Combinatorics on Adjacency Graphs and Incidence Pseudographs

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Regular orthogonal grid in $n$D

**2D:** pixel
- points: 4- and 8-adjacencies
  [Rosenfeld 1970]
- cells: 1- and 0-adjacencies
  [Artzy/Frieder/Herman 1981]

**3D:** voxel, frontier faces
- 6-, 18- and 26-adjacencies

![Diagram of grid models]
Menger sponge \quad \text{Euler characteristic } = ?
Structure of this Talk

1. (Oriented) adjacency graphs
   generalization of adjacencies in digital images
   2D grid point model

   unlabeled nodes
   symmetric and irreflexive adjacency relation

2. Incidence Pseudographs
   generalization of Euclidean complexes (poset topologies)
   $nD$ grid cell (incidence) model, for $n \geq 1$

   nodes labeled by dimension $0, 1, \ldots, n$
   symmetric and reflexive incidence relation
Aleksandrov-Hopf 1935

Khalimsky 1986

homeomorphic poset topologies for 2D

Kovalevsky 1989

\[ m \times n \text{ picture grid and } (m+1) \times (n+1) \text{ frontier grid} \]

``maximum-label rule''

Voss 1993

incidence relations in \( n \)-D grid
ORIENTED ADJACENCY GRAPHS \([S, A, \xi]\)

[Voss/Klette 1986]

countable set \(S\), adjacency relation \(A\) (irreflexive, symmetric)
local cyclic order \(\xi\)

\begin{itemize}
  \item \(A(p)\) is finite for any node \(p\) in \(S\)
  \item \([S, A]\) is a connected undirected graph (finite or infinite)
  \item any finite subset \(M\) of \(S\) has at most one infinite complementary component
  \item any directed edge generates a periodic path (a \textit{circuit})
\end{itemize}

note: generalizes oriented 2D tilings or 2D combinatorial maps
local circular order $\xi(p) = [a, b, c, d, e]$
of all points in the adjacency set $A(p)$
the undirected graph needs not to be planar (as in 2D tilings) and not to be finite (as in 2D combinatorial maps)

- LEFT: numberings of local circular orders
- RIGHT: drawing convention: clockwise order of outgoing edges

\[ \xi(a) = [c, b, d] \quad \xi(b) = [e, d, a] \quad \xi(c) = [d, a, e] \]
\[ \xi(d) = [c, e, b, a] \quad \xi(e) = [b, d, c] \]

directed edge \((d, a)\) generates circuit \(\xi(d, a) = <d, a, c, e, b>\) 
\(\xi(a, d) = <a, d, c>, \ldots\)
not an oriented adjacency graph (infinite paths)

cycle = generated circuit
oriented adjacency graph \([S, A, \xi]\): 
\[\sum_{p \in S} \nu(p) = 2\alpha_1 \quad \sum_{\rho} \lambda(\rho) = 2\alpha_1\]

\[\begin{align*}
\alpha_0 &= \text{card} (S) \\
\alpha_1 &= \text{card} (A) \\
\nu(p) &= \text{card}(A(p)) \\
\lambda(\rho) &= \text{length of cycle } \rho \\
\alpha_2 &= \# \text{ cycles}
\end{align*}\]

\[\begin{align*}
\alpha_0 &= 6 \\
\alpha_1 &= 9 \\
\alpha_2 &= 3 \\
\chi &= 0
\end{align*}\]

\[\begin{align*}
\alpha_0 &= 5 \\
\alpha_1 &= 10 \\
\alpha_2 &= 3 \\
\chi &= -2
\end{align*}\]  

Euler characteristic \(\chi = \alpha_0 - \alpha_1 + \alpha_2\)
combinatorial maps: each directed edge = two darts

Heffter 1895, Edmonds 1960, Tutte 1963

anti-clockwise

\[ \alpha = (1, -1)(2, -2)(3, -3)(4, -4)(5, -5)(6, -6)(7, -7)(8, -8)(9, -9) \]
\[ \sigma = (5, 3, 1)(-4, 6, -9)(-7, 8, -2)(-1, 2, 4)(-6, 5, 7)(-8, 9, -3) \]
\[ \varphi = \sigma \circ \alpha = (1, 2, -7, -6, -9, -3)(-1, 5, 7, 8, 9, -4)(-2, 4, 6, -5, 3, -8) \]

clockwise

\[ \sigma = (2, 10, -7, -1)(-2, 3, -9, -8)(-3, 4, -6, -10)(9, -4, 5, 7)(8, 6, -5, 1) \]
\[ \varphi = \sigma \circ \alpha = (-2, 10, -3, -9, -4, -6, -5, 7, -1, 8)(1, 2, 3, 4, 5)(6, -10, -7, 9, -8) \]
Theorem on Euler Characteristics

\[ \chi \leq 2 \quad \text{for any finite oriented adjacency graph, and even} \]

\[ \text{[Voss/Klette 1986]} \]

finite: **planar** iff \( \chi = 2 \)

infinite: **planar** iff any non-empty finite connected subgraph planar
8-adjacency fails basic topological rules:

\[ \alpha_0, \alpha_1, \alpha_2 = (4,6,2) \quad (6,11,3) \quad (8,16,4) \quad (12,26,6) \quad (2n,5n-4,n) \]

\[ \chi : \quad 0 \quad -2 \quad -4 \quad -8 \quad -2(n-2) \]

\[ \alpha_0, \alpha_1, \alpha_2 = (8,16,4)(12,29,7) \quad (16,42,10) \quad (24,68,16) \quad (4n,13n-10,3n-2) \quad -2(3n-4) \]

\[ \chi : \quad -4 \quad -10 \quad -16 \quad -28 \]
$M \subseteq S$ generates **constrained local circular orders** $\xi_M(a) = [b,c,d]$

$<b,a,c>$ is cycle in $[S, A, \xi]$ : **atomic cycle**

$<a,b,c,d>$ and $<d,c,a>$ are not cycles in $[S, A, \xi]$ : **border cycles**
\([S, A, \xi_M]:\) 8 atomic cycles
2 border cycles
undirected invalid edges assigned to a border cycle
Note: Euler characteristic of graphs (with +1 for the `"exterior"`)
Let \( [S, A, \xi] \) be a planar oriented adjacency graph.

Let \( M \) be a non-empty finite connected proper subset of \( S \).

By deleting all undirected invalid edges assigned to one of the border cycles of \( M \), \( [S, A, \xi] \) splits into at least two non-connected substructures.

Note: partitioning by tracing border cycles is the aim in image segmentation, and this theorem is a theoretical justification for this approach.

**Boundary** = all assigned invalid edges
the uniquely defined **outer border cycle** of $M$ separates one (infinite) *background component* and a finite number of *improper holes* from $M$

any **inner border cycle** of $M$ separates a finite number of *proper holes* from $M$
proper hole
improper hole
background component
atomic cycles
border cycles
proper hole
improper hole
background component
proper hole
improper hole
tiling = planar oriented (finite or infinite) adjacency graph
regular tiling = \( v(p) \) and \( \lambda(\rho) \) are constants

left: \( v = 3, \lambda = 6, \alpha_0 = 49, \alpha_1 = 59, \alpha_2 = 12, \ l = 52, \ k = 29, \ f = 11 \)
middle: \( v = 4, \lambda = 4, \alpha_0 = 23, \alpha_1 = 30, \alpha_2 = 9, \ l = 28, \ k = 32, \ f = 8 \)
right: \( v = 6, \lambda = 3, \alpha_0 = 18, \alpha_1 = 32, \alpha_2 = 16, \ l = 19, \ k = 44, \ f = 15 \)

\[ l = \text{length of (outer) border cycle} \]
\[ k = \# \text{invalid edges assigned to border cycle} \]
\[ f = \alpha_2 - 1 \]
We have (for any regular tiling or outer border cycle):

\[ k = v + \frac{\nu}{\lambda} l \]

Examples previous slide:

\[ 29 = 3 + \frac{3}{6} \times 5 \quad \text{“hexagonal grid”} \]
\[ 32 = 4 + \frac{4}{4} \times 28 \quad \text{“orthogonal grid”} \]
\[ 44 = 6 + \frac{6}{3} \times 19 \quad \text{“triagonal grid”} \]
What about inner border cycles?
one inner, one outer

two inner, two outer
[Voss 1986]: total curvature theorem

\[ M = \text{finite connected subset of an infinite regular tiling } S_{v,\lambda} \]

for any border cycle: \[ \pm 1 = \frac{k}{v} - \frac{l}{\lambda} \]

**outer border cycle:** defined by positive sign

**inner border cycle:** defined by negative sign

(Provides an efficient test after border tracing: inner or outer?)
[Voss 1986]: Generalized Pick Theorem

\( M = \) finite connected subset of an infinite regular tiling \( S_{v,\lambda} \) without proper holes, then

for the (outer) border cycle: \( \alpha_0 = \frac{v}{\lambda} f + l/2 + 1 \)

\( M = \) finite connected subset of an infinite regular tiling \( S_{v,\lambda} \) then

for any inner border cycle: \( \alpha_0 = \frac{v}{\lambda} f - l/2 + 1 \)

(see Georg Pick’s area theorem \( A = i + b/2 - 1 \) from 1899 for the orthogonal grid)
\[ \alpha_0 = 22 \quad f = 10 \quad l = 22 \]
\[ 22 = 10 + \frac{22}{2} + 1 \]

**outer border cycle**: set is connected, no proper hole, but one improper hole

\[ \alpha_0 = 5 \quad f = 12 \quad l = 16 \]
\[ 5 = 12 - \frac{16}{2} + 1 \]

**inner border cycle** defining two proper holes
3D  [Yip/Klette 2002]: angles in an isothetic connected polyhedron

\[(H_A + H_G) - (H_C + H_E) - 2(H_{D1} + H_{D2}) = 8\]

\(H_A, H_c, H_{D1}, H_{D2}, H_E, H_G\)

= \# A, C, D1, D2, E, G angles of polyhedron H, respectively

\(H_A - H_G < 0\) iff inner border

\(H_A - H_G > 0\) iff outer border

geometric proof, not just graphs
Generalization of oriented adjacency graphs to 3D?

(there are 3D combinatorial maps)

Alternative option:

spatial subdivisions (combinatorial topology)
R. Descartes (Cartesius): one convex polyhedron with $\alpha_0$, $\alpha_1$, $\alpha_2$

L. Euler

\[
\alpha_0 - \alpha_1 + \alpha_2 = 2
\]

first proof: 1794 by A.-M. Legendre

A. Cauchy 1813: $D$ polyhedral cells within one convex polyhedron

\[
\alpha_0 - \alpha_1 + \alpha_2 = D + 1
\]

A.-J. Lhuilier 1812: $b$ `bubbles’, $t$ `tunnels’ and $p$ `entrances/exits’

\[
\alpha_0 - \alpha_1 + \alpha_2 = 2 (b - t + 1) + p
\]

\textbf{wrong}
Note: 0-cells (vertices) and 1-cells (edges) shown in this figure by polygons (for graphical reasons)
incidence structure $G = [S, I, \text{dim}]$

- countable set $S$
- incidence relation $I$ on $S$ (reflexive and symmetric)
- function $\text{dim} : S \rightarrow \{0, 1, ..., m\}$
  defining classes of $i$-nodes $c$ by $\text{dim}(c) = i$

$\text{ind}(G) = \text{maximum value of } \text{dim}(c)$

**principal node** $c$ if $\text{dim}(c) = \text{ind}(G)$

all principal nodes = core of $G$

**marginal node** otherwise

$c_1 A_i c_2$ iff $c_1 \neq c_2$
  and ex. $i$-node $c$
  $c_1 \in I(c) \land c \in I(c_2)$

- $i$-adjacent, $i$-connected, $i$-path,
- $i$-components, complementary $i$-components

**adjacent** iff ex. $i$ and $i$-adjacent
let $m = \text{ind}(G)$

$G = [S, I, \text{dim}]$ is incidence pseudograph iff

$I_1$: $I(c)$ always finite

$I_2$: set of principal nodes in $G$ is $(m-1)$-connected

$I_3$: $M \subseteq S$ finite: at most one infinite complementary $(m-1)$-component of principal node

$I_4$: $c' \in I(c)$ and $c \neq c'$ then $\text{dim}(c) \neq \text{dim}(c')$

$I_5$: $\text{dim}(c) < m$ then $c$ incident with at least one $m$-node

[Klette 2003]
“Repair of incompleteness”

**BLUE**

all incident 2-cells of those 1-cells are in the region

➢ take them in as well

**RED**

now, all incident 1- and 2-cells of those 0-cells are in the region

➢ take them in as well
completion $M^+$ of $M$

(a) $M \subseteq M^+$

(b) if $c' \in M^+$ for all $c' \in I(c)$ with $\dim(c') > \dim(c)$
then $c \in M^+$

$M$ complete iff $M = M^+$

component $C$ of set $M$

= nonempty core, core connected, $C$ is complete
   every node in $C$ is incident with a principal node in $C$
closed

neither closed nor open
region \( M = \) finite component
\[
\begin{align*}
c & \in M & \text{inner node} & \text{iff} & I(c) \subseteq M \\
\text{otherwise:} & & \text{border node}
\end{align*}
\]
inner set \( M^\triangledown \) and border \( \delta M \)

POSET TOPOLOGY [Aleksandrov 1953]
\[
\begin{align*}
M & \subseteq S & \text{closed} \\
& & c \in M & \text{and} & c' \in I(c) & \text{with} & \dim(c') < \dim(c) \\
& & \text{then} & c' \in M
\end{align*}
\]
\[
\begin{align*}
M & \subseteq S & \text{open} & \text{iff} & \overline{M} = S \setminus M & \text{is closed}
\end{align*}
\]
(note: if closed or open, then complete)

frontier of a set = border of its closure
Binary pictures: 4-adjacency (open sets) and 8-adjacency (closed sets)
Proposed dual use: [Duda/Hart/Munson 1967], [Rosenfeld 1970]

Multi-level pictures: apply maximum-label rule

This is a **general and simple approach** for topologically sound 2D or 3D image analysis (without “Photoshop-type surprises”).
frontier

border (of open set)
incidence counts

\[ a_{ij}(c) = \begin{cases} 
\text{card}\{c' \in S : \text{dim}(c') = j \land \{c, c'\} \in I\} & \text{if } i = \text{dim}(c) \\
0 & \text{otherwise}
\end{cases} \]

MATCHING THEOREM

\[ \sum_{c \in S} a_{ij}(c) = \sum_{c \in S} a_{ji}(c) \quad \text{for} \quad 0 \leq i, j \leq n \]

incidence grid = regular incidence pseudograph

\[ a_{ij}(c) = a_{ij} \quad \text{for all } c \in S, \text{ with } \text{dim}(c) = i \]
Euler characteristic (Def.)

\[ \chi(M) = \sum_{i=0}^{n} (-1)^i \alpha_i \]

from Matching Theorem:

\[ \frac{\chi(M)}{\alpha_k} = \sum_{i=0}^{n} (-1)^i \frac{a_{ki}}{a_{ik}} \]

for finite n-dim. incidence grids

\[ \chi(M) = 2 \]
a node $c \in S$ is **invalid** w.r.t. $M$ iff

$c \not\in M$ but there is $c' \in M$ with $c' \in I(c)$

**boundary** of $M = \text{set of all invalid nodes}$

boundary of a closed region ≠ its (border = frontier)
boundary of an open region = border (frontier) of its closure
boundary counts

\[ b_{ij}^M (c) = \begin{cases} 
\text{card}\{c' \in I(c) : \text{dim}(c') = j \land c' \text{ invalid}\} \\
0 & \text{otherwise}
\end{cases} \]

if \( i = \text{dim}(c) \) and \( c \in M \)

\[ b_{ij}^M = \sum_{c \in S} b_{ij}^M (c) \]

total boundary counts for \( M \)
Basic formula for incidence counts:

\[
a_{ij} = \begin{cases} 
2^{j-i} \binom{n-i}{n-j} & \text{if } i < j \\
1 & \text{if } i = j \\
2^{i-j} \binom{i}{j} & \text{if } i > j 
\end{cases}
\]

n-dimensional incidence grids

independently shown by

B. Rosenfeld, I. Jaglom 1971 (Russia)
R. Klette 1972 (Germany)
H. S. M. Coxeter 1973 (Canada)
Corollary: \[
\sum_{j=0}^{n} (-1)^j \frac{a_{ij}}{a_{ji}} = 0
\] [Voss 1993]

**REGION MATCHING THEOREM**

*M* open or closed region in *n*-dim. IG

\[\alpha_i a_{ij} - b_{ij} = \alpha_j a_{ji} \quad \text{for} \quad i < j \quad \text{if closed, or for} \quad i > j \quad \text{if open}\]

\[\alpha_i a_{ij} = \alpha_j a_{ji} \quad \text{for} \quad i = j\]

\[\alpha_i a_{ij} + b_{ji} = \alpha_j a_{ji} \quad \text{for} \quad i > j \quad \text{if closed, or for} \quad i < j \quad \text{if open}\]

[Voss 1993] for open regions

[Klette 2003] for closed regions
Let $M$ be a finite union of pairwise disjoint closed (or pairwise disjoint open) regions.

The Euler characteristic of $M$ is

$$\chi(M) = \frac{1}{2n} \cdot \sum_{i=1}^{n} (-1)^{i+1} b_{i,i-1}$$

for open regions

and

$$\chi(M) = \frac{1}{2n} \cdot \sum_{i=0}^{n-1} (-1)^{i+1} b_{i,i+1}$$

for closed regions

[Voss 1993] for open regions
[Klette 2003] for closed regions
Obviously, Poincare-formula and known Euler characteristic do not provide Betti-numbers

[Li/Klette 2006] (see IMA Preprint 2113)

calculation of $\beta_1$ (first Betti number = number of tunnels)

(in subquadratic time !)

for any finite 2-connected set of 3-cells
Euler characteristic known. Let $m(t)$ = number of tunnels at iteration $t$
$m(0) = 9, m(1) = 209, \ldots$ (good for testing, because $\beta_0 = 1, \beta_2 = 0$)