Paley Wiener Theorems and Point Clouds in Scattering and Vision: Announcement of Results

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ABSTRACT

A: We announce recent results on point energies, scar defects, separation and mesh norm for optimal $N \geq 1$ arrangements of points on a class of $d \geq 1$-dimensional compact sets embedded in $\mathbb{R}^n$, $n \geq 1$, which interact through a Riesz potential. We announce (1) local Paley Wiener theorems for a class of functions $f : \mathbb{C}^n \to \mathbb{C}$, $n = 2, 3$, whose restrictions to the $n-1$ dimensional unit sphere, are analytic. (2) We then apply our results in (1) to determine tight bounds for the support of a source or scattering potential from far field data.

Motivation

In what follows we are interested in two fundamental problems of vision and scattering which deal with the following tasks: Task 1: **Learning** mathematical methods in order to study meaningful descriptions of a finite number of given discrete objects. Task 2: **Learning** mathematical methods in order to study complicated structure by way of local methods or discretization. Regarding Task 1, we study point energies, scar defects, separation and mesh norm for optimal $N \geq 1$ arrangements of points on a class of d-dimensional compact sets embedded in $\mathbb{R}^n$, $n \geq 1$, which interact through a Riesz potential. Regarding Task 2, we study local Paley Wiener theorems which determine bounds for supporting scattering networks assuming only smoothness of complex valued local sources f with some specified growth for large argument

Point Clouds-Good Distribution

The problem of uniformly distributing points on spheres (more generally, on compact sets in $\mathbb{R}^n$, $n \geq 1$) is an interesting and difficult problem. It is folklore, that such problems were discussed already by Carl Friedrich Gauss in his famous Disqvistiones arithmaticae, although it is most likely that similar problems appeared in mathematical writings even before that time.
For $d \geq 1$, let S^d denote the d-dimensional unit sphere in $\mathbb{R}^{d+1}$, given by

$$x_1^2 + \cdots + x_{d+1}^2 = 1.$$

For $d = 1$, the problem is reduced to uniformly distributing N points on a circle, and equidistant points provide an obvious answer. For $d \geq 2$, the problem becomes much more difficult; in fact, there are numerous criteria for uniformity, resulting in different optimal configurations on the sphere. Many constructions of "well-distributed" point sets have been given in the literature. These include constructions of generalized spiral points, low-discrepancy point sets in the unit cube, which can be transformed via standard parametrizations, constructions given by integer solutions of the equation $x_1^2 + \cdots + x_{d+1}^2 = N$ projected onto the sphere, rotations of certain subgroups applied to points on the sphere, finite field constructions of point sets based on finite fields and associated combinatorial designs.

Electrostatic Potentials and Riesz Points

We are interested in studying certain arrangements of $N \geq 1$ points on a class of d-dimensional compact sets A embedded in $\mathbb{R}^n$. We assume that these points interact through a power law (Riesz) potential $V = r^{-s}$, where $s > 0$ and r is the Euclidean distance in $\mathbb{R}^n$.
For a compact set $A \subset \mathbb{R}^n$, $s > 0$, and a set $\omega_N = \{x_1, \ldots, x_N\}$ of distinct points on A, the discrete *Riesz s-energy* associated with ω_N is given by

$$E_s(A, \omega_N) := \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-s}.$$

Let $\omega_N^* := \{x_1^*, \ldots, x_N^*\} \subset A$ be a configuration, for which $E_s(A, \omega_N)$ attains its minimal value; that is,

$$\mathcal{E}_s(A, N) := \min_{\omega_N \subset A} E_s(A, \omega_N) = E_s(A, \omega_N^*).$$

We shall call such minimizing configurations *s-extremal configurations*. It is well-known that, in general, s-extremal configurations are not always unique. For example, in the case of S^d, they are invariant under rotations. A natural physical interpretation of minimal energy problem on the sphere is the electron problem, which asks for distributions of electrons in stable equilibrium.

Electrostatic Potentials and Riesz Points II

Natural questions that arise in studying the discrete Riesz energy are:

(1) What is the asymptotic behavior of $\mathcal{E}_s(A, N)$, as $N \to \infty$?

(2) How are s-extremal configurations distributed on A for large N?

The answers to these questions essentially depend on the relation between s and the Hausdorff dimension $d_H(A)$ of A. We demonstrate this fact with the following two examples.

Continuing Electrostatic Potentials and Riesz Points II

Example: Interval The interval $[-1, 1]$, $d_H([-1, 1]) = 1$: The case $s = 1$ is the critical value in the sense that s-extremal configurations are distributed on $[-1, 1]$ differently for $s < 1$ and $s \geq 1$. Indeed, for $0 < s < 1$, the limiting distribution of s-extremal configurations has an arcsine-type density and, for $s \geq 1$, the limiting distribution is the uniform distribution on $[-1, 1]$. Concerning the minimal energies, they again behave differently for $s < 1$, $s = 1$, and $s > 1$. With $e_s := [\sqrt{\pi}\Gamma(1 + s/2)] / [\cos(\pi s/2)\Gamma((1 + s)/2)]$,

$$\mathcal{E}_s([-1, 1], N) \sim \begin{cases} (1/2)N^2 e_s\,, & s < 1, \\ (1/2)N^2 \ln N, & s = 1, \\ (1/2)^s \zeta(s) e(s) N^{1+s}, & s > 1, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta function.

Example: The unit sphere The unit sphere S^d, $d_H(S^d) = d$: Here again, there are three cases to consider: $s < d$, $s = d$, and $s > d$. In all cases, the limiting distribution of s-extremal configurations is given by the normalized area measure σ_d on S^d, which is natural due to rotation invariance, but the asymptotic behavior of $\mathcal{E}_s(S^d, N)$ is quite different. With $\tau_{s,d}(N)$ denoting N^2 if $s < d$, $N^2 \ln N$ if $s = d$, and $N^{1+s/d}$ if $s > d$, the limit $\lim_{N \to \infty} \mathcal{E}_s(S^d, N)/\tau_{s,d}(N)$ exists and is known in the first two cases.

Electrostatic Potentials and Riesz Points III

The dependence of the distribution of s-extremal configurations over A and the asymptotics for minimal discrete s-energy on s can be explained using potential theory. Indeed, for a probability Borel measure ν on A, its s-energy integral is defined to be

$$I_s(A, \nu) := \int_{A \times A} |x - y|^{-s} d\nu(x) d\nu(y),$$

which can be finite or infinite. For a set $\omega_N = \{x_1, \ldots, x_N\} \subset A$, let

$$\nu^{\omega_N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}$$

denote the normalized counting measure of ω_N (so that $\nu^{\omega_N}(A) = 1$). Then the discrete Riesz s-energy, associated with ω_N, can be written as

$$E_s(A, \omega_N) = \frac{N^2}{2} \int_{x \neq y} |x - y|^{-s} d\nu^{\omega_N}(x) d\nu^{\omega_N}(y).$$

where the integral represents a discrete analog of the s-energy integral.
If $s < d_H(A)$, then the energy integral is minimized uniquely by the *equilibrium measure* ν_s^A. On the other hand, the normalized counting measure $\nu^{\omega_N^*}$ of an s-extremal configuration minimizes the discrete energy integral over all sets ω_N on A. Thus, one can reasonably expect that, for N large, $\nu^{\omega_N^*}$ is "close" to ν_s^A and, therefore, the minimal discrete s-energy $\mathcal{E}_s(A, N)$ is close to $(1/2)N^2 I_s(A, \nu_s^A)$.
If $s \geq d_H(A)$, then the energy integral diverges for every measure ν. Thus, $\mathcal{E}_s(A, N)$ must grow faster than N^2. Concerning the distribution of s-extremal points over A, the interactions are strong enough to force points to stay away from each other as far as possible since the closest neighbors are now dominating. So, s-extremal points distribute themselves over A in an equally spaced manner.

Point Energies, Separation, and Mesh Norm for Optimal Riesz Points on d-Rectifiable Sets

We say that a set A belongs to the class $\mathcal{A}^d$ if, for some $n \geq d$, $A \subset \mathbb{R}^n$ and

(1) $H^d(A) > 0$ and

(2) A is a finite union of bi-Lipschitz images of compact sets in $\mathbb{R}^d$, that is

$$A = \bigcup_{i=1}^{m} \phi_i(K_i),$$

where each $K_i \subset \mathbb{R}^d$ is compact and $\phi_i : K_i \to \mathbb{R}^n$ is bi-Lipschitz on K_i, $i = 1, \ldots, m$.

Here and throughout, $H^d(\cdot)$ denotes the d-dimensional Hausdorff measure in $\mathbb{R}^n$.
For a collection $\omega_N = \{x_1, \ldots, x_N\}$ of distinct points on a set $A \subset \mathbb{R}^n$, let

$$\delta(A, \omega_N) := \min_{i \neq j} |x_i - x_j|, \qquad \rho(A, \omega_N) := \max_{x \in A} \min_{1 \leq j \leq N} |x - x_j|.$$

The quantity $\delta(A, \omega_N)$ is called the *separation radius* and gives the minimal distance between points in ω_N, while the *mesh norm* $\rho(A, \omega_N)$ means the maximal radius of a "cap" on A, which does not contain points from ω_N. We also define the point energies of the points in ω_N by

$$E_{j,s}(A, \omega_N) := \sum_{i \neq j} |x_j - x_i|^{-s}, \quad j = 1, \ldots, N.$$

Point Energies, Separation, and Mesh Norm for Optimal Riesz Points on d-Rectifiable Sets II

Throughout, $C, c > 0$ are constants, different in general, independent of N but depending on A, s, d.

We have: **Theorem: Damelin-Maymeskul** Let $A \in \mathcal{A}^d$ and $s > d$. Then, for all $1 \leq j \leq N$,

$$E_{j,s}(A, \omega_N^*) \leq CN^{s/d}.$$

Corollary: Damelin-Maymeskul For $A \in \mathcal{A}^d$, $s > d$, and any s-extremal configuration ω_N^* on A,

$$\delta(A, \omega_N^*) \geq cN^{-1/d}.$$

We note that this is the best possible lower estimate on the separation radius. Concerning the mesh norm $\rho(A, \omega_n^*)$ of s-extremal configurations, we have:

Theorem: Damelin-Maymeskul Let $A \in \mathcal{A}^d$, $s > d$, and let ω_N^* be an s-extremal configuration on A. Then

$$\rho(A, \omega_N^*) \leq CN^{-1/d}.$$

Corollary: Damelin-Maymeskul For $s > d$ and any s-extremal configuration ω_N^* on $A \in \mathcal{A}^d$,

$$c \leq \frac{\max_{1 \leq j \leq N} E_{j,s}(A, \omega_N^*)}{\min_{1 \leq j \leq N} E_{j,s}(A, \omega_N^*)} \leq C.$$

Thus, for $A \in \mathcal{A}^d$ and $s > d$, all point energies in an s-extremal configuration are asymptotically of the same order, as $N \to \infty$.

Local Paley Wiener Theorems I

We consider the mathematical problem of determining the support of a compactly supported, square integrable function $F : \mathbb{R}^n \to \mathbb{C}$, $n = 2, 3$ from far field data given by a function $f : \mathbb{C}^n \to \mathbb{R}$. Here and in what follows, $\mathbb{R}^n$ will denote $n \geq 1$ dimensional Euclidean space consisting of all n tuples of real numbers. $\mathbb{C}^n$ will be defined analogously by way of complex numbers. Typically, in inverse scattering theory, the function f takes the form of a *radiation pattern* in the case of an inverse source problem and the *scattering amplitude* in the case of an inverse scattering problem. The function F is a *source* or *scattering potential*. In network problems, f may be taken as a far field *network*. Typically, in the applications we have in mind, the function f is smooth globally with some specified growth at infinity and methods to recover F from f are referred to as *Paley Wiener* theorems [PW] in the literature. Although useful in global network problems, the global smoothness typically assumed falls short for applications to local problems where all that might be known is information on local nodes or local data. In what follows, we will consider what we call *generalised Paley Wiener* theorems [GPW], which determine bounds for supporting functions F assuming only smoothness of local sources f with specified growth at infinity. Throughout, we shall say that a function $f : \mathbb{C}^n \to \mathbb{C}$ is of exponential type $a > 0$, if uniformly for every $z \in \mathbb{C}$,

$$|f(z)| = O\left(e^{a|z|}\right)$$

Finally, given $f : \mathbb{C}^n \to \mathbb{C}$, by f_A, we will always mean the restriction of f to a proper subset A of $\mathbb{C}^n$.

The Crux of the Matter

Suppose, approri, that we know that for every vector $\mathbf{z} \in S^1$, the function f is given by an integral

$$f(\mathbf{z}) = \int_\tau F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n(x)$$

for some bounded set $\tau \in \mathbb{R}^n$ and bounded, compactly supported $F : \mathbb{R}^n \to \mathbb{C}$ with support in τ. In practice, $\mathbf{z}$ is the unit vector in the direction where f is measured and k is an absolute real constant. The "inverse support" problem, studies the problem of determining bounds for the support of the set τ, assuming the model above. For many applications, the above assumption on the given f is strong and often not obvious from the given data. Our aims are twofold. Firstly, we provide necessary and sufficient conditions so that the above assumption holds. In particular, we will show that provided the restriction of f to the unit sphere S^{n-1} is analytic and f satisfies a growth condition of exponential type, then the above assumption holds for some compact set $\tau \in \mathbb{R}^n$ and bounded $F : \mathbb{R}^n \to \mathbb{C}$ with compact support in τ. We also describe the smallest supports sets for which our results are best possible. Results of this type are typically known in the literature as Paley Wiener Theorems [PW]. We announce generalizations of [PW] in that we do not assume that f is entire nor that its restriction to $\mathbb{R}^n$ is square integrable. The later assumptions are basic in [PW]. Indeed, we show that (1) a growth condition of f of exponential type and (2) an assumption that the restriction of f to S^{n-1} is analytic are enough to deduce the above. In [PW], the function F obtained is square integrable on $\mathbb{R}^n$, compactly supported but not necessarily bounded.

Continuing The Crux of the Matter

In the above sense, we are using smoothness properties of f only on S^{n-1} to establish the Fourier transform relationship above. Square integrability and analyticity of the restriction of f off the given sphere are not required, nor used in our results.
The new theorems developed here, which we will call generalized Paley Wiener Theorems [GPW], are inherently different from the conventional [PW] as they take into account so called *non-radiating sources*, which play an important role in inverse source and scattering problems. Our method of proof also differs considerably from the proof of conventional [PW] in that the mapping $f \to F$ cannot be investigated via a Fourier integral. Indeed, in place of a Fourier integral , we employ *angular spectra* or *plane wave* expansions of outgoing wave solutions to the Helmholtz equation which replace the square integrabilty assumptions of the restriction of f to $\mathbb{R}^n$.

Classical PW in one variable

Theorem: [PW1] A function $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type with its restriction $f_{\mathbb{R}} \in L_2(\mathbb{R})$ iff

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_a^b F(x) e^{-ixz} dx, \; z \in \mathbb{C}.$$

for some $F \in L_2(\mathbb{R})$ with support in some $[a, b]$, $a \neq b$, $a, b \in \mathbb{R}$. Moreover, $[a, b]$ is the smallest set containing the support of F and

$$a = -\limsup_{y \to \infty} \frac{\log|f(-iy)|}{y}, \; b = \limsup_{y \to \infty} \frac{\log|f(iy)|}{y}.$$

Local PW: Announcement of Results

Theorem: Damelin, Devaney, Luke [GPW1] Let $n = 2, 3$, $f : \mathbb{C}^n \to \mathbb{C}^n$ be a function whose restiction $f_{S^{n-1}}$ is analytic and let k, a be positive numbers. Then the following are equivalent:

(a) Given any vector $\mathbf{z} \in S^{n-1}$,

$$f(z) = \int_{B_a} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} d^n x,$$

where F is a bounded function supported in the closed ball $B_a \in \mathbb{R}^n$ with centre 0 and radius a.

(b) For any fixed real vector $\mathbf{u} \in S^{n-1}$.

$$|f(\mathbf{z})| = O\left(e^{ka(\mathbf{u}.\Im(\mathbf{z}))}\right), \; \mathbf{u}.\Im(\mathbf{z}) \to \infty, \; \mathbf{z} \in \mathbb{C}^n.$$

Note that in the statement of [GPW1], compared to that of [PW1], we use smoothness properties of f only on S^{n-1} to establish our result and square integrability and analyticity of the restriction of f off the given sphere is not required nor used in our results. The function F obtained is both bounded and compactly supported in $\mathbb{R}^n$.
[GPW1] stated above does not, in general, yield the smallest support volume for the function F. However, the smallest *convex* support volume for this function can be obtained in analogy with [PW2]. This is contained in

Theorem: Damelin, Devaney, Luke [GPW2] Let $n = 2, 3$, k a positive constant and $f : \mathbb{C}^n \to \mathbb{C}^n$ be a function whose restiction $f_{S^{n-1}}$ is analytic. Then the following two statements are equivalent:

(a) For any fixed real vector $\mathbf{u} \in S^{n-1}$,

$$|f(\mathbf{z})| = O\left(e^{k\rho(\mathbf{u})(\mathbf{u}.\Im(\mathbf{z}))}\right), \; \mathbf{u}.\Im(\mathbf{z}) \to \infty, \; \mathbf{z} \in \mathbb{C}^n.$$

(b) For every vector $\mathbf{z} \in S^{n-1}$,

$$f(\mathbf{z}) = \int_{\tau_c} F(\mathbf{x}) e^{-ik(\mathbf{z}.\mathbf{x})} dx$$

where F is a bounded function supported in a convex region τ_c having support function $\rho(\mathbf{u}, \tau_c)$ for any vector $\mathbf{u} \in S^{n-1}$.

All papers below are on the webpage of S. B. Damelin: http://www.ima.umn.edu/~ damelin

References

[(1)] . B. Damelin and V. Maymeskul, *On Point Energies, Separation Radius and Mesh Norm for s-Extremal Configurations on Compact Sets in R^n*, Journal of Complexity, Volume 21(6)(2005), pp 845-863.

[(2)] . B. Damelin and V. Maymeskul, *On Separation Radius and Mesh Norm for s-Extremal Configurations on Compact Sets in R^n-II*, submitted.

[(3)] . B. Damelin and V. Maymeskul, *Minimal Discrete Energy Problems and Numerical Integration on Compact Sets in Euclidean Spaces*, Algorithms for Approximation, Springer, to appear.

[(4)] . B. Damelin, Y. Ma, O. Masoud and N. Papanikolopoulos, *Automated Activity Recognition via Supervised Manifold Learning, Point Clouds and Diffusion Mappings*, submitted.

[(5)] . J. Devaney, S. B. Damelin and R. Luke, *A Paley Wiener theorem for analytic functions on the unit sphere and its applications to inverse source and scattering problems*, Manuscript.