# Gröbner Basis Methods in Integer Programming

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# Linear Programs

Matrix $A = [\mathbf{a}_1\, \mathbf{a}_2\, \cdots\, \mathbf{a}_n] \in \mathbb{Z}^{d\times n}$

Cost vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$

Constraint vector $\mathbf{b} = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$

The linear program $\text{LP}_{A,\mathbf{c}}(\mathbf{b}) := \min\{\mathbf{c}\cdot\mathbf{u} : A\mathbf{u} = \mathbf{b},\, \mathbf{u} \geq \mathbf{0}\}$

Feasible solution $P_{\mathbf{b}} := \{\mathbf{u} \geq \mathbf{0} : A\mathbf{u} = \mathbf{b}\}$

Feasibility $P_{\mathbf{b}} \neq \emptyset$ iff $\mathbf{b} \in \mathbb{R}_{\geq 0}A := \text{cone}(A)$

# Optimal Solutions

Recall LP$_{A,\mathbf{c}}(\mathbf{b})$ $:=$ min$\{\mathbf{c}\cdot\mathbf{u} \,:\, A\mathbf{u}=\mathbf{b},\, \mathbf{u}\geq\mathbf{0}\}$

Optimal solution all $\mathbf{u}^* \in P_{\mathbf{b}}$ s.th. $\mathbf{c}\cdot\mathbf{u}^* \leq \mathbf{c}\cdot\mathbf{u}$ for all $\mathbf{u}\in P_{\mathbf{b}}$

Optimal value the actual value of $\mathbf{c}\cdot\mathbf{u}^*$

Problem Given $\mathbf{b}$, decide if a feasible $\mathbf{u}\in P_{\mathbf{b}}$ is optimal.

A cost vector $\mathbf{c}\in\mathbb{R}^n$ is **generic** if LP$_{A,\mathbf{c}}(\mathbf{b})$ has a unique optimal solution for every $\mathbf{b}\in$ cone$(A)$.

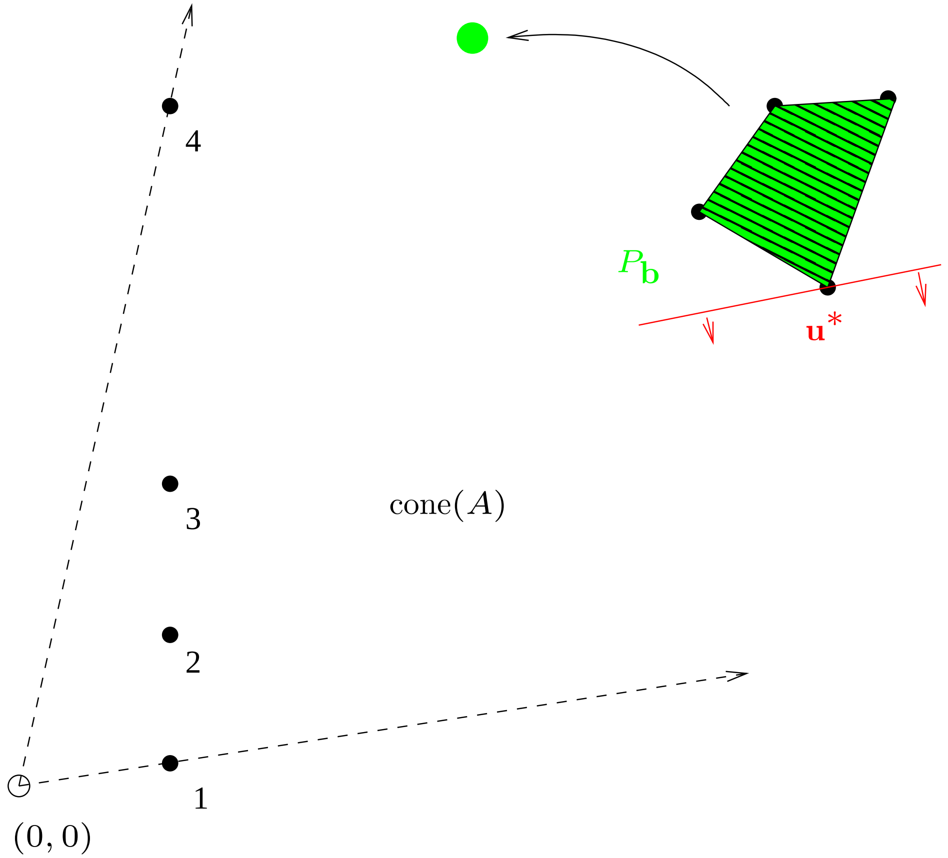
# The coin example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 10 & 25 \end{bmatrix}$$

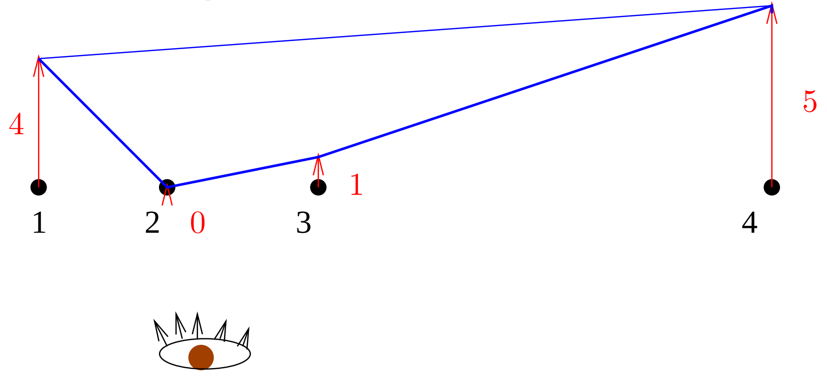
$$\mathbf{c} = (4, 0, 1, 5)$$

$$\mathbf{b} = (3, 27)$$

$$\mathbf{u}^* = (0, \tfrac{3}{5}, \tfrac{12}{5}, 0)$$



# Regular subdivisions



Regular subdivisions lift each vector $\mathbf{a}_i \in \mathbb{Z}^d$ into $\mathbb{R}^{d+1}$ by a height of $c_i$. Record the maximal faces (facets) $\sigma$ of the **lower hull** of the convex hull of the lifted configuration.

Notation $\Delta_{\mathbf{c}}(A)$ or $\Delta_{\mathbf{c}}$ when $A$ is unambiguous.

Maximal faces (facets) for $\Delta_{\mathbf{c}}$ are $\{\{1,2\},\{2,3\},\{3,4\}\}$.

Triangulations When all facets of $\Delta_{\mathbf{c}}$ have cardinality $d$.

# Regular subdivisions and optimal solutions

Support of a vector supp$(\mathbf{v}) := \{\, i \,:\, v_i \neq 0 \,\}$

[WalkupWets] Given $\mathbf{b} \in \mathbb{R}^d$, $\mathbf{u}$ is an optimal solution to LP$_{A,\mathbf{c}}(\mathbf{b})$

**iff**

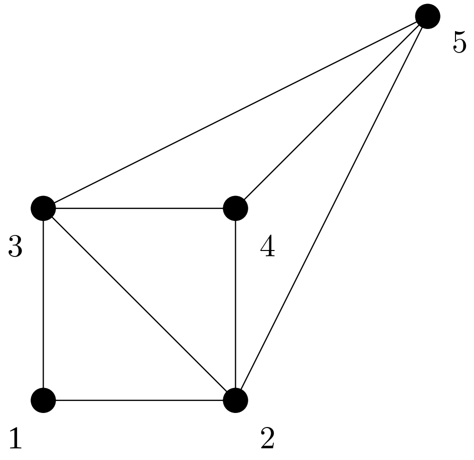
$$\mathbf{u} \geq 0, \;\; A\mathbf{u} = \mathbf{b}, \;\; \text{and supp}(\mathbf{u}) \subseteq \text{ some facet of } \Delta_{\mathbf{c}}.$$

Coin example supp$(\mathbf{u}^*) \subseteq$ one of $\{\{1,2\},\{2,3\},\{3,4\}\}$ So, $\mathbf{u} = (0,2,3,0)$ is an optimal solution to LP$_{A,\mathbf{c}}(A\mathbf{u})$. $(12,3,0,1)$ is not.

# Regular subdivisions: another example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\mathbf{c} = (3, 0, 0, -1, 0)$$



$\Delta_{(3,0,0,-1,0)}$

Optimal solutions to $\mathrm{LP}_{A,\mathbf{c}}(\mathbf{b})$ are all $\mathbf{u}^* \geq 0$ with

$$\mathrm{supp}(\mathbf{u}^*) \subseteq \{\{1,2,3\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}$$

# Circuits and test sets for LP$_{A,\mathbf{c}}(\mathbf{b})$

Problem What if I presented you with a feasible solution $\mathbf{u}$ to LP$_{A,\mathbf{c}}(\mathbf{b})$. How could you find an optimal $\mathbf{u}^*$ from $\mathbf{u}$ ?

Unique representation $\mathbf{v} = \mathbf{v}^+ - \mathbf{v}^-$ uniquely with $\mathbf{v}^+, \mathbf{v}^- \geq \mathbf{0}$ and supp$(\mathbf{v}^+) \cap$ supp$(\mathbf{v}^-) = \emptyset$.

Circuits of $A$ The minimal linear dependencies of the columns of $A$. i.e.

$$\mathcal{C}_A = \{\mathbf{v}^+ - \mathbf{v}^- \ : \ A\mathbf{v}^+ = A\mathbf{v}^-, \ \mathbf{v}^+, \mathbf{v}^- \in \mathbb{N}^n, \ \text{supp}(\mathbf{v}) \text{ minimal}, \ \text{gcd}_i(v_i) = 1\}$$

Note $\mathcal{C}_A$ only depends on $A$; no mention of $\mathbf{c}$ or $\mathbf{b}$.

# The coin example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 10 & 25 \end{bmatrix}, \quad \mathbf{c} = (4,0,1,5), \quad \mathbf{b} = (3,27).$$

$$\mathcal{C}_A = \{ \; (0,3,0,1) - (0,0,4,0),\; (5,0,4,0) - (0,9,0,0), \\ (5,0,0,1) - (0,6,0,0),\; (5,0,0,3) - (0,0,8,0) \, \}$$

Note We've written the vectors with $\mathbf{c} \cdot \mathbf{v}^+ > \mathbf{c} \cdot \mathbf{v}^-$. For example, $\mathbf{v}_1^+ - \mathbf{v}_1^- = (0,3,0,1) - (0,0,4,0)$ has $5 = (4,0,1,5) \cdot (0,3,0,1) > (4,0,1,5) \cdot (0,0,4,0) = 4$

We can also write the circuits as

$$\mathcal{C}_A = \{ (0,3,-4,1),\; (5,-9,4,0),\; (5,-6,0,1),\; (5,0,-8,3) \}$$

# The coin example (cont'd)

$\mathcal{C}_A = \{\,(0,3,-4,1),\ (5,-9,4,0),\ (5,-6,0,1),\ (5,0,-8,3)\,\}$

$\mathbf{u} = (2,0,0,1)$ is feasible and non-optimal solution to LP$_{A,(4,0,1,5)}(3,27)$.

$$\begin{aligned} (2,0,0,1) \quad &\longrightarrow \quad (2,0,0,1) - \frac{1}{3}(5,0,-8,3) \;=\; (\frac{1}{3},0,\frac{8}{3},0) \\ &\longrightarrow \quad (\frac{1}{3},0,\frac{8}{3},0) - \frac{1}{15}(5,-9,4,0) \;=\; (0,\frac{3}{5},\frac{12}{5},0) \end{aligned}$$

The feasible solution $\mathbf{u}^* = (0,\frac{3}{5},\frac{12}{5},0)$ is optimal.

# Summary of Linear Programs

Family of linear programs $\mathsf{LP}_{A,\mathbf{c}} := \{\, \mathsf{LP}_{A,\mathbf{c}}(\mathbf{b}) \ :\ \mathbf{b} \in \mathsf{cone}(A) \,\}$

Let $\mathcal{O}_{\mathbf{c}}^{\mathsf{LP}}$ denote the set of all optimal solutions to $\mathsf{LP}_{A,\mathbf{c}}$.

$$\mathbf{u} \in \mathcal{O}_{\mathbf{c}}^{\mathsf{LP}} \textbf{ iff } \mathsf{supp}(\mathbf{u}) \subseteq \sigma \text{ for some } \sigma \in \Delta_{\mathbf{c}}$$

$$\mathbf{u} \notin \mathcal{O}_{\mathbf{c}}^{\mathsf{LP}} \textbf{ iff } \exists\, \alpha > 0,\ \mathbf{v}^{+} - \mathbf{v}^{-} \in \mathcal{C}_A \text{ with } \mathbf{c}\cdot\mathbf{v}^{+} > \mathbf{c}\cdot\mathbf{v}^{-} \text{ s.t.}$$

$$\mathbf{u} - \alpha(\mathbf{v}^{+} - \mathbf{v}^{-}) \geq \mathbf{0}$$

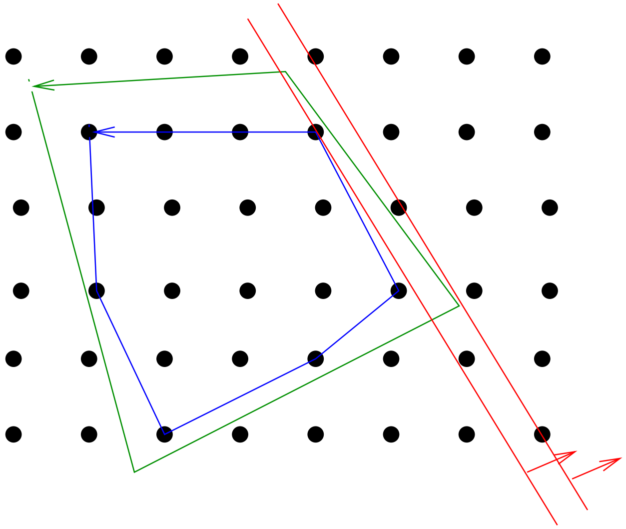
$\mathcal{C}_A$ is a test set for the family of integer programs $\mathsf{LP}_{A,\mathbf{c}}$. It generalizes the simplex algorithm – see [SturmfelsThomas].

# Integer Programs

The integer program $\mathsf{IP}_{A,\mathbf{c}}(\mathbf{b}) := \min\{\mathbf{c}\cdot\mathbf{u} : A\mathbf{u} = \mathbf{b},\ \mathbf{u}\in\mathbb{N}^n\}$

Feasibility $\mathsf{IP}_{A,\mathbf{c}}(\mathbf{b})$ has feasible solutions **iff** $\mathbf{b}\in\mathbb{N}A$.

Assume $\mathbf{c}\in\mathbb{R}^n$ is **generic**, i. e. $\mathsf{IP}_{A,\mathbf{c}}(\mathbf{b})$ has a unique optimal solution for every $\mathbf{b}\in\mathbb{N}A$.



# The family of Integer Programs

Family of integer programs $\text{IP}_{A,\mathbf{c}} := \{\, \text{IP}_{A,\mathbf{c}}(\mathbf{b}) \; : \; \mathbf{b} \in \mathbb{N}A \,\}$

The set of optimal solutions Let $\mathcal{O}_{\mathbf{c}}^{\text{IP}}$ denote the set of all optimal solutions to the family $\text{IP}_{A,\mathbf{c}}$.

Problem Decide if a given $\mathbf{u} \in \mathbb{N}^n$ is in $\mathcal{O}_{\mathbf{c}}^{\text{IP}}$.

Problem Find test sets for $\text{IP}_{A,\mathbf{c}}$.

# The toric ideal

Linear algebra for LP$_{A,\mathbf{c}}$ analysis.

Polynomial algebra for IP$_{A,\mathbf{c}}$ analysis.

Exponent vectors Given $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$, $\mathbf{x}^{\mathbf{v}} := x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$.

The toric ideal of $A$

$$I_A = \langle\, \mathbf{x}^{\alpha} - \mathbf{x}^{\beta} \;:\; A\alpha = A\beta\,;\; \alpha,\, \beta \in \mathbb{N}^n \,\rangle \subseteq \mathbf{k}[x_1, x_2, \ldots, x_n]$$

For the coin example (using Macaulay 2): $I_A = \langle x_3^4 - x_2^3 x_4, x_2^6 - x_1^5 x_4 \rangle$.

# Initial ideals and Gröbner bases

Let $f = \sum r_i \mathbf{x}^{\mathbf{v}_i}$, $\mathbf{c} \in \mathbb{R}^n$ and $I$ an ideal in $\mathbf{k}[\mathbf{x}] := \mathbf{k}[x_1, x_2, \ldots, x_n]$.

Initial term of a polynomial (with respect to $\mathbf{c}$)

$\mathrm{in}_{\mathbf{c}}(f)$ is the sum of all the $r_i \mathbf{x}^{\mathbf{v}_i}$'s for which $\mathbf{c} \cdot \mathbf{v}_i$ is **maximal**.

Initial ideal of $I$ (with respect to $\mathbf{c}$)

$$\mathrm{in}_{\mathbf{c}}(I) \;=\; \langle\, \mathrm{in}_{\mathbf{c}}(f) \;:\; f \in I \,\rangle \;\subseteq\; \mathbf{k}[\mathbf{x}]$$

Gröbner basis of $I$ (with respect to $\mathbf{c}$) [Buchberger]

$$\mathcal{G}_{\mathbf{c}}(I) \;=\; \{\, g_1, g_2, \ldots, g_s \in I \;:\; \mathrm{in}_{\mathbf{c}}(I) \;=\; \langle\, \mathrm{in}_{\mathbf{c}}(g_1), \mathrm{in}_{\mathbf{c}}(g_2), \ldots, \mathrm{in}_{\mathbf{c}}(g_s) \,\rangle \,\} \subseteq \mathbf{k}[\mathbf{x}]$$

# The coin example

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 10 & 25 \end{bmatrix}$, $\mathbf{c} = (4,0,1,5)$ and $I_A = \langle\, x_3^4 - x_2^3 x_4,\ x_2^6 - x_1^5 x_4 \,\rangle$.

$\mathrm{in}_{(4,0,1,5)}(x_2^3 x_4 - x_3^4) = x_2^3 x_4$ since $x_2^3 x_4 \leftrightarrow (0,3,0,1)$ and $x_3^4 \leftrightarrow (0,0,4,0)$

and $(4,0,1,5) \cdot (0,3,0,1) = 5 > 4 = (4,0,1,5) \cdot (0,0,4,0)$.

$$\mathcal{G}_{(4,0,1,5)}(I_A) \;=\; \{x_2^3 x_4 - x_3^4,\ x_1^5 x_3^4 - x_2^9,\ x_1^5 x_4 - x_2^6\}$$

$$\mathrm{in}_{(4,0,1,5)}(I_A) \;=\; \langle x_1^5 x_3^4,\ x_1^5 x_4,\ x_2^3 x_4 \rangle$$

# Gröbner bases and initial ideals of toric ideals

Reduced Gröbner basis of $I_A$ consists of binomials [EisenbudSturmfels]

$$g_i = \mathbf{x}^{\mathbf{v}_i^+} - \mathbf{x}^{\mathbf{v}_i^-} \text{ or, as a vector, } \mathbf{g}_i = \mathbf{v}_i^+ - \mathbf{v}_i^- \text{ with } \mathbf{v}_i^+, \mathbf{v}_i^- \in \mathbb{N}^n$$

$$\mathcal{G}_{\mathbf{c}}(I_A) = \{\mathbf{v}_i^+ - \mathbf{v}_i^- \ : \ i = 1, \ldots, s\}$$

$$\text{and} \quad \mathbf{c} \cdot \mathbf{v}_i^+ > \mathbf{c} \cdot \mathbf{v}_i^- \text{ for each } i = 1, \ldots, s.$$

The toric initial ideal is $\text{in}_{\mathbf{c}}(I_A) \ = \ \langle\, \mathbf{x}^{\mathbf{v}_1^+}, \ldots, \mathbf{x}^{\mathbf{v}_s^+} \,\rangle$.

$\mathcal{N}_{\mathbf{c}}^{\text{IP}} = \bigcup_{i=1}^{s} (\mathbf{v}_i^+ + \mathbb{N}^n)$, where $\mathcal{N}_{\mathbf{c}}^{\text{IP}} = \mathbb{N}^n \backslash \mathcal{O}_{\mathbf{c}}^{\text{IP}}$.

# Gröbner bases as test sets

[ContiTraverso] The reduced Gröbner basis $\mathcal{G}_{\mathbf{c}}$ is a test set for $\mathrm{IP}_{A,\mathbf{c}}(\mathbf{b})$

$$\mathbf{u} \in \mathcal{N}_{\mathbf{c}}^{\mathrm{IP}} \Leftrightarrow \exists\ \mathbf{v}_i^+ - \mathbf{v}_i^- \in \mathcal{G}_{\mathbf{c}} \text{ s.t. } \mathbf{u} - (\mathbf{v}_i^+ - \mathbf{v}_i^-) \in \mathbb{N}^n$$

$$\mathcal{G}_{(4,0,1,5)} = \{(0,3,0,1)-(0,0,4,0),\ (5,0,4,0)-(0,9,0,0),\ (5,0,0,1)-(0,6,0,0)\}$$

$$\begin{aligned} (12,2,5,3) \quad &\longrightarrow \quad (12,2,5,3) - 2(5,-6,0,1) \;=\; (2,14,5,1) \\ &\longrightarrow \quad (2,14,5,1) - (0,3,-4,1) \;=\; (2,11,9,0) \end{aligned}$$

$\mathbf{u}^* = (2,11,9,0)$ is an optimal solution $\mathrm{IP}_{A,(4,0,1,5)}(22,147)$.