Gröbner Basis Methods in Integer Programming

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Secondary fan

Fix $A = [\mathbf{a}_1\, \mathbf{a}_2\, \ldots\, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$

Recall $\mathcal{O}_{\mathbf{c}}^{\mathsf{LP}}$ is precisely $\mathbf{u} \geq \mathbf{0}$ s.t. $\mathrm{supp}(\mathbf{u}) \subseteq$ some facet of $\Delta_{\mathbf{c}}$.

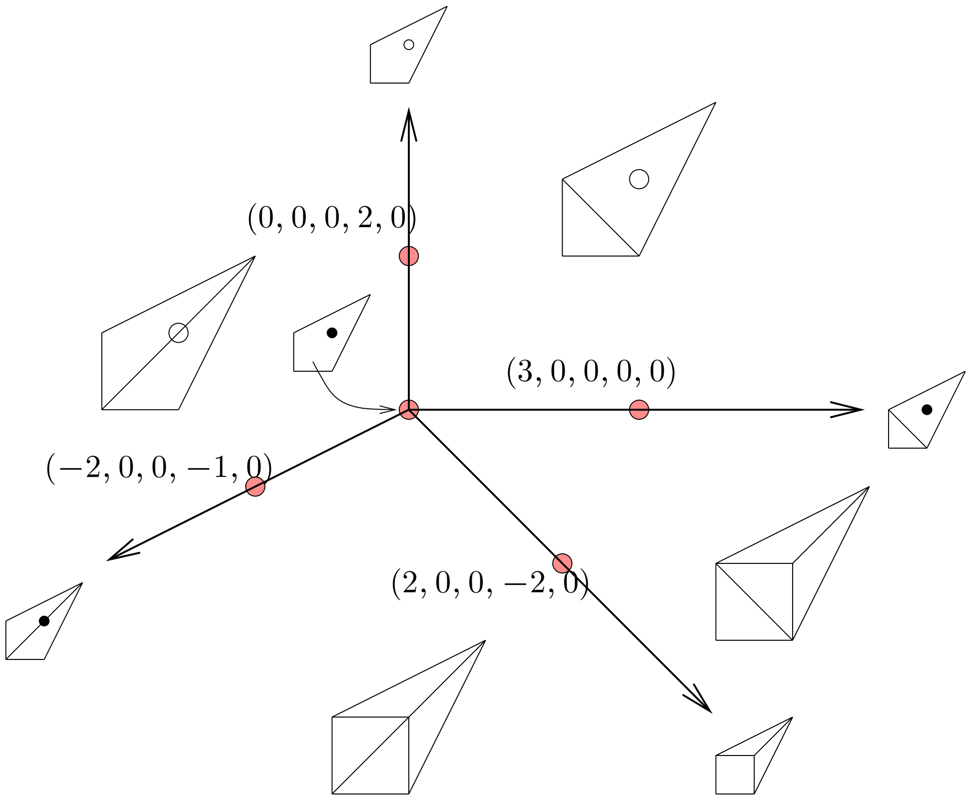
Hence $\mathcal{O}_{\mathbf{c}}^{\mathsf{LP}} = \mathcal{O}_{\mathbf{c}'}^{\mathsf{LP}} \Leftrightarrow \Delta_{\mathbf{c}} = \Delta_{\mathbf{c}'}$.

The secondary cell of $\mathbf{c}$ [GKZ] $\mathbf{C}[\mathbf{c}] := \{\, \mathbf{c}' \in \mathbb{R}^n \,:\, \Delta_{\mathbf{c}'} = \Delta_{\mathbf{c}} \,\}$

Secondary fan $\cup_{\mathbf{c} \in \mathbb{R}^n} \mathbf{C}[\mathbf{c}]$ has finitely many cells and partitions $\mathbb{R}^n$. The full dimensional cells correspond to triangulations.

The secondary fan of the 3×5 example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$



Gröbner fan

Recall The non-optimal solutions $\mathcal{N}_{\mathbf{c}}^{\mathsf{IP}}$ for $\mathsf{IP}_{A,\mathbf{c}}$ are precisely

$$\mathbf{u} \in \mathbb{N}^n \text{ s.t. } \mathbf{x}^{\mathbf{u}} \in \mathsf{in}_{\mathbf{c}}(I_A).$$

Hence $\mathcal{O}_{\mathbf{c}}^{\mathsf{IP}} = \mathcal{O}_{\mathbf{c}'}^{\mathsf{IP}} \Leftrightarrow \mathsf{in}_{\mathbf{c}}(I_A) = \mathsf{in}_{\mathbf{c}'}(I_A)$.

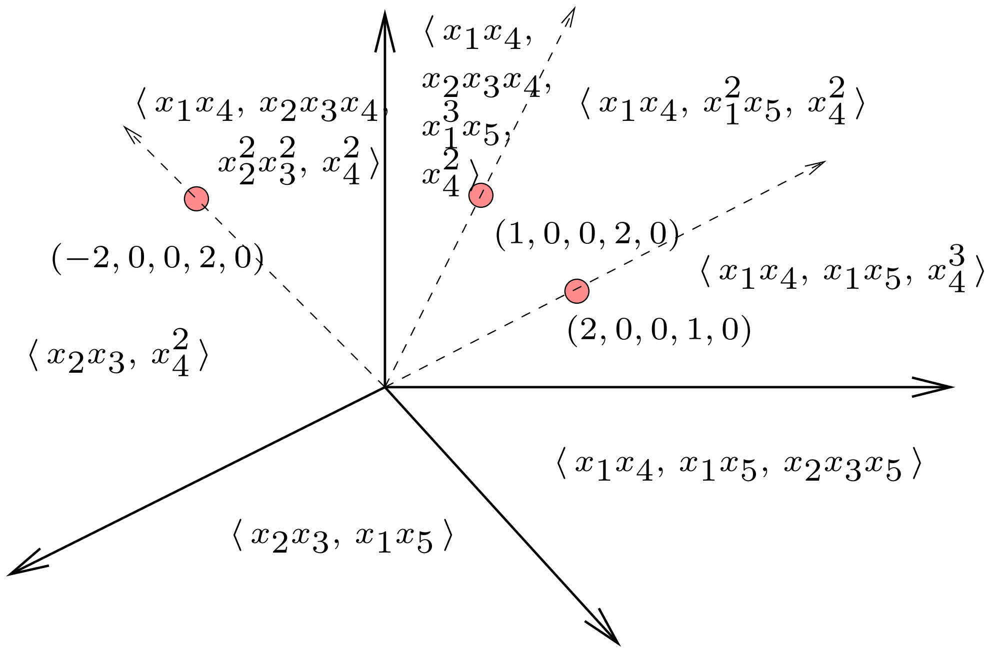
The Gröbner cell of $\mathbf{c}$ [MorRob]

$$\mathcal{K}_{\mathbf{c}} := \{ \mathbf{c}' \in \mathbb{R}^n : \mathsf{in}_{\mathbf{c}}(I_A) = \mathsf{in}_{\mathbf{c}'}(I_A) \}$$

Gröbner fan $\cup_{\mathbf{c} \in \mathbb{R}^n} \mathcal{K}_{\mathbf{c}}$ has finitely many cells and partitions $\mathbb{R}^n$. The full dimensional cells correspond to monomial initial ideals.

The Gröbner fan of the 3×5 example

All the initial ideals were computed using CaTS [Jensen]



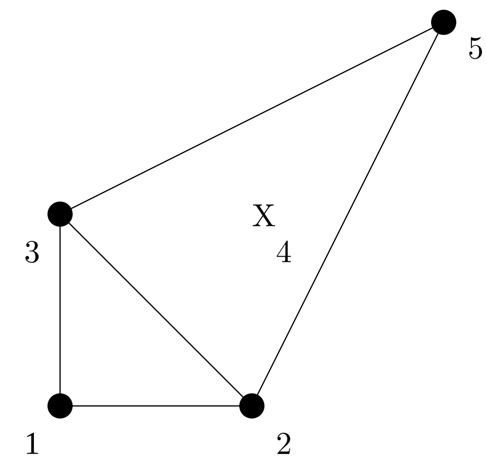
Gröbner fan refines secondary fan If $\mathbf{c} \in \mathbb{R}^n$ then $\mathcal{K}_\mathbf{c} \subseteq \mathbf{C}[\mathbf{c}]$. That is, if $\text{in}_\mathbf{c}(I_A) = \text{in}_{\mathbf{c}'}(I_A)$ then $\Delta_\mathbf{c} = \Delta_{\mathbf{c}'}$

Radicals of initial ideals

minimal non-faces $\{\,\{1,5\},\ \{4\}\,\}$

$\mathrm{in}_{(3,0,0,1,0)}(I_A)\ =\ \langle\, x_1x_4,\ x_1x_5,\ x_4^2\,\rangle$

Radical $\sqrt{I} = \{f\ :\ f^n \in I \text{ for some } n\}$



$\Delta_{(3,0,0,1,0)} = \{\,123,\ 235\,\}$

[Sturmfels] Let $\mathbf{c}$ be generic. Let $\tau \subseteq [n]$ and $\mathbf{x}_\tau := \prod_{i\in\tau} x_i$.

$$\sqrt{\mathrm{in}_{\mathbf{c}}(I_A)}\ =\ \langle\, \mathbf{x}_\tau\ :\ \tau \text{ a minimal non-face of } \Delta_{\mathbf{c}}\,\rangle$$

3×5 example $\sqrt{\mathrm{in}_{(3,0,0,1,0)}(I_A)}\ =\ \sqrt{\langle\, x_1x_4,\ x_1x_5,\ x_4^2\,\rangle}\ =\ \langle\, x_1x_5,\ x_4\,\rangle$.

Total Dual Integrality (TDI)

Question When do we have the good fortune of $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ having an **integer** optimum for all $\mathbf{b} \in \text{cone}(A) \cap \mathbb{Z}^d$. That is, $\text{LP}_{A,\mathbf{c}} = \text{IP}_{A,\mathbf{c}}$. In this case, we say $\mathbf{y}A \leq \mathbf{c}$ is TDI.

Unimodular triangulation Every facet σ of $\Delta_{\mathbf{c}}$ satisfies $\det(A_\sigma) = \pm 1$.

If $\mathbf{c}$ generic then $\mathbf{y}A \leq \mathbf{c}$ is TDI **iff** $\Delta_{\mathbf{c}}$ is a unimodular triangulation.

[Stu] $\Delta_{\mathbf{c}}$ is a unimodular triangulation **iff** $\sqrt{\text{in}_{\mathbf{c}}(I_A)} = \text{in}_{\mathbf{c}}(I_A)$

Equivalently, $\sqrt{\text{in}_{\mathbf{c}}(I_A)} = \text{in}_{\mathbf{c}}(I_A)$ **iff** $\text{in}_{\mathbf{c}}(I_A)$ is **squarefree**. That is,

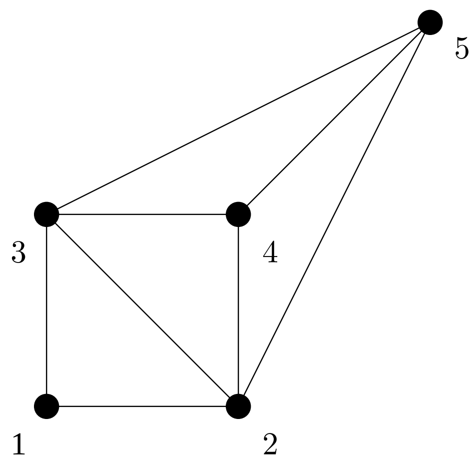
$$\text{in}_{\mathbf{c}}(I_A) = \langle \mathbf{x}^{\mathbf{v}_1^+}, \mathbf{x}^{\mathbf{v}_2^+}, \ldots, \mathbf{x}^{\mathbf{v}_s^+} \rangle \text{ with each } \mathbf{v}_i^+ \in \{0,1\}^n.$$

Drop the genericity of $\mathbf{c}$ [O'S.Seb] $\mathbf{y}A \leq \mathbf{c}$ is TDI **iff** $\langle \mathbf{x}^{\mathbf{v}} : \mathbf{v} \in \mathcal{O}_{\mathbf{c}}^{\text{IP}} \rangle$ is a squarefree initial ideal.

The 3×5 example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

The system $\mathbf{y}A \leq (3,0,0,-1,0)$ is TDI.



$\Delta_{(3,0,0,-1,0)}$

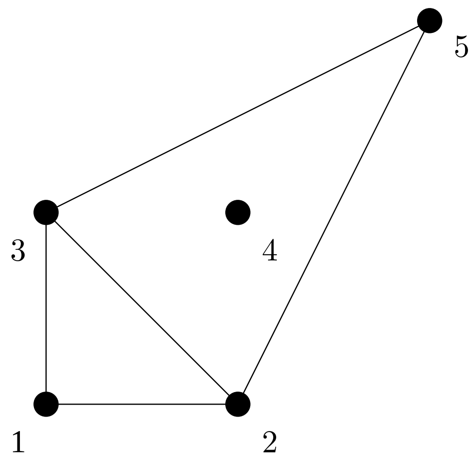
$$\det(A_{\{1,2,3\}}) = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

and $\det(A_{\{2,3,4\}}) = \det(A_{\{2,4,5\}}) = \det(A_{\{3,4,5\}}) = \pm 1$.

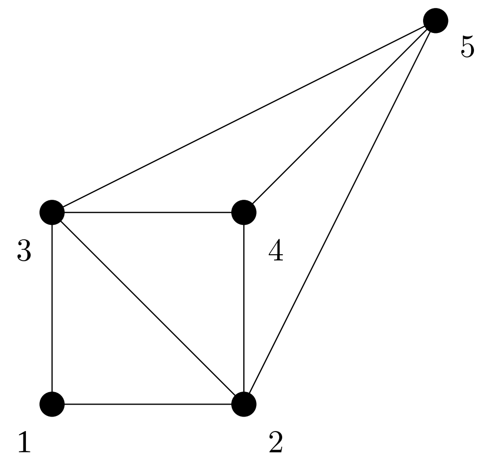
$$\text{in}_{(3,0,0,-1,0)}(I_A) = \langle\, x_1x_4,\, x_1x_5, x_2x_3x_5 \,\rangle$$

Detecting TDI

Refinements $\Delta_{\mathbf{c}'}$ refines $\Delta_{\mathbf{c}}$ if for every $\tau \in \Delta_{\mathbf{c}}$ $\exists$ $\sigma \in \Delta_{\mathbf{c}'}$ with $\sigma \subseteq \tau$.



$\Delta_{(3,0,0,0,0)} = \{\, 123,\ 2345 \,\}$

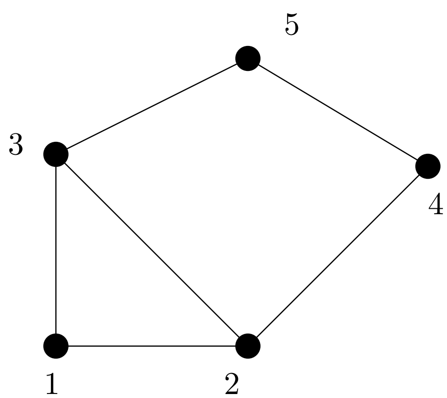


$\Delta_{(3,0,0,-1,0)} = \{\, 123, 234, 245, 345 \,\}$

[O'S.,Seb] If $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^n$ with $\Delta_{\mathbf{c}'}$ refining $\Delta_{\mathbf{c}}$ and $\Delta_{\mathbf{c}'}$ is a unimodular triangulation then $\mathbf{y}A \leq \mathbf{c}$ is TDI.

Example The system $\mathbf{y}A \leq (3,0,0,0,0)$ is TDI since $\Delta_{(3,0,0,-1,0)}$ is unimodular and it refines $\Delta_{(3,0,0,0,0)}$.

Weak Perfect Graph Theorem



$$M(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad \begin{array}{l} A(G) = [M^T \,|\, -I_n] \\ \mathbf{c}(G) = (\mathbf{1} \,|\, \mathbf{0}) \end{array}$$

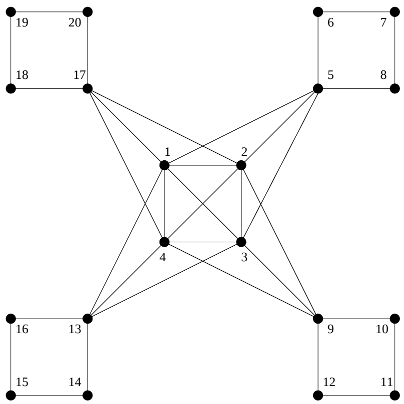
WPGT [Lovász],[Fulkerson] G perfect **iff** $\mathbf{y}A(G) \leq \mathbf{c}(G)$ is TDI.

[O'S.,Seb] $\exists\, \mathbf{c}'(G)$ with $\Delta_{\mathbf{c}'(G)}$ refining $\Delta_{\mathbf{c}(G)}$ and $\Delta_{\mathbf{c}'(G)}$ is a unimodular triangulation.

Remark The above strengthens WPGT by picking out the integer optima in a **uniform** manner, given by $\Delta_{\mathbf{c}'(G)}$.

Computations: initial ideals and triangulations

Problem Given a large A say (41×21) and $\mathbf{c}$ coming from a combinatorial optimization problem how can we **practically** detect $\mathbf{y}A \leq \mathbf{c}$ TDI ?



$\text{in}_{\mathbf{c}'(G)}(I_{A(G)})$ has 61 squarefree generators – computed using Macaulay 2 (< 1 sec).

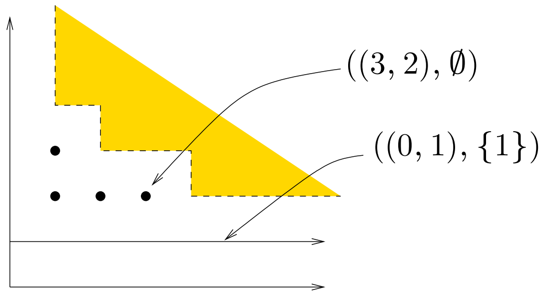
Using TOPCOM (days) $\Delta_{\mathbf{c}'(G)}$ has roughly 5×10^5 facets, each of dimension 21.
Confirm $\det(A_\sigma) = \pm 1$ for each facet $|\sigma| = 21$.

Max-flow Min-cut: A close cousin of WPGT Do max-flow min-cut (clutter) matrices also have a strong perturbation in the sense of WPGT ?

Identifying optimal sets $\mathcal{O}_{\mathbf{c}}^{\mathsf{IP}}$

Problem Characterize monomial ideals M for which $M = \text{in}_{\mathbf{c}}(I_A)$ for some A and $\mathbf{c}$. (**Assume** from now on that $\mathbf{c}$ generic.)

Standard pair decomposition of $\mathcal{O}_{\mathbf{c}}^{\mathsf{IP}}$



$(\mathbf{u},\sigma)$ std pair $\Rightarrow \mathbf{u} \in \mathbb{N}^n,\ \sigma \subseteq [n]$

- $\text{supp}(\mathbf{u}) \cap \sigma = \emptyset$
- $(\mathbf{u},\sigma) \subseteq \mathcal{O}_{\mathbf{c}}^{\mathsf{IP}}$
- $(\mathbf{u},\sigma) \not\subseteq (\mathbf{u}',\sigma')$

[STV] Std. pair decomposition is unique.
[STV] $(\cdot,\tau)$ std pair **iff** $P_\tau = \langle x_j : j \notin \tau \rangle$ **associated prime** of $\text{in}_{\mathbf{c}}(I_A)$.
[Stu] $(\cdot,\sigma)$ std pair $\Rightarrow \sigma \in \Delta_{\mathbf{c}} \Rightarrow |\sigma| \leq d$.
[Stu] σ a facet of $\Delta_{\mathbf{c}} \Rightarrow \#$ std. pairs $(\cdot,\sigma) = \det(A_\sigma)$.

The Chain Theorem

QuestionWhich non-maximal σ's appear?

Chain Theorem [Hoş,Tho] If $(\cdot, \sigma)$ std. pair with $|\sigma| < d$ then there exists $(\cdot, \sigma')$ std. pair with $\sigma \subset \sigma'$ and $|\sigma' \backslash \sigma| = 1$.

Example If $d = 3$ and $(1, \{2\})$ is a standard pair of $\text{in}_{\mathbf{c}}(I_A)$ then so must either $(\cdot, \{1, 2\})$ $(\cdot, \{2, 3\})$, or possibly both.

Gomory's Group Relaxations

Group relaxation of $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ w.r.t. $\sigma \subseteq [n]$:

$$G^{\sigma}(\mathbf{b}) = \min\{\mathbf{cu} : A\mathbf{u} = \mathbf{b},\ \mathbf{u} \in \mathbb{Z}^n,\ \mathbf{u}_{\overline{\sigma}} \geq \mathbf{0}\}$$

[HoşTho] $G^{\sigma}(\mathbf{b})$ bounded $\Leftrightarrow \sigma \in \Delta_{\mathbf{c}}$.

$G^{\sigma}(\mathbf{b})$ solves $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ if opt($G^{\sigma}(\mathbf{b})$) in $\mathbb{N}^n$.

$G^{\sigma}(\mathbf{b})$ minimally solves $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ if $G^{\sigma}(\mathbf{b})$ solves $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ but $G^{\tau}(\mathbf{b})$ does not for all $\tau \in \Delta_{\mathbf{c}}$, $\tau \supset \sigma$.

[HoşTho] $G^{\sigma}(\mathbf{b})$ minimally solves $\text{IP}_{A,\mathbf{c}}(\mathbf{b}) \Leftrightarrow$ opt($\text{IP}_{A,\mathbf{c}}(\mathbf{b})$) lies on a standard pair $(\cdot, \sigma)$.

[HoşTho] Polyhedral theory: standard polytopes.

Gomory families

[HoşTho] The family of integer programs $\text{IP}_{A,\mathbf{c}}$ Gomory family if $\forall\ \mathbf{b}$, $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ minimally solved by $G^{\sigma}(\mathbf{b})$ for some facet σ of $\Delta_{\mathbf{c}}$.

$\mathbf{y}A \leq \mathbf{c}$ TDI $\Rightarrow$ $\text{IP}_{A,\mathbf{c}}$ Gomory family.

Problem Classify all A for which there exists $\mathbf{c}$ such that $\text{IP}_{A,\mathbf{c}}$ Gomory family

Some progress [HoşTho] A is Δ-normal matrix.

Mind the gap

Integer Programming Gap [HoşStu] $\mathrm{gap}_{A,\mathbf{c}}(\mathbf{b}) := \mathrm{IP}_{A,\mathbf{c}}(\mathbf{b}) - \mathrm{LP}_{A,\mathbf{c}}(\mathbf{b})$

$$\mathrm{gap}_{A,\mathbf{c}} := \max_{\mathbf{b}}(\mathrm{IP}_{A,\mathbf{c}}(\mathbf{b}) - \mathrm{LP}_{A,\mathbf{c}}(\mathbf{b}))$$

[HoşStu] $\sum_{\mathbf{b}} \mathbf{t}^{\mathbf{b}} s^{\mathrm{gap}_{A,\mathbf{c}}(\mathbf{b})}$ (gen. function for gap) can be written as a short rational function and its degree in $s = \mathrm{gap}_{A,\mathbf{c}}$.

Tools used Standard pair decompositions, group relaxations and more.