Optimization over Polynomials with Moment Matrices and SOS: Further Topics

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IMA Tutorial: Algebraic Algorithms in Optimization

How to reduce the size of the SDP relaxations (MOMt) and (SOSt)?

Exploit the structure of the problem:

- Equations
- Sparsity
- Symmetry

Exploit equations: The 0/1 instance

Assume the equations $h_i := x_i^2 - x_i = 0$ are present in the description of (POP)

On the moment side:

$$M_t(h_i y) = 0 \Longrightarrow y_{\alpha+2e_i} = y_\alpha \ \forall \alpha$$
$$\Longrightarrow y_\alpha = y_{\text{supp}(\alpha)} \ \forall \alpha$$

Hence: (MOMt) involves only variables y_α with $\alpha \in \{0,1\}^n$

$\rightsquigarrow$ reduce number of variables and (MOMt) is exact at $t = n$

On the SOS side: work in the quotient space $\mathbb{R}[x]/I$

Extension to general ideal I - See [Lasserre, La, Jibetean-La, Parrilo,..]

Example: The Stable Set Problem

The **stability number** $\alpha(G)$ of graph $G = (V, E)$:

$$\max \sum_{i \in V} x_i \ \text{ s.t. } x_i x_j = 0 \ (ij \in E), \ x_i^2 = x_i \ (i \in V)$$

- **Moment relaxation of order** 1= Shor relaxation:

$$\max \sum_{i \in V} y_i \ \text{ s.t. } M_1(y) \succeq 0, \ y_{ij} = 0 \ (ij \in E), \ y_{\emptyset} = 1$$

$\rightsquigarrow$ Lovász' **theta number**

- **Finite convergence** in $\alpha(G)$ steps

Example: The Max-Cut Problem

$G = (V, E)$ graph with edge weights $w \in \mathbb{R}^E$.

Max-Cut: $\max \sum_{ij \in E} w_{ij}(1 - x_i x_j)$ s.t. $x_i^2 = 1$ $(i \in V)$

• **Moment relaxation of order** $t = 1$= Shor relaxation:

$$\max \sum_{ij \in E} w_{ij}(1 - X_{ij}) \text{ s.t. } X \succeq 0, \; X_{ii} = 1 \; (i \in V)$$

$\leadsto$ Goemans-Williamson **0.878**-approximation algorithm

• **Finite convergence** in n steps

Proven: $n/2$ steps are necessary

Conjecture: $n/2$ steps are sufficient

Exploit sparsity

$$p = \sum_\alpha p_\alpha x^\alpha$$

$\rightsquigarrow$ **Newton polytope**: $N(p) = \text{conv}(\alpha \mid p_\alpha \neq 0)$

Fact: [Reznick 1978] $p = \sum_i f_i^2 \Longrightarrow N(f_i) \subseteq \frac{1}{2}N(p)$

$\rightsquigarrow$ reduce the search space for the f_i's

Example: $p = (x^4+1)(y^4+1)(z^4+1) + 2x - z + 5$

$N(p)$: cube with vertices $(0,0,0)$ and $(4,4,4)$

$\frac{1}{2}N(p)$: cube with vertices $(0,0,0)$ and $(2,2,2)$

Hence, f_i uses at most $3^3 = 27$ instead of $\binom{n+d}{d} = \binom{3+6}{6} = 84$ monomials

Exploit sparsity (continued)

Minimize $p = p_1(x_1 \dots x_3 x_4 x_5) + p_2(x_4 x_5 x_6 \dots x_8)$

s.t. $x \in K : \begin{cases} f_j(x_1 \dots x_3 x_4 x_5) \geq 0 \ (j \in J_1) & \longrightarrow K_1 \\ g_j(x_4 x_5 x_6 \dots x_8) \geq 0 \ (j \in J_2) & \longrightarrow K_2 \end{cases}$

If Putinar's assumption holds for K_1, K_2, one can give a more economical SOS certificate: $p > 0$ on $K \Longrightarrow$

$$p = \underbrace{s_0 + \sum_j s_j f_j}_{s_0, s_j \text{ SOS in } \mathbb{R}[x_1 \dots x_3 x_4 x_5]} + \underbrace{t_0 + \sum_j t_j g_j}_{t_0, t_j \text{ SOS in } \mathbb{R}[x_4 x_5 x_6 \dots x_8]}$$

See [Lasserre (06), Kojima, Kim, Waki, Muramatsu (05,06), Grimm, Netzer, Schweighofer (06), Nie (06), ...]

Exploit symmetry

Lot of recent work about symmetry reduction using invariant theory and block-diagonalization of matrix $*$-algebras:

Gatermann-Parrilo (2004), J. Pure and Applied Algebra

de Klerk-Pasechnik-Schrijver (2006), Math. Programming

Schrijver (2005), IEEE Trans. Inform. Theory

La (2006), Math. Programming

Jansson-Lasserre-Riener-Theobald (2006)

Bachoc-Vallentin (2006)

Bosse (2006)

Many other interesting issues

- **How to improve the numerical stability ?**

Hankel matrices have a poor condition number ...

Idea: [Roh-Vandenberghe], [Löfberg-Parrilo], [de Klerk-den Hertog-Elabwabi]

Use another basis (instead of the monomial basis)

Use interpolation points for representing polynomials

- **Estimates on the quality of the MOM/SOS relaxations ?**

[de Klerk-La-Parrilo]: yes for minimization on the simplex

- **Degree bounds on the SOS relaxations ?**

Results by Schweighofer, Nie,..