JUMP-DIFFUSIONS

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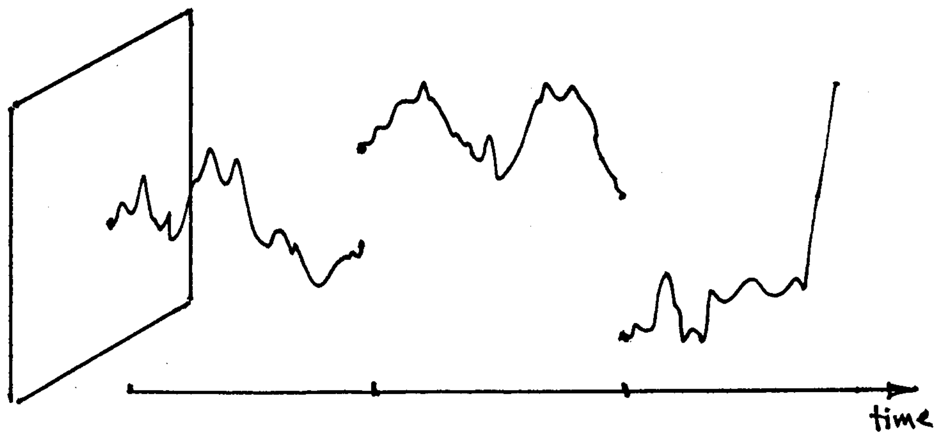


Jump-diffusion: piecewise continuous Itô process in $\mathbb{R}^d$

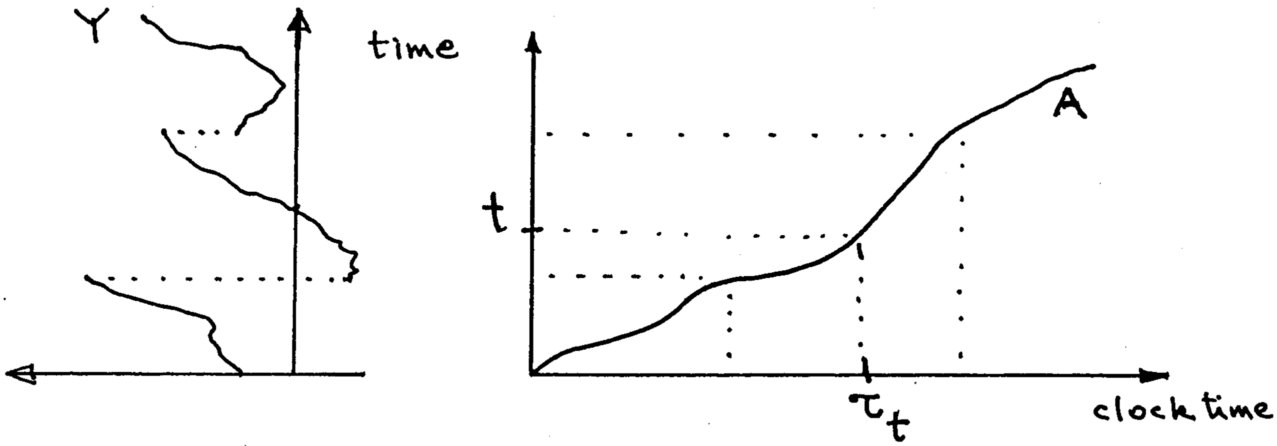
$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t c(X_s)\,dW_s$$

$$+ \int_{[0,t]\times\mathbb{R}_+} N(ds,du)\; j(X_{s-},u)$$

W – Wiener in $\mathbb{R}^d$

N – Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$

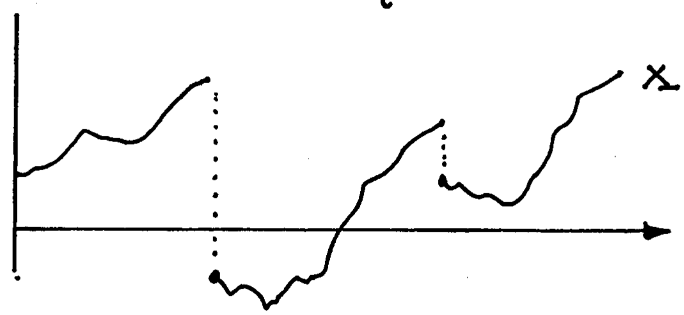
mean = Lebesgue.



X Itô

$A_s = \int_0^s a(X_u)\,du$

$Y_t = X_{\tau_t}$

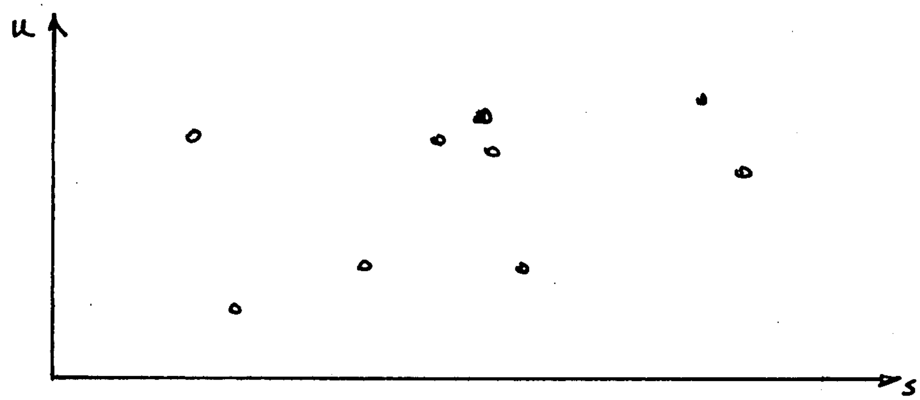


$\Updownarrow$

Y Hunt process in $\mathbb{R}^d$

piecewise cont's semimartingale

Poisson N



For each ω, the measure $N_\omega : A \mapsto N_\omega(A)$

is purely atomic, each atom has weight one

$$\mathbb{P}\{ N(A) = k \} = \frac{e^{-a}\, a^k}{k!}, \qquad k = 0, 1, \dots$$

$$a = \text{Leb}\, A .$$

$\Updownarrow$

$$\mathbb{P}\{ N(A) = 0 \} = e^{-\text{Leb}\, A} .$$

$\Downarrow$ $N(A), \dots, N(B)$ are independent whenever $A, \dots, B$ are disjoint

Lévy kernel

$$\mathbb{E}^x \sum_{s \le t} f(X_{s-}, X_s) \qquad \begin{array}{l} f \geq 0 \\ f(x,x) = 0 \end{array}$$

$$= \mathbb{E}^x \int_{[0,t] \times \mathbb{R}_+} N(ds, du)\ f(X_{s-}, X_{s-} + j(X_{s-}, u))$$

$$= \mathbb{E}^x \qquad ds \quad du$$

$$= \mathbb{E}^x \int_0^t ds \int_{\mathbb{R}_+} du\ f(X_s, X_s + j(X_s, u))$$

$$= \mathbb{E}^x \int_0^t ds \int_{\mathbb{R}^d} L(X_s, dy)\ f(X_s, y)$$

where

$$L(x,B) = \mathrm{Leb}\,\{u \in \mathbb{R}_+ : \quad x + j(x,u) \in B\}$$

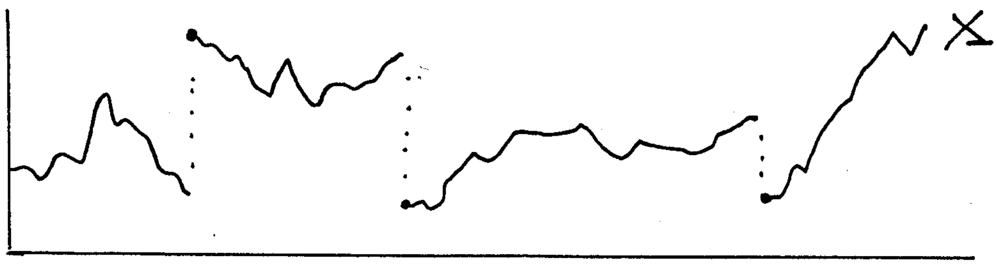
$$= \text{"rate" of jumps from } x \text{ into } B$$

Piecewise cont'y $\Rightarrow$ $\quad k(x) = L(x, \mathbb{R}^d) < \infty$

$$L(x,B) = k(x)\ K(x,B)$$

$k(x)$ = rate of jumps from x

$K(x,B)$ = prob (jump lands in B | given jump from x)



$$dX_t = b(X_t)\, dt + c(X_t)\, dW_t + \int_{u=0}^{\infty} N(dt, du)\, j(X_{t-}, u)$$

Transition semigroup: $P_t f(x) = \mathbb{E}^x f(X_t)$

THM. (Forward equations)

$$\frac{d}{dt} P_t = P_t G$$

where

$$Gf(x) = b(x) f'(x) + \tfrac{1}{2} c(x)^2 f''(x)\, dt + \int_{\mathbb{R}} L(x, dy)\, [f(y) - f(x)],$$

$$L(x, dy) = \text{rate of jumps from } x \text{ into } dy = k(x)\, K(x, dy)$$

Proof Outline

$$df(X_t) = b(X_t) f'(X_t)\,dt + \tfrac{1}{2} c(X_t)^2 f''(X_t)\,dt + \text{mart}$$

$$+ \int_{u=0}^{\infty} N(dt, du)\,[f(X_{t-} + j(X_{t-}, u)) - f(X_{t-})] .$$

$\Rightarrow$

$$d\,\mathbb{E}^x f(X_t) = \mathbb{E}^x \Big[b(X_t) f'(X_t) + \tfrac{1}{2} c(X_t)^2 f''(X_t)$$

$$+ \int L(X_t, dy)\,[f(y) - f(X_t)] \Big]\, dt$$

$$= \mathbb{E}^x [Gf(X_t)]\, dt$$

where

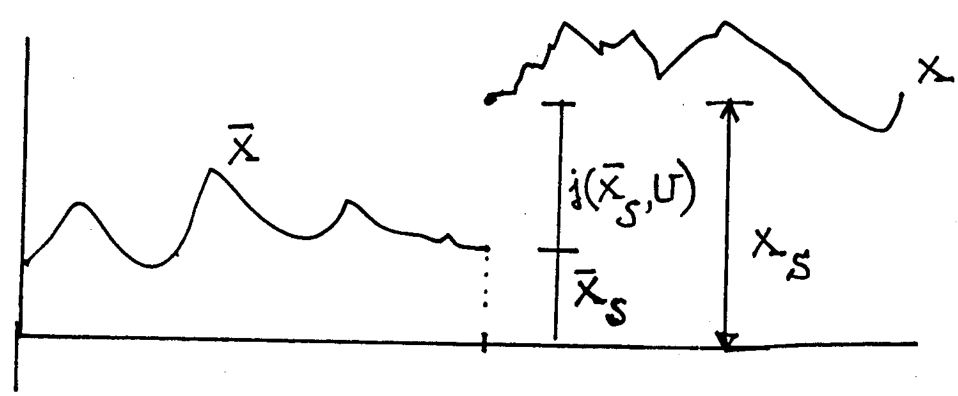
$$Gf(x) = b(x) f'(x) + \tfrac{1}{2} c(x)^2 f''(x) + \int L(x, dy)\,[f(y) - f(x)] .$$

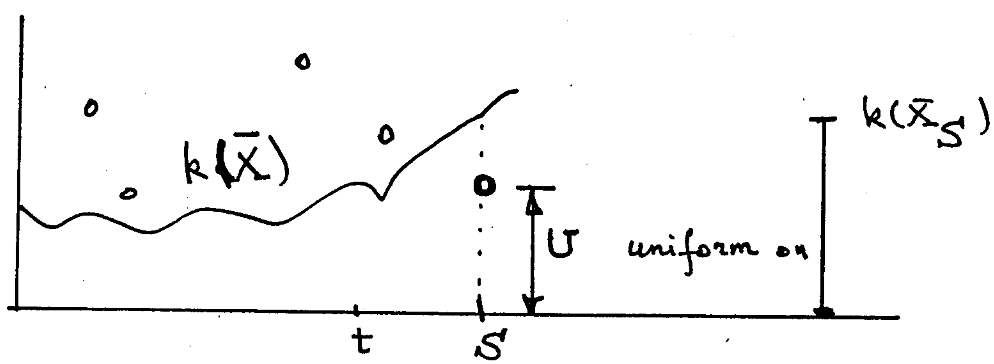
$$\frac{d}{dt} P_t f(x) = P_t G f(x) .$$

Meaning of $dX = b(X)\,dt + c(X)\,dW + \int_{u=0}^{\infty} N(dt,du)\, j(X_{t-},u)$

Define

$$d\bar{X} = b(\bar{X})\,dt + c(\bar{X})\,dW$$

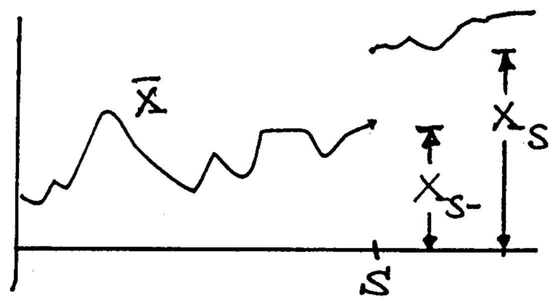




$$\hat{P}_t f(x) = \mathbb{E}^x\, f\circ X_t\, 1_{\{S>t\}}$$

$$= \mathbb{E}^x\, (f\circ\bar{X}_t)\left(\exp - \int_0^t k(\bar{X}_s)\,ds\right)$$

The first jump



$$\mathbb{P}^x\{ S > t \mid \bar{X} \} = \exp_- \int_0^t k\circ\bar{X}_s \, ds$$

$$X_{S-} = \bar{X}_S$$

$$\mathbb{P}^x\{ X_S \in dz \mid X_{S-} = y \} = K(y, dz)$$

- $\mathbb{P}^x\{ X_{S-} \in dy,\ X_S = dz,\ S \in dt \}$

$$= \mathbb{E}^x \left(\exp_- \int_0^t k\circ\bar{X}_s \, ds \right) (k\circ\bar{X}_t)\, dt \;\; I(\bar{X}_t, dy)\, K(y, dz)$$

$$= \mathbb{E}^x \left(\cdots\cdots\cdots \right) I(\bar{X}_t, dy)\, L(y, dz)\, dt$$

$$= \hat{P}_t(x, dy)\, L(y, dz)\, dt$$

- $Q(x, dz, dt) = \mathbb{P}^x\{ X_S \in dz,\ S \in dt \} = \hat{P}_t L(x, dz)\, dt$

$$Q(x, dy, ds) = \mathbb{P}^x\{X_S \in dy, S \in dt\} = \hat{P}_t L(x, dy)\, dt$$

$$\hat{P}_t f(x) = \mathbb{E}^x f \circ \bar{X}_t 1_{\{S > t\}} = \mathbb{E}^x f \circ \hat{X}_t$$

semigroup for $\hat{X}$, the pro. $\bar{X}$ killed at rate k.

Semigroup for X

$$P_t f(x) = \mathbb{E}^x f \circ X_t$$

$$= \mathbb{E}^x f \circ X_t 1_{\{S > t\}} + \mathbb{E}^x f \circ X_t 1_{\{S \le t\}}$$

$$= \hat{P}_t f(x) + \int_{\mathbb{R}^d \times [0,t]} Q(x, dy, ds)\, P_{t-s} f(y)$$

Potentials:

$$U_\alpha = \hat{U}_\alpha + Q_\alpha U_\alpha \Rightarrow U_\alpha = R_\alpha \hat{U}_\alpha$$

where

$Q_\alpha = \hat{U}_\alpha L$. $\leftarrow$ sub-Markovian kernel

$R_\alpha = \sum_{n=0}^\infty (Q_\alpha)^n$ $\longleftarrow$ potential for Q_α.

Given: $b(x)$ drift rate
$c(x)$ volatility
$k(x)$ killing rate
$K(x,B)$ jump kernel

Known things for the diffusion $\hat{X}$ killed:

Generator: $\hat{G}f = bf' + \frac{1}{2}c^2 f'' - kf$

Semigroup: $\hat{P}_t = e^{t\hat{G}}$

Resolvent: $\hat{U}_\alpha = \int_0^\infty dt\, e^{-\alpha t}\, \hat{P}_t = (\alpha I - \hat{G})^{-1}$

Embedded Markov renewal process

Semimarkov kernel $Q(x,dz,dt) = \int \hat{P}_t(x,dy)\, k(y)\, K(y,dz)\, dt$

Laplace transform $Q_\alpha(x,dz) = \int \hat{U}_\alpha(x,dy)\, k(y)\, K(y,dz)$

Potential $R_\alpha = \sum_n (Q_\alpha)^n = (I - Q_\alpha)^{-1}$

Original Process X

Resolvent: $U_\alpha = \hat{U}_\alpha + Q_\alpha U_\alpha$

$U_\alpha = R_\alpha \hat{U}_\alpha = (\alpha I - \hat{G} - L)^{-1}$

Semigroup P_t

Hitting times and places

Fix b. $\qquad T_b = \inf\{t : X_t > b\}$.

A

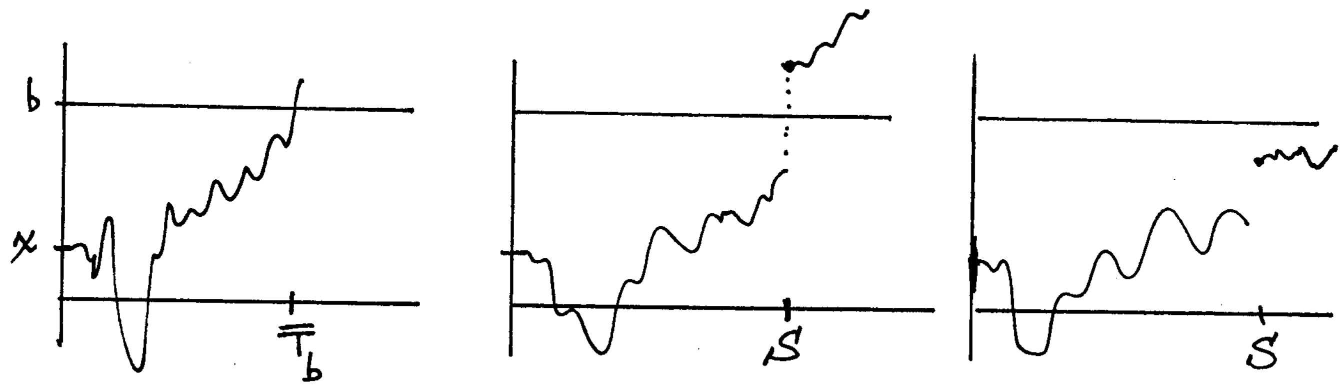
B

want $\quad H(x,t) = \mathbb{P}^x\{ T_b \le t,\ X_{T_b -} \in A,\ X_{T_b} \in B \}$

H satisfies a Markov renewal equation:

$$H(x,t) = h(x,t) + \int_{(-\infty, b] \times [0,t]} Q(x; dz, ds)\, H(z, t-s)$$

where h and Q are known ...



$$H(x,t) = \mathbb{P}^x\{T_b \le t,\ X_{T_b-} \in A,\ X_{T_b} \in B\}$$

$$= \int_{[0,t]} \mathbb{P}^x\{\overline{T}_b \in ds,\ S > s\}\ I(b,A)\, I(b,B)$$

$$+ \int_{[0,t]\times A} \mathbb{P}^x\{S \in ds,\ \overline{X}_s \in dy,\ \overline{T}_b > s\}\ K(y,B)$$

$$+ \int_{[0,t]\times(-\infty,b]} \mathbb{P}^x\{S \in ds,\ \overline{X}_s \in dy,\ \overline{T}_b > s\} \int_{(-\infty,b]} K(y,dz)\ H(z,t-s)$$

$$= \cdots + \cdots + \int_{[\ \ ]\times[\ \ ]} Q(x,dz,ds)\ H(z,t-s)$$