Shape space smoothing splines for planar landmark data

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Motivation

In many disciplines it is of interest to study the shape of an object, where the effects of translation, rotation and scale can be ignored [1]. We focus on the case where a time series of shapes is available, and we wish to fit a smooth curve to the data, describing the trend in the shapes over time. An example of this type of application includes the study of human movement, where markers are observed on a volunteer who performs a task over time.

FIGURE 1: The human movement recording equipment

Since the shape space is not a flat Euclidean space, we cannot simply apply the standard methods of linear regression and spline fitting directly to the shape data. If the sample has little variability, the problem can be transfered to a tangent space. However, we shall be interested in situations where the shape change is large, and then the tangent space approximation is no longer appropriate.

Pre-shape and shape space

For a configuration in the plane with k labelled vertices, its pre-shape is what is left after the effects of translation and scaling are removed, and this can be represented by a complex row vector $z = (z_1, \ldots, z_{k-1})$ with $\| z \|^2 = \sum |z_j|^2 = 1$. The pre-shape space is

$$\mathscr{S}_2^k = \{z \in \mathbb{C}^{k-1} : \| z \| = 1\}.$$

The shape space Σ_2^k of configurations with k labelled vertices in the plane is then the quotient space of $\mathscr{S}_2^k$ by the rotation group $SO(2)$ and we shall use $[z]$ to denote the shape of z. For any given point z in $\mathscr{S}_2^k$, the tangent space of $\mathscr{S}_2^k$ at z is

$$\tau_z(\mathscr{S}_2^k) = \{v \in \mathbb{C}^{k-1} : \Re(z\, v^*) = 0\},$$

where v^* denotes the transposed complex conjugate of v and $\Re(\cdot)$ denotes the real part of a complex number. The horizontal subspace of $\tau_z(\mathscr{S}_2^k)$, with respect to the quotient map from $\mathscr{S}_2^k$ to Σ_2^k, can be expressed as

$$\mathscr{H}_z(\mathscr{S}_2^k) = \{v \in \mathbb{C}^{k-1} : z\, v^* = 0\}. \qquad (1)$$

The horizontal subspace is isometric to the tangent space of Σ_2^k at the shape of z, and more straightforward to work with than directly on the shape tangent space.

In the triangle case ($k = 3$), since Σ_2^3 is isometric with the 2-dimensional sphere of radius 1/2, we can use a highly novel method [2] for fitting *spherical smoothing splines* to spherical data based on the techniques of *unrolling and unwrapping* onto an appropriate tangent space. We consider the extension of this method for $k > 3$ points.

Unrolling and unwrapping

Let us consider a continuous and piecewise differentiable path $\Gamma(t)$, $t \in [t_0, t_n]$, in Σ_2^k. If we think of this path as being marked by wet ink, then its *unrolling* onto $\mathscr{T} = \tau_{\Gamma(t_0)}(\Sigma_2^k)$, the tangent space to Σ_2^k at the starting point $\Gamma(t_0)$, is the trace $\Gamma^\dagger$ left on $\mathscr{T}$ after $\mathscr{T}$ has been rolled without slipping or twisting along Γ such that, at time t, $\mathscr{T}$ touches Σ_2^k at $\Gamma(t)$. The point, in $\mathscr{T}$, at which $\mathscr{T}$ touches Σ_2^k at time t is $\Gamma(t)^\dagger$. The *unwrapping* at time $t \in [t_0, t_n]$, with respect to the base path Γ, of a shape $[w] \in \Sigma_2^k$ is the point $[w]^\dagger$ in $\mathscr{T}$ such that the tangent vector $[w]^\dagger - \Gamma(t)^\dagger$ preserves the direction and distance of the shape $[w] \in \Sigma_2^k$ when it is seen at time t by a person walking along Γ at the same speed as that of the rolling of $\mathscr{T}$ along Γ ([3]).

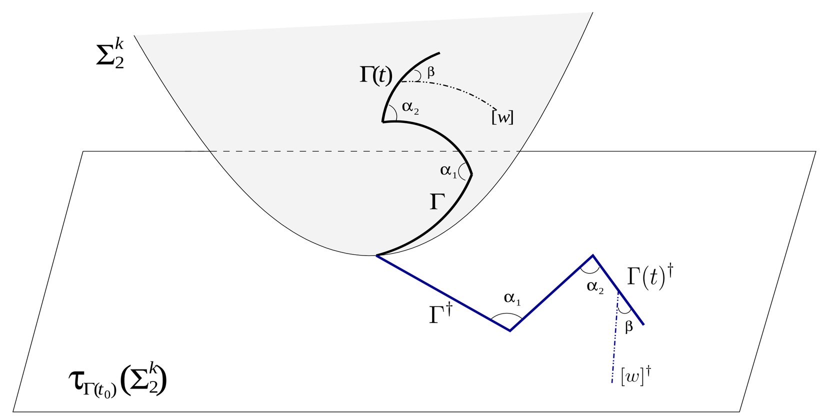

FIGURE 2: $\Gamma^\dagger$ is the unrolling of the piecewise geodesic path Γ onto $\tau_{\Gamma(t_0)}(\Sigma_2^k)$ and $[w]^\dagger$ is the unwrapping at time t, with respect to Γ, of point $[w]$ in Σ_2^k.

Smoothing spline fitting

For a given data set $\{v_j : j = 0, \ldots, n\}$ in $\mathbb{R}^m$, where v_j is observed at time $t_j \in T, j = 0, \ldots, n$, the (cubic) spline in $\mathbb{R}^m$ fitted to this data set with smoothing parameter λ is the function $f(\cdot, \lambda) : T \to \mathbb{R}^m$ that minimizes

$$\sum_{j=0}^{n} \|f^*(t_j, \lambda) - v_j\|^2 + \lambda \int_T \|(f^*)''(t, \lambda)\|^2 dt$$

among all $\mathscr{C}^2$-functions, where T is a time interval.

The following algorithm involves spline fitting in the horizontal sub-space to the preshape sphere (which is equivalent to fitting a shape space spline). Let $\gamma(\cdot, \lambda)$ be the horizontal lift of the shape-space spline $\Gamma(\cdot, \lambda)$.

Algorithm

step 1 Rotate successively the corresponding pre-shapes such that all $z_j\, z_{j+1}^* > 0$. Obtain $\mathscr{D} = \{[z_j] : 0 \leqslant j \leqslant n\}$ and take $\Gamma_{\mathscr{D}}$ to be the piecewise geodesic such that $\Gamma_{\mathscr{D}}(t_j) = [z_j]$.

step 2 Construct a grid of points in $[t_0, t_n]$ with mesh δ, such that

$$t_0 = t_{01} < t_{02} < \cdots < t_1 = t_{11} < \cdots < t_{ij} < \cdots < t_{n1} = t_n$$

with $t_{i1} = t_i$ such that the difference between successive t_{ij} is less than or equal to δ.

step 3 Take initially $\mathscr{D}_1$ as $\mathscr{D}_1 = \{\Gamma_{ij} = \Gamma_{\mathscr{D}}(t_{ij}) : ij\}$.

step 4 Unwrap the data $\mathscr{D}$ at corresponding times into $\tau_{\Gamma_{\mathscr{D}_1}(t_0)}(\Sigma_2^k)$, with respect to the base path $\Gamma_{\mathscr{D}_1}$, and denote by $\mathscr{D}^\dagger$ the corresponding data in the horizontal subspace $\mathscr{H}_{\gamma_{\mathscr{D}_1}(t_0)}(\mathscr{S}_2^k)$.

step 5 Treating $\mathscr{D}^\dagger$ as a set of points in $\mathbb{C}^{k-1}$, fit the *smoothing spline* $f^\dagger(\cdot, \hat\lambda)$ (with smoothing parameter $\hat\lambda$) to $\mathscr{D}^\dagger$ and denote

$$\mathscr{D}_2^\dagger = \{w_{ij}^\dagger = f^\dagger(t_{ij}, \hat\lambda) : ij\} \subset \mathscr{H}_{\gamma_{\mathscr{D}_1}(t_0)}(\mathscr{S}_2^k).$$

step 6 Wrap $\mathscr{D}_2^\dagger$ back at the corresponding times, with respect to $\Gamma_{\mathscr{D}_1}$, to get a set of pre-shapes of $\mathscr{D}_2 = \{w_{ij} : ij\}$ in $\mathscr{S}_2^k$.

step 7 Rotate successively the set of pre-shapes of $\mathscr{D}_2$ to get $\{w_{ij} : ij\}$ such that all $w_{ij}\, w_{i,j+1}^* > 0$, and take $\Gamma_{\mathscr{D}_2}$ to be the piecewise geodesic such that $\Gamma_{\mathscr{D}_2}(t_{ij}) = [w_{ij}]$.

step 8 If $\max\{\rho(\Gamma_{\mathscr{D}_2}(t_{ij}), \Gamma_{\mathscr{D}_1}(t_{ij})) : ij\} \geqslant \epsilon$, replace $\mathscr{D}_1$ by $\mathscr{D}_2$ and go to step 4. Otherwise stop.

An appropriate value for $\hat\lambda$ can be determined by applying a cross-validation procedure.

Simulation

We consider some simulated data generated in the shape space of configurations with $k = 4$ landmarks. Initially a set of 18 equally spaced observations were taken along a true piecewise geodesic path in shape space. The true path has only three geodesic components each of length $\pi/2.5 = 1.256$. Unit size configurations corresponding to these 18 shapes are displayed in the first two rows of the left plot of Figure 3. Then independent identically distributed isotropic Gaussian noise was added to each landmark of the unit size configurations, with standard deviation 0.05, and the perturbed configurations are also displayed in the left plot of Figure 3. Note that the diameter of Σ_2^4 is $\pi/2$, and so our data are highly dispersed. We also assume that these generated shapes are observed at equal time intervals. The shape space spline fitted by our method (see Figure 3) fits the data very well since the maximum distance between the data points and those fitted is around 2% of the diameter of the data.

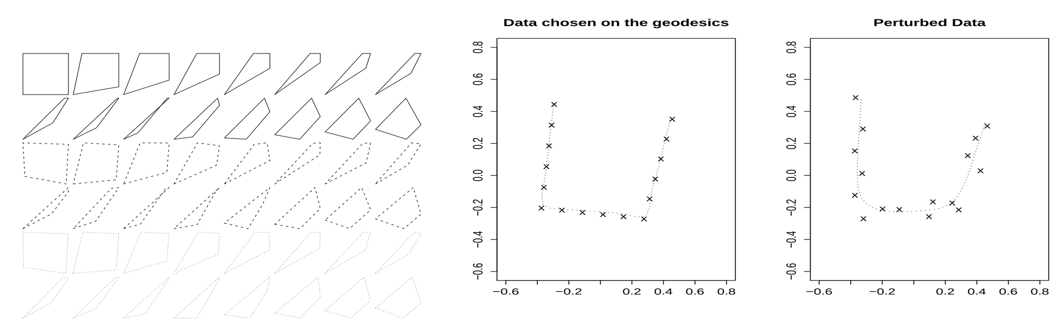

FIGURE 3: (left) —— shapes chosen on the true piecewise geodesic path, – – the perturbed data, ··· shapes fitted to perturbed data. (middle/right) The first two principal component scores of the unrolled paths in the tangent space of the first data point.

Human movement data

We now consider a particular application in the study of human movement data, consisting of $k = 4$ landmarks (lower back, shoulder, wrist and index finger) moving in time. We consider 10 equally spaced time points for each individual movement who undergoes a pointing movement to a target on a table. The data were collected by Dr James Richardson, Université Paris Sud, France. We concentrate on the shapes of the configurations in the plane of the table, and five curves are available.

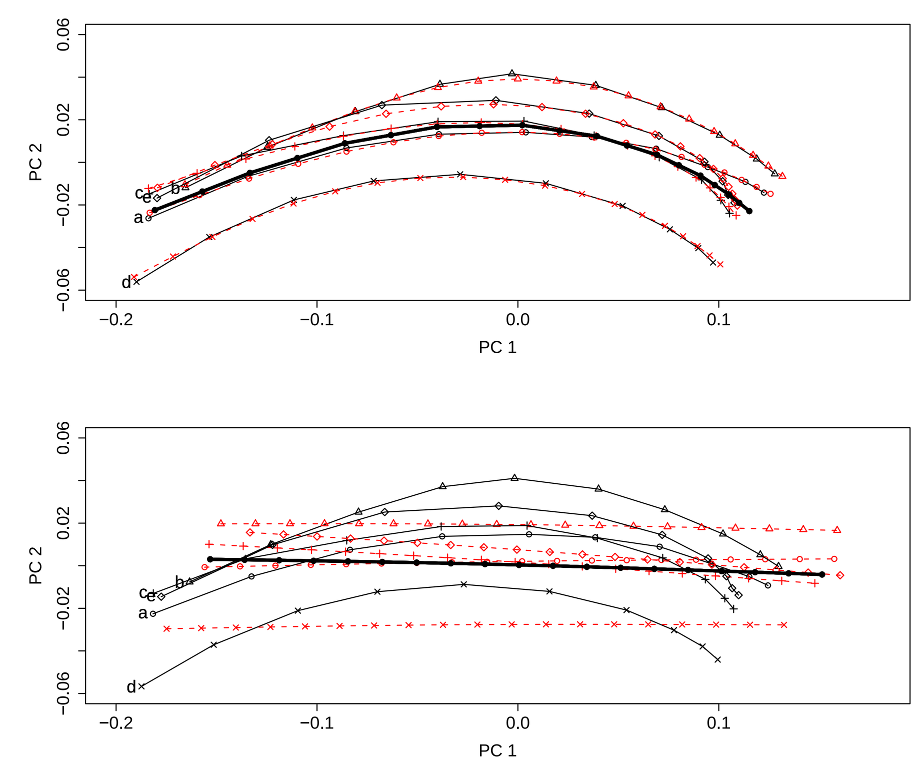

FIGURE 4: The first two principal component scores of the unrolling of the human movement data (black) and fitted spline (red) paths with respect to the fitted mean path (thick black). Above $\hat\lambda = 0.0023$ (obtained by cross-validation) and below $\hat\lambda = 1$ (close to geodesics).

In Figure 4 we consider fitted paths which illustrate a range of possible behaviour. If we let the smoothing parameter $\lambda \to \infty$, then the unrolling of the shape-space spline onto the tangent space at its initial point will be a line segment through the origin, and so the shape-space spline fitted to the data is a geodesic.

References

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[3] Le, H. (2003). Unrolling shape curves. *J. London Math. Soc*, **68**, 511-526.

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