d}^d \cap F_{x_{d+1}}) = \text{rk}P[x]\}. \quad (1)$$

Furthermore, we can recursively solve a family of equations for  $X$  using  $P$ .

**Open Problem.** Can one find a finite set of rules for moving dots in a 3-d permutation array which determines the  $c_{uv}^w$ 's analogous to one of the many Littlewood-Richardson rules?

**Recent Progress.** Izzet Coskun's Mondrian tableaux.

# Generalizations of Schubert Calculus for $G/B$

## 1997-2007: A Highly Productive Decade.

$$\left\{ \begin{array}{l} \mathbf{A: } GL_n \\ \mathbf{B: } SO_{2n+1} \\ \mathbf{C: } SP_{2n} \\ \mathbf{D: } SO_{2n} \\ \text{Semisimple Lie Groups} \\ \text{Kac-Moody Groups} \\ \text{GKM Spaces} \end{array} \right\} \times \left\{ \begin{array}{l} \text{cohomology} \\ \text{quantum} \\ \text{equivariant} \\ \text{K-theory} \\ \text{eq. K-theory} \end{array} \right\}$$

Contributions from: Bergeron, Billey, Brion, Buch, Carrell, Ciocan-Fontaine, Coskun, Duan, Fomin, Fulton, Gelfand, Goldin, Graham, Griffeth, Guillemin, Haibao, Haiman, Holm, Huber, Kirillov, Knutson, Kogan, Kostant, Kresh, S. Kumar, A. Kumar, Lam, Lascoux, Lenart, Miller, Peterson, Pitti, Postnikov, Ram, Robinson, Shimozono, Sottile, Sturmfels, Tamvakis, Vakil, Winkle, Yong, Zara...

See also A. Yong's slides on "Enumerative Formulas in Schubert Calculus"

<http://math.berkeley.edu/ayong/slides.html>

# Some Recommended Further Reading

1. “Schubert Calculus” by S. L. Kleiman; Dan Laksov The American Mathematical Monthly, Vol. 79, No. 10. (Dec., 1972), pp. 1061-1082.
2. “Young Tableaux” by William Fulton, London Math. Soc. Stud. Texts, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1997.
3. “The Symmetric Group” by Bruce Sagan, Wadsworth, Inc., 1991.
4. “Numerical Schubert calculus” by Birkett Huber, Frank Sottile, and Bernd Sturmfels, Journal of Symbolic Computation, 26, (1998) pp. 767-788.
5. “Determining the Lines Through Four Lines” by Michael Hohmeyer and Seth Teller, Journal of Graphics Tools, 4(3):11-22, 1999.
6. “Flag arrangements and triangulations of products of simplices” by Sara Billey and Federico Ardila, to appear in Adv. in Math.
7. Recent work of Frank Sottile and Anton Leykin here at IMA

<http://www.math.tamu.edu/~sottile/pages/Flags/Data/F37/We7W2W21.91.html>