Orthogonal Wavelet Frames for Color Image Compression

Eric Weber

Iowa State University

with Brody Johnson and Ghann Bhatt

Abstract: We develop a vector valued discrete wavelet transform (VDWT) using the concept of orthogonal frames. This VDWT is designed to process vector valued data, such as stereo audio or color images. We describe the construction of the VDWT, and present preliminary data on its performance on image compression versus the standard scalar valued discrete wavelet transform.

Vector Valued Data: Our main motivation is color image processing, in particular, image compression. We shall describe the main idea for such data. However, in the technical discussion below, we will consider stereo audio.

A (digital) color image consists of three matrices (or arrays), whose entries describe the pixel intensities for each of red, green, and blue. For a specific pixel, if we take the three color intensities together, then the image data is a single matrix with entries which are three dimensional vectors.

Mathematically, we model the image (of size $N \times M$) by the Hilbert space

$$l^2(\mathbb{Z}_N \times \mathbb{Z}_M) \oplus l^2(\mathbb{Z}_N \times \mathbb{Z}_M) \oplus l^2(\mathbb{Z}_N \times \mathbb{Z}_M) \simeq l^2(\mathbb{Z}_N \times \mathbb{Z}_M, \mathbb{C}^3).$$

Vector Valued Data (cont'd): The essential idea is to construct a wavelet transform for the space $l^2(\mathbb{Z}_N \times \mathbb{Z}_M, \mathbb{C}^3)$. This is accomplished by constructing three wavelet transforms, one for each summand $l^2(\mathbb{Z}_N \times \mathbb{Z}_M)$ which can then be <u>combined</u> into a single transform by summing the appropriate outputs. The result is a wavelet transform which correlates the different color channels.

Definition 1. For a finite dimensional Hilbert space H, by a *Parseval frame*, we mean any finite set of vectors $\{v_j\}_{j\in\mathbb{J}} \subset H$ such that for every $v \in H$,

$$\sum_{j\in\mathbb{J}} \langle v, v_j \rangle v_j = v.$$

Vector Valued Data (cont'd):

Definition 2. For a second set of vectors $\{w_j\}_{j\in\mathbb{J}} \subset H$, we say $\{v_j\}$ and $\{w_j\}$ are *orthogonal* if for every $v \in H$,

$$\sum_{j\in\mathbb{J}} \langle v, v_j\rangle w_j = 0.$$

The essential underlying result:

Theorem 1. *Suppose $\{v_j\}, \{w_j\} \subset H$; $\{v_j \oplus w_j\}_{j\in\mathbb{J}}$ is a Parseval frame for $H \oplus H$ if and only if $\{v_j\}$ and $\{w_j\}$ are both Parseval frames and are also orthogonal.*

Thus, in essence, $\{v_j \oplus w_j\}$ is a "vector valued frame" for $H \oplus H$.

Orthogonal Wavelet Frames:

Theorem 2. *Suppose* $\{\psi_1, \ldots, \psi_r\}$ *and* $\{\eta_1, \ldots, \eta_r\}$ *generate affine Bessel sequences in* $L^2(\mathbb{R})$*; they are orthogonal if and only if*

1.
$$\sum_{k=1}^{r} \sum_{j\in\mathbb{Z}} \widehat{\psi}_k(2^j\xi)\overline{\widehat{\eta}_k(2^j\xi)} = 0 \ a.e.\ \xi;$$

2. *for every* q *odd,*
$$\sum_{k=1}^{r} \sum_{j=0}^{\infty} \widehat{\psi}_k(2^j\xi)\overline{\widehat{\eta}_k(2^j(\xi+q))} = 0 \ a.e.\ \xi.$$

Orthogonal Wavelet Frames (cont'd):

Theorem 3. *Suppose* $\phi \in L^2(\mathbb{R})$ *be a refinable function which satisfies the conditions of the unitary extension principle; let* $m(\xi)$ *be the associated low pass filter. Let* $m_1(\xi), \ldots, m_r(\xi)$*,* $n_1(\xi), \ldots, n_r(\xi)$ *be elements of* $L^2([0,1))$ *such that the following matrix equations hold:*

1. *the matrix*

$$B(\xi) = \begin{pmatrix} m(\xi) & m(\xi + 1/2) \\ m_1(\xi) & m_1(\xi + 1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi + 1/2) \end{pmatrix}$$

satisfies $B^*(\xi)B(\xi) = I_2$ *for almost every* ξ*.*

2. *the matrix*

$$C(\xi) = \begin{pmatrix} m(\xi) & m(\xi + 1/2) \\ n_1(\xi) & n_1(\xi + 1/2) \\ \vdots & \vdots \\ n_r(\xi) & n_r(\xi + 1/2) \end{pmatrix}$$

satisfies $C^*(\xi)C(\xi) = I_2$ *for almost every* ξ.

3. *the matrices*

$$B_0(\xi) = \begin{pmatrix} m_1(\xi) & m_1(\xi + 1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi + 1/2) \end{pmatrix} \quad C_0(\xi) = \begin{pmatrix} n_1(\xi) & n_1(\xi + 1/2) \\ \vdots & \vdots \\ n_r(\xi) & n_r(\xi + 1/2) \end{pmatrix}$$

satisfy $C_0^*(\xi)B_0(\xi) = 0$ *for almost every* ξ.

Let $\widehat{\psi}_k(2\xi) = m_k(\xi)\widehat{\phi}(\xi)$ *and* $\widehat{\eta}_k(2\xi) = n_k(\xi)\widehat{\phi}(\xi)$. *Then* $\{\psi_1, \ldots, \psi_r\}$ *and* $\{\eta_1, \ldots, \eta_r\}$ *generate orthogonal Parseval wavelet frames.*

Orthogonal Wavelet Frames (cont'd):

The following is a recipe for constructing the wavelet frames.

Theorem 4. *Suppose* $K(\xi)$ *is a* $N \times N$ *paraunitary matrix with 1/2-periodic entries* $a_{i,j}(\xi)$*; let* $K_j(\xi)$ *denote the* j*-th column. Let* $m(\xi), n(\xi)$ *be the low pass/high pass filters for any orthonormal wavelet basis, with scaling function* ϕ*. For each* $j = 1, \ldots, N$*, define the new filters via*

$$\begin{pmatrix} m_1^j(\xi) \\ \vdots \\ m_N^j(\xi) \end{pmatrix} := K_j(\xi) n(\xi).$$

Then, for $j = 1, \ldots, N$*, the wavelet frames generated by* $\{\psi_l^j : l = 1, \ldots, N\}$ *obtained via*

$$\widehat{\psi}_l^j(2\xi) = m_l^j(\xi)\widehat{\phi}(\xi) \tag{1}$$

are Parseval and pairwise orthogonal.

Orthogonal Wavelet Frames (cont'd):

The paraunitary matrix $K(\xi)$ is "orthogonalizing" the wavelet frames for us, and can be one of two (useful) possibilities:

1. $K(\xi)$ is constant; equivalently its entries are scalars–we call this scalar orthogonalization.

2. $K(\xi)$ has entries which are trigonometric polynomials–we call this polynomial orthogonalization.

The initial filters $m(\xi)$ and $n(\xi)$ in Theorem 4 are called the base filters.

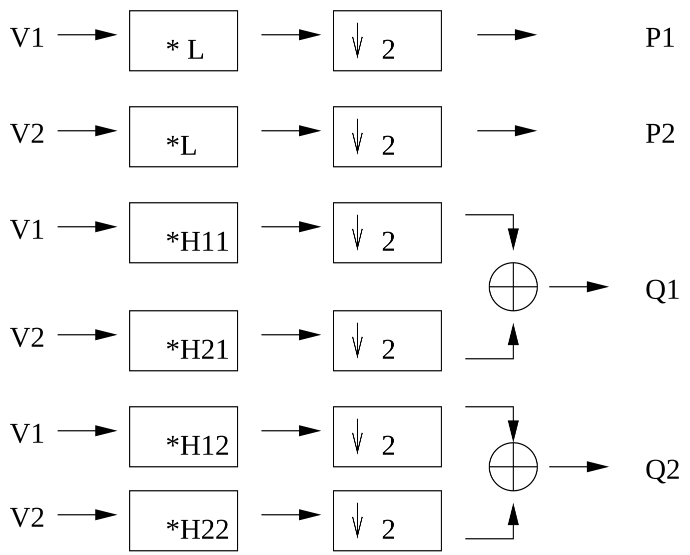
Vector Valued Discrete Wavelet Transform:

Description

We now describe our VDWT. For simplification, we will describe the VDWT on two-dimensional valued data in one spatial dimension, $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ (e.g. stereo audio). Suppose $L \in l^2(\mathbb{Z})$ is a low pass filter corresponding to a scaling function on $L^2(\mathbb{R})$ which satisfies the assumptions of the unitary extension principle. Suppose that $H11, H12 \in l^2(\mathbb{Z})$ and $H21, H22 \in l^2(\mathbb{Z})$ are two pairs of high pass filters whose Fourier transforms satisfy the conditions of Theorem 3. If $V1 \oplus V2 \in l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$, the first stage output of the VDWT is given by filtering $V1$ and $V2$ via a convolution and downsampling, along with the summation of the appropriate pairs of high pass outputs. Graphically, it is given in Figure 1.

Vector Valued Discrete Wavelet Transform:

Figure 1



First Stage Output of VDWT

Vector Valued Discrete Wavelet Transform (cont'd):

Remarks:

1. We only "orthogonalize" the high pass filters–a technical obstruction prevents the low pass filters from being orthogonalized as well.

2. In Figure 1, the redundancy of the filters $H11, H12, H21, H22$ are eliminated after summing the outputs.

3. Why is it vector valued? Heuristically, by summing the high pass filter outputs, any correlation among channels is being amplified, and anticorrelation is being attenuated.

Preliminary Results:

Picture	Method	Threshhold	Comp. Ratio	SNR
Lena	D4, none	15	11.78	30.64
"	D4, scalar	15	14.50	30.84
"	D4, poly.	15	13.31	30.68
"	D4, none	50	45.81	26.14
"	D4, scalar	50	54.53	26.89
"	D4, poly.	50	54.14	26.75
Pepper	D4, none	15	11.21	30.59
"	D4, scalar	15	13.29	32.38
"	D4, poly.	15	12.76	31.65

Preliminary Results (cont'd):

Description of the data:

- Lena and peppers are two standard images in the wavelet community.

- Our base filters are Daubechies 4; we compress images using no orthogonalization (each color channel compressed individually), then using scalar orthogonalization, and polynomial orthogonalization.

- The compression ratio calculates

$$\frac{\text{number of pixels} \times 3}{\text{non-zero (retained) coefficients}}.$$

  No quantization or encoding is performed.

- The VDWT uniformly outperforms the individual compression in *both* compression ratio and SNR of the reconstructed image.