Overdetermined elliptic boundary problems

Jukka Tuomela
University of Joensuu
Finland

Theory of Boundary Problems

joint work with Katya Krupchyk and Nikolai Tarkhanov

Practice of Boundary Problems

joint work with Bijan Mohammadi

In theory there is no difference

between theory and practice.

In practice there is.

K. Krupchyk, J. Tuomela: *Elliptic boundary problems*, to appear in LMS J. Comp. Math.

K. Krupchyk, N. Tarkhanov, J. Tuomela: *Elliptic quasicomplexes in Boutet de Monvel algebra*, preprint 2006/12 Universität Potsdam.

B. Mohammadi, J. Tuomela, *Simplifying numerical solution of constrained PDE systems through involutive completion*, Math. Model. Numer. Anal. vol. 39, 2005, 909–929.

Operators and (Quasi)complexes

Consider the operator

$$Ay = \sum_{|\mu| \leq q} a_\mu(x) \partial^\mu y = f \quad , \quad x \in \Omega \subset \mathbb{R}^n$$

Let $E_0 = \Omega \times \mathbb{R}^m$ and $E_1 = \Omega \times \mathbb{R}^k$, $k \geq m$.

Symbol of A is a map $\sigma A : E_0 \to E_1$ given for fixed ξ by

$$\sigma A(x, \xi) = \sum_{|\mu| = q} a_\mu(x) \xi^\mu$$

To analyse overdetermined systems we need (quasi)complexes.

$$\mathcal{C} \quad : \qquad 0 \longrightarrow V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \cdots$$

$\mathcal{C}$ is a complex (resp. quasicomplex) if

$$A_{i+1}A_i = 0 \quad \text{(resp. compact)}$$

Let $\mathcal{H}_i = \mathsf{ker}(A_{i+1})/\mathsf{im}(A_i)$. Complex is

(i) *Fredholm*, if $\mathsf{dim}(\mathcal{H}_i) < \infty$ and

(ii) *exact*, if $\mathsf{dim}(\mathcal{H}_i) = 0$.

To each complex $\mathcal{C}$ we associate its symbol complex:

$$\sigma\mathcal{C} \quad : \quad 0 \longrightarrow E_0 \xrightarrow{\sigma A_0} E_1 \xrightarrow{\sigma A_1} E_2 \xrightarrow{\sigma A_2} \cdots$$

Examples

(i) de Rham complex is Fredholm and its symbol complex exact, $\mathsf{dim}(\mathcal{H}_i)$ depends on topology of Ω. $\mathsf{dim}(\mathcal{H}_1)$ and $\mathsf{dim}(\mathcal{H}_2)$ have actually a direct physical interpretation in electromagnetics.

(ii) elasticity complex is Fredholm and its symbol complex exact, $\mathsf{dim}(\mathcal{H}_i)$ depends on ??
Physical interpretation of $\mathsf{dim}(\mathcal{H}_i)$?

Starting with σA_0 the determination of $\sigma\mathcal{C}$ is free resolution of modules which

(i) is finite because of Hilbert's theorem and

(ii) can be computed using Gröbner bases.

In case of constant coefficients the computation of $\mathcal{C}$ is essentially the same.

We have used Singular.

Relevant complex is the *compatibility* complex, i.e.

$$A_i y = f$$

has a solution only if

$$A_{i+1} f = 0$$

Example again de Rham complex.

So the strategy of studying overdetermined problems is:

– find the compatibility complex and the corresponding symbol complex(es)

ellipticity

A_0 is *elliptic* in Ω, if σA_0 is injective for all real $\xi \neq 0$.

$\mathcal{C}$ is *elliptic*, if A_0 is elliptic and $\sigma\mathcal{C}$ is exact.

Theorem If Ω is a compact manifold without boundary, then elliptic complexes are Fredholm.

It is essential that A_0 is involutive:

$$By = \nabla \times y + y = 0 \qquad\qquad A_0 y = \begin{pmatrix} \nabla \times y + y \\ \nabla \cdot y \end{pmatrix} = 0$$

B is not elliptic (not even DN–elliptic), while A_0 is elliptic and its compatibility operator is

$$A_1 = \big(\nabla \cdot, -1\big)$$

Why Fredholm operators/complexes?

Recall that Fredholm operators remain Fredholm and their index/Euler characteristic is invariant under compact perturbations.

$$\chi(A_0) = \mathsf{dim}\big(\mathsf{ker}(A_0)\big) - \mathsf{dim}\big(\mathsf{coker}(A_0)\big)$$

Adding lower order terms to DOs is a compact perturbation and the symbol doesn't change if one adds lower order terms.

Hence Fredholm operators/complexes is the largest class for which “general theory” is possible, and to study the finite dimensional “residue” $\mathsf{dim}(\mathcal{H}_i)$ requires case by case analysis.

Why quasicomplexes?

For Fredholm complexes we can similarly define

$$\chi(\mathcal{C}) = \sum_{i=0}^{n} (-1)^i \mathsf{dim}(\mathcal{H}_i)$$

But now the condition $A_{i+1}A_i = 0$ is not stable with respect to perturbations.

However, for quasicomplexes we have

Theorem 1. *It is possible to define Fredholm property and χ for quasicomplexes. These are stable with respect to compact perturbations.*

Idea of proof: we can associate to each quasicomplex a certain complex, and use this complex to prove the stated properties.

Note that individual $\mathsf{dim}(\mathcal{H}_i)$ are *not* invariant.

Example

If one adds lower order terms to

$$A_0 y = \begin{pmatrix} \nabla \times y + y \\ \nabla \cdot y \end{pmatrix} \quad , \quad A_1 \begin{pmatrix} f \\ g \end{pmatrix} = \nabla \cdot f - g$$

then the composition $A_1 A_0$ is still a first order operator, i.e. compact in appropriate Sobolev spaces (Rellich's Theorem).

Boundary

Let Ω be a smooth manifold (or domain in $\mathbb{R}^n$) with smooth boundary Γ. Let E_i (resp. F_i) denote (finite dimensional) bundles on Ω (resp. Γ). Further let V_i (resp. W_i) denote some function spaces on Ω (resp. Γ).

Boundary conditions are given by

$$T_0 y = \sum_\mu t_\mu(x)\partial^\mu y = g \quad , \quad x \in \Gamma$$

Cauchy-Kovalevskaia-Riquier-Janet-Cartan-Kähler way of studying boundary (or initial) conditions is apparently irrelevant for “real” boundary value problems.

The boundary value problem is a map

$$\begin{pmatrix} A_0 \\ T_0 \end{pmatrix} \quad : \quad V_0 \longrightarrow \begin{matrix} V_1 \\ \oplus \\ W_1 \end{matrix}$$

and compatibility operators are of the form

$$\begin{pmatrix} A_i & 0 \\ T_i & S_i \end{pmatrix} \quad : \quad \begin{matrix} V_i \\ \oplus \\ W_i \end{matrix} \longrightarrow \begin{matrix} V_{i+1} \\ \oplus \\ W_{i+1} \end{matrix}$$

Such operators are *DB–operators*.

Example where $S_i \neq 0$.

In Maxwell system a natural boundary condition for electrical conductor is

$$n \times B = J$$

Hence we must require that $S_1 J = \nabla \cdot J = 0$ (conservation of charge).

Boutet de Monvel operators

L. Boutet de Monvel: *Boundary problems for ΨDOs*, Acta Math. 1971.

$$\mathcal{A} = \begin{pmatrix} P+G & K \\ T & S \end{pmatrix} \quad : \quad \begin{matrix} H^{s_i}(E_i) \\ \bigoplus \\ H^{r_i}(F_i) \end{matrix} \longrightarrow \begin{matrix} H^{s_{i+1}}(E_{i+1}) \\ \bigoplus \\ H^{r_{i+1}}(F_{i+1}) \end{matrix}$$

P is ΨDO on Ω and S is ΨDO on Γ.

T, K, G are called *trace*, *Poisson* and *singular Green* operators

Why such operators? Classically P is a parametrix if

$$AP - \mathsf{id} \quad \text{and} \quad PA - \mathsf{id} \quad \text{are compact}$$

For manifolds without boundary

$$A \text{ is } DO \implies P \text{ is } \Psi DO$$

Parametrices are needed in proving actual existence results (useful also for other reasons).

Important point is that ΨDOs is an algebra.

But now we have a (quasi)complex $\mathcal{C}$:

$$0 \longrightarrow V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} \cdots \xrightarrow{A_{n-1}} V_n \longrightarrow 0$$

The sequence

$$0 \longleftarrow V_0 \xleftarrow{P_1} V_1 \xleftarrow{P_2} \cdots \xleftarrow{P_n} V_n \longleftarrow 0$$

is called a *parametrix* of the (quasi)complex $\mathcal{C}$, if

$$A_{i-1}P_i + P_{i+1}A_i - \mathsf{id}_{V_i} \quad \text{are compact}$$

If A_i *are DB–operators,*

then P_i *are BdM–operators.*

Moreover BdM–operators is an algebra.

So how to define ellipticity for BdM (quasi)complexes?

We need a second symbol complex, called the *boundary symbol complex*. The symbol complex defined previously can then be called *interior symbol complex*.

Then we say that a BdM (quasi)complex is *elliptic*, if *both* symbol complexes are exact.

Then we can prove

Theorem 2. *Elliptic BdM (quasi)complex has a parametrix. Moreover this parametrix is also a quasicomplex.*

Theorem 3. *Elliptic BdM (quasi)complex is Fredholm in appropriate Sobolev spaces.*

Let us return to differential equations.

(i) When analysing complexes, important technical tool is the laplacian of the complex

$$L_i = A_{i-1}A^*_{i-i} + A^*_i A_i \quad : \quad V_i \to V_i$$

To use this A_0 has to be involutive.

In fact we need to reduce to first order operators; L_i are then second order operators.

(ii) In the definition of the boundary symbol complex, the involutivity also plays a role.

(iii) Boundary symbol complex can be constructed (at least partly) with syzygies and Gröbner bases.

Some numerical aspects

Unfortunately the completion to involutive form behaves strangely with respect to boundary conditions.

Example (inspired by Stokes problem)

$$\mathcal{S} \quad : \quad \begin{cases} y^1_{20} - y^2 = 0 \\ y^1_{02} + y^2 = 0 \end{cases} \quad , \quad \mathcal{S}_I \quad : \quad \begin{cases} y^1_{20} - y^2 = 0 \\ y^1_{02} + y^2 = 0 \\ \Delta y^2 = 0 \end{cases}$$

Actually $\mathcal{S}_I$ is just formally integrable.

Why this example?

Stokes problem is DN–elliptic, but not elliptic.

The above system is the simplest nontrivial DN–elliptic system.

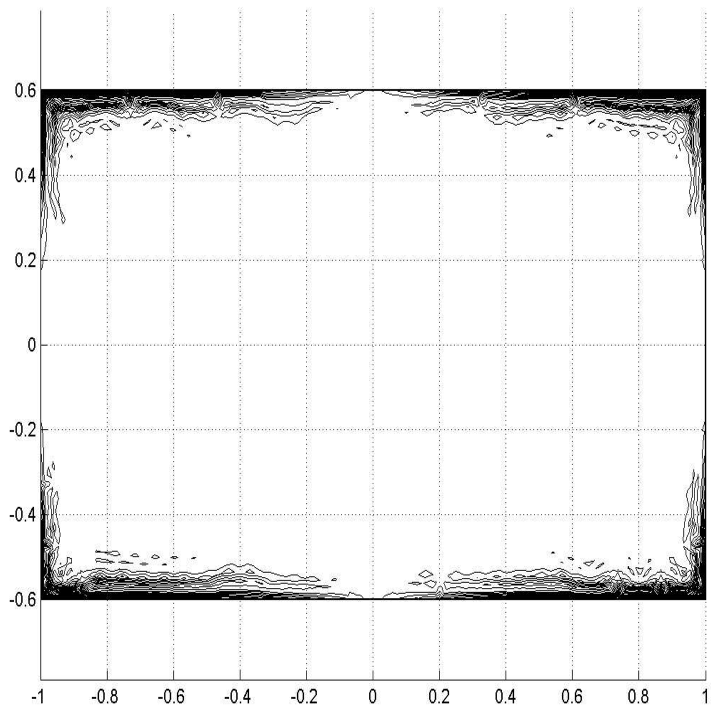

Figure 1: y^2 using $\mathcal{S}$.

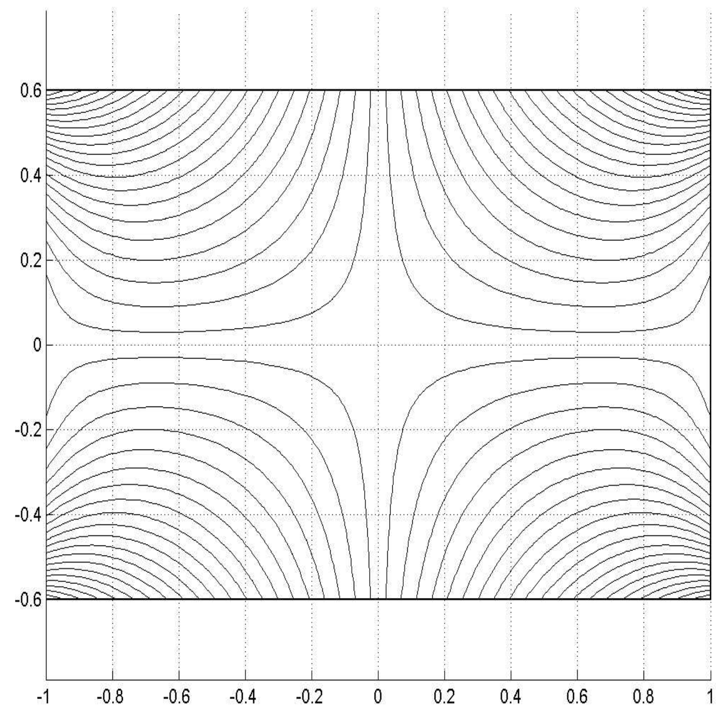

Figure 2: y^2 using $\mathcal{S}_I$.

Computations with FEMLAB.

For y^1 there was no essential difference between $\mathcal{S}$ and $\mathcal{S}_I$ (compare velocity and pressure in Stokes).

Any FE spaces worked for involutive system. However, probably one would need some more special spaces for initial problem (compare *inf-sup condition* for Stokes problem).

Actually same conclusion holds for Stokes problem itself.

Now according to standard theory (Agmon, Douglis, Nirenberg) $\mathcal{S}$ is elliptic if values of y^1 are given on Γ.

According to "overdetermined" theory $\mathcal{S}_I$ is elliptic if values of y^2 are given on Γ.

Resolution of paradox

$\mathcal{S}$ and $\mathcal{S}_I$ are defined in different Sobolev spaces. So

(i) from purely mathematical point of view everything is settled

(ii) but from modelling point of view it is strange that different boundary conditions are required in initial and involutive models.

How we actually computed

$$A_0 y = \begin{pmatrix} y^1_{20} - y^2 \\ y^1_{02} + y^2 \\ \Delta y^2 \end{pmatrix} = 0 \quad , \quad A_1 f = f^1_{02} - f^2_{20} + f^3$$

$$0 \longrightarrow V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} V_2 \longrightarrow 0$$

$$A_0 y + A_1^T z = 0 \quad \Longleftrightarrow \quad \begin{cases} y^1_{20} - y^2 + z_{02} = 0 \\ y^1_{02} + y^2 - z_{20} = 0 \\ \Delta y^2 + z = 0 \end{cases}$$

This is elliptic in standard sense. However, confusingly we need to impose boundary conditions for y^1 **and** y^2 (also for z)...

While z is an artificial variable it can be used in error control.

Resulting system with z is numerically better than direct least squares approach (although we haven't made detailed comparisons).

Square system $\Longrightarrow$ standard software.

Perspectives & problems

- more constructive study of the second symbol complex

- how to resolve the problem of boundary conditions?