Geometry “à la Cartan” revisited:

Hamilton-Jacobi theory in moving frames

Roman Smirnov[1]

Department of Mathematics and Statistics

Dalhousie University

July 28, 2006

I will review the Hamilton-Jacobi theory of orthogonal separation of variables in the context of the Cartan geometry, in particular, its most valuable asset, - the method of moving frames. The central concept in this setting is that of frames of eigenvectors (eigenforms) of Killing two-tensors which provides a natural presentation of the theory in terms of principal bundles. Eisenhart (implicitly) employed this idea in 1934 to study orthogonal separation of variables in Euclidean 3-space for geodesic Hamiltonians. I will show how the corresponding problem for natural Hamiltonians can be solved with the aid of a more general version of the moving frames method than the one used by Eisenhart (joint work with J.T. Horwood and R.G. McLenaghan).

As an application, the approach outlined above together with symmetry methods will be used to determine a new class of maximally superintegrable and multi-separable potentials in Euclidean 3-space. These potentials given by a formula depending on an arbitrary function do not appear in Evans' classification of 1990. A particular example of such a potential is the potential of the Calogero-Moser system (joint work with P. Winternitz).

IMA Summer Program
“Symmetries and Overdetermined Systems of Partial Differential Equations”
organized by
Michael Eastwood and Willard Miller, Jr.

[1] http://www.mathstat.dal.ca/∼smirnov

1 Generalized Killing tensor equations

Let $(M, \boldsymbol{g})$ be an m-dimensional (pseudo-) Riemannian manifold of constant curvature.

Definition 1. *A symmetric contravariant tensor* $\boldsymbol{K}$ *of valence* p *defined on* $(M, \boldsymbol{g})$ *is said to be a* generalized Killing tensor (GKT) of order n *if and only if*

$$[[\dots[\boldsymbol{K}, \boldsymbol{g}], \boldsymbol{g}], \dots, \boldsymbol{g}] = 0 \quad (n+1 \text{ brackets}), \tag{1}$$

where $[\ ,\ ]$ *denotes the Schouten bracket [21].*

We are interested in the vector spaces of solutions to (1). When $n = 0$ we have the standard *Killing tensors*, when $n = 0$ and $p = 1$ - *Killing vectors*.

Let $\mathcal{K}_n^p(M)$ denotes the vector space of the generalized Killing tensors of valence p and order n defined on $(M, \boldsymbol{g})$ (i.e., the space of solutions to (1)).

Nikitin-Prylypko-Eastwood (NPE) formula [5, 6, 19]:

$$d = \dim \mathcal{K}_n^p(M) = \frac{n+1}{m} \binom{p+m-1}{m-1} \binom{p+n+m}{m-1}, \tag{2}$$

Remark 1. The *Delong-Takeuchi-Thompson (DTT)* [4, 25, 26] formula derived for the case $n = 1$ is a particular case of (2).

2 Examples: $\mathcal{K}_0^2(\mathbb{E}^2)$ and $\mathcal{K}_0^2(\mathbb{E}^3)$

Example: $\mathcal{K}_0^2(\mathbb{E}^2)$

Solving the Killing tensor equation $[\boldsymbol{K}, \boldsymbol{g}] = 0$ in Cartesian coordinates $\boldsymbol{x} = (x_1, x_2)$ yields:

$$\begin{aligned} \boldsymbol{K} &= (\beta_1 + 2\beta_4 x_2 + \beta_6 x_2^2)\partial_1 \odot \partial_1 \\ &\quad +(\beta_3 - \beta_4 x_1 - \beta_5 x_2 - \beta_6 x_1 x_2)\partial_1 \odot \partial_2 \\ &\quad +(\beta_2 + 2\beta_5 x_1 + \beta_6 x_1^2)\partial_2 \odot \partial_2, \end{aligned} \tag{3}$$

where $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ and $\odot$ denotes the symmetric tensor product. The components of the generic Killing tensor given by (3) are as follows:

$$\begin{aligned} K^{11} &= \beta_1 + 2\beta_4 x_2 + \beta_6 x_2^2 \\ K^{12} = K^{21} &= \beta_3 - \beta_4 x_1 - \beta_5 x_2 - \beta_6 x_1 x_2 \\ K^{22} &= \beta_2 + 2\beta_5 x_1 + \beta_6 x_1^2 \end{aligned} \tag{4}$$

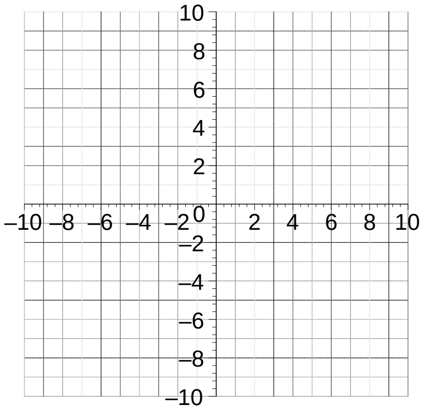
Consider the following generators of the Lie algebra of $SE(2)$: $\boldsymbol{X_1} = \partial_1$, $\boldsymbol{X_2} = \partial_2$ and $\boldsymbol{R} = x_2\partial_1 - x_1\partial_2$. Then the formula (3) can be rewritten as follows:

$$\boldsymbol{K} = A^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j + B^{\ell}\boldsymbol{R} \odot \boldsymbol{X}_{\ell} + C\boldsymbol{R} \odot \boldsymbol{R}, \quad i, j, \ell = 1, 2, \tag{5}$$

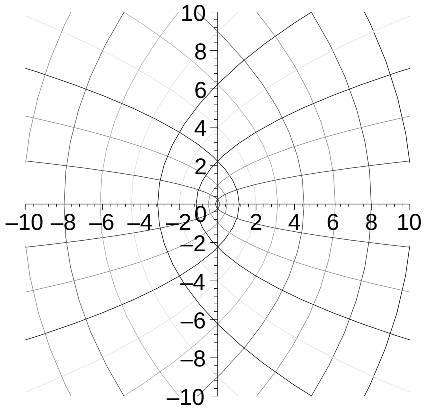
where $A^{ij} = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_3 & \beta_2 \end{pmatrix}$, $B^{\ell} = \begin{pmatrix} \beta_4 \\ \beta_5 \end{pmatrix}$, $C = \beta_6$.

Orthogonal coordinate systems in $\mathbb{E}^2$

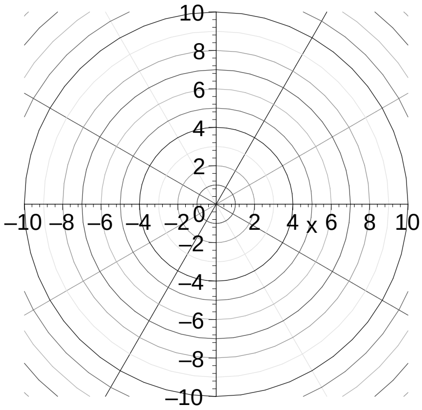
There are four coordinate webs $\mathbb{E}^2$ generated by *non-trivial* (characteristic) elements of the vector space $\mathcal{K}_0^2(\mathbb{E}^2)$:



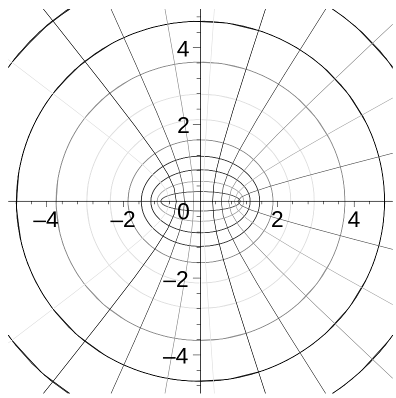
I: Cartesian coordinates
$x_1 = u,\ x_2 = v$



II: Parabolic coordinates
$x_1 = 1/2(u^2 - v^2),\ x_2 = uv$



III: Polar coordinates
$x_1 = u\cos v,\ x_2 = u\sin v$



IV: Elliptic-hyperbolic coordinates
$x_1 = k\cosh u\cos v,\ x_2 = k\sinh u\sin v$

Figure 1: Families of confocal conics

Example: $\mathcal{K}_0^2(\mathbb{E}^3)$

Solving the Killing tensor equation $[\boldsymbol{K},\boldsymbol{g}]=0$ in Cartesian coordinates $\boldsymbol{x}=(x,y,z)$ yields (the notations below are compatible with those adapted in [2]):

$$\begin{aligned}
K^{11} &= a_1 - 2b_{12}z + 2b_{13}y + c_2z^2 + c_3y^2 - 2\gamma_1 yz,\\
K^{22} &= a_2 - 2b_{23}x + 2b_{21}z + c_3x^2 + c_1z^2 - 2\gamma_2 zx,\\
K^{33} &= a_3 - 2b_{31}y + 2b_{32}x + c_1y^2 + c_2x^2 - 2\gamma_3 xy,\\
K^{23} &= \alpha_1 + b_{31}z - b_{21}y + (b_{22}-b_{33})x + (\gamma_3 z + \gamma_2 y - \gamma_1 x)x - c_1yz,\\
K^{31} &= \alpha_2 + b_{12}x - b_{32}z + (b_{33}-b_{11})y + (\gamma_1 x + \gamma_3 z - \gamma_2 y)y - c_2zx,\\
K^{12} &= \alpha_3 + b_{23}y - b_{13}x + (b_{11}-b_{22})z + (\gamma_2 y + \gamma_1 x - \gamma_3 z)z - c_3xy.
\end{aligned} \tag{6}$$

Note

$$\beta_1 = b_{22}-b_{33}, \quad \beta_2 = b_{33}-b_{11}, \quad \beta_3 = b_{11}-b_{22}, \tag{7}$$

hence $\beta_1+\beta_2+\beta_3=0 \Rightarrow d = \dim \mathcal{K}_0^2(\mathbb{E}^3) = 20$ (as expected by (2)). Generators of the Lie algebra of $SE(3)$:

$$\boldsymbol{X}_i = \frac{\partial}{\partial x^i}, \qquad \boldsymbol{R}_i = \epsilon^k{}_{ji} x^j \boldsymbol{X}_k, \tag{8}$$

for $i=1,2,3$, where ϵ_{ijk} is the Levi-Civita permutation tensor.

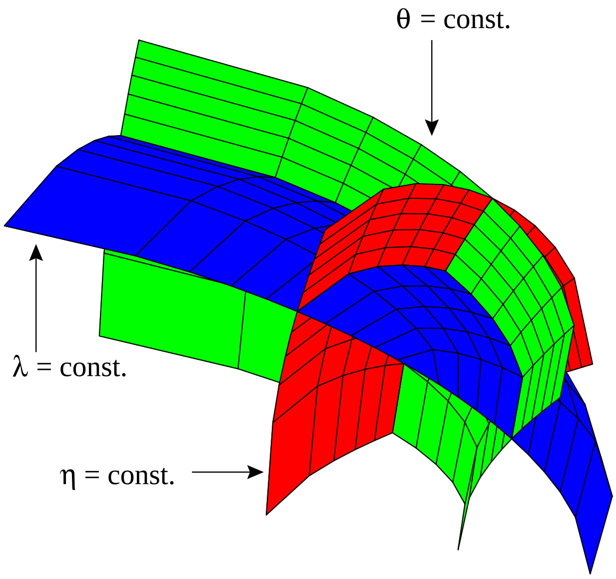
$$A^{ij} = \begin{pmatrix} a_1 & \alpha_3 & \alpha_2 \\ \alpha_3 & a_2 & \alpha_1 \\ \alpha_2 & \alpha_1 & a_3 \end{pmatrix}, \quad B^{ij} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad C^{ij} = \begin{pmatrix} c_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & c_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & c_3 \end{pmatrix} \tag{9}$$

In view of the above, (6) can be re-written as follows:

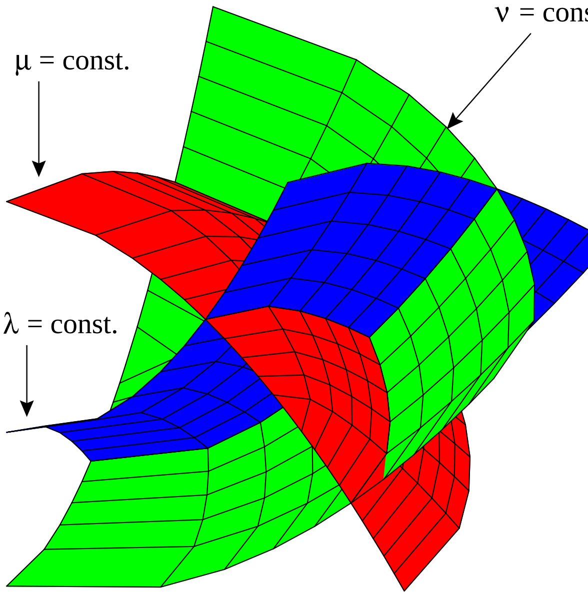
$$\boldsymbol{K} = A^{ij}\boldsymbol{X}_i \odot \boldsymbol{X}_j + 2B^{ij}\boldsymbol{X}_i \odot \boldsymbol{R}_j + C^{ij}\boldsymbol{R}_i \odot \boldsymbol{R}_j. \tag{10}$$

Orthogonal coordinate systems in $\mathbb{E}^3$

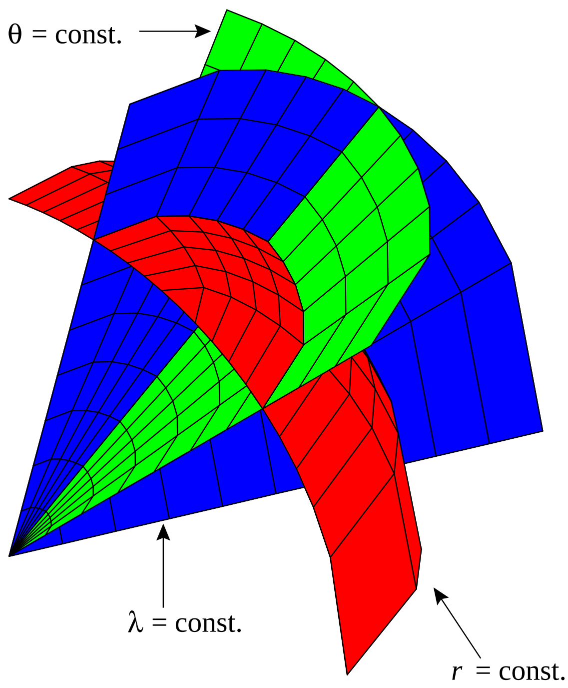
There are 11 coordinate webs $\mathbb{E}^3$ generated by characteristic (that is with *normal* or *surface-forming* eigenvectors) elements of the vector space $\mathcal{K}^3_0(\mathbb{E}^2)$:



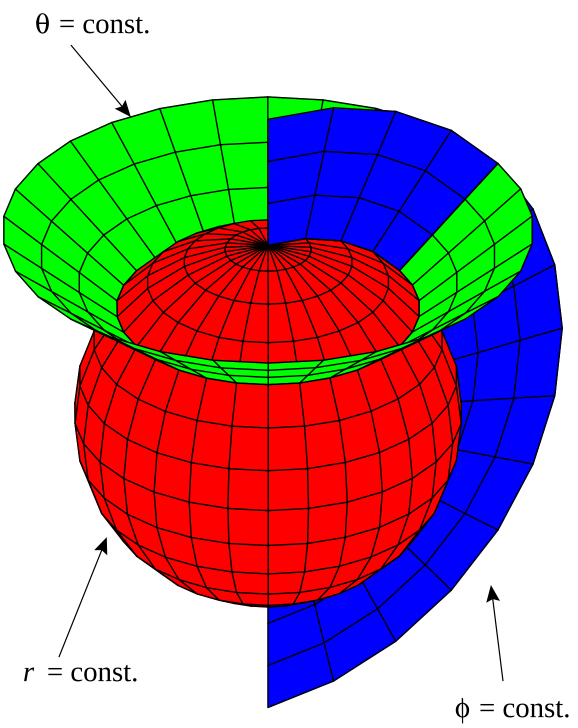
Ellipsoidal



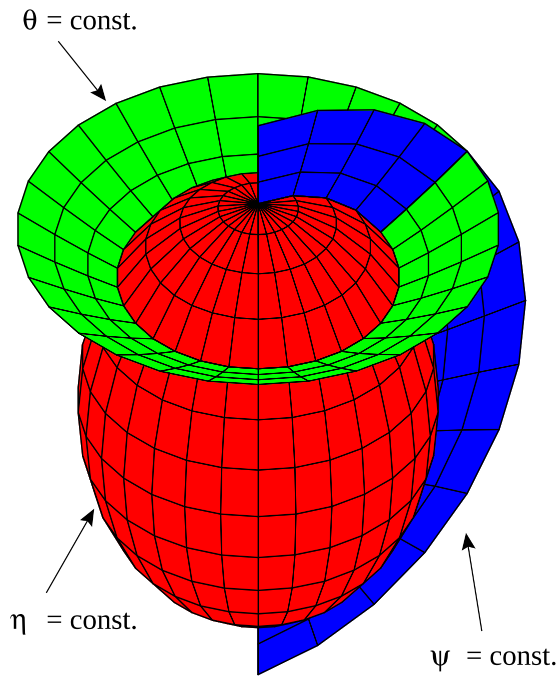
Paraboloidal



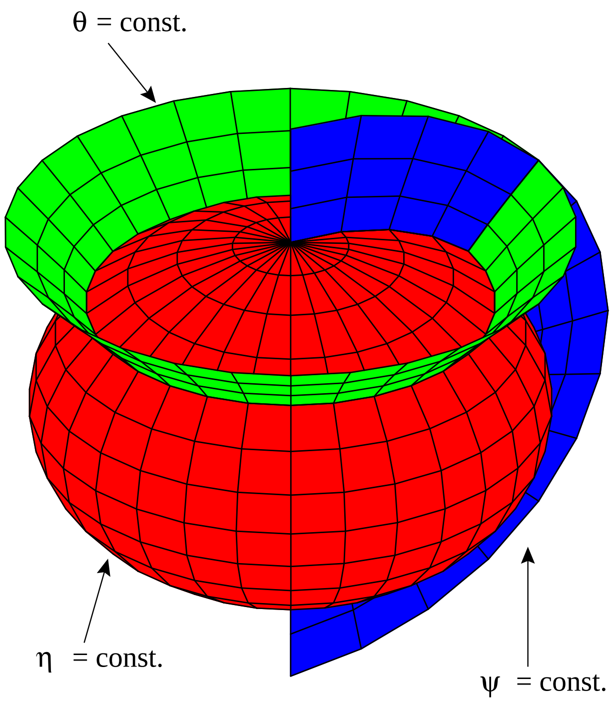
Conical



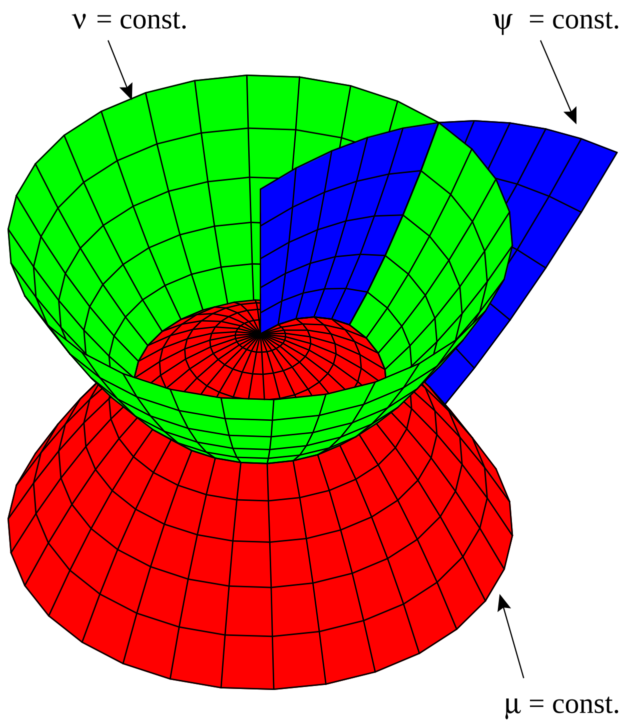
Spherical



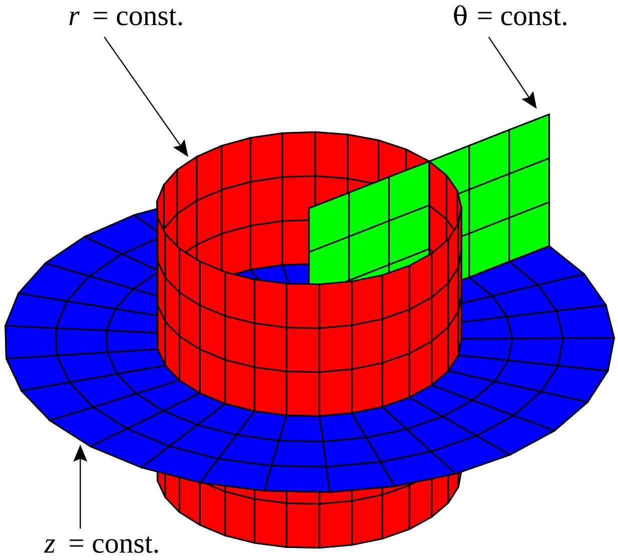
Prolate Spheroidal



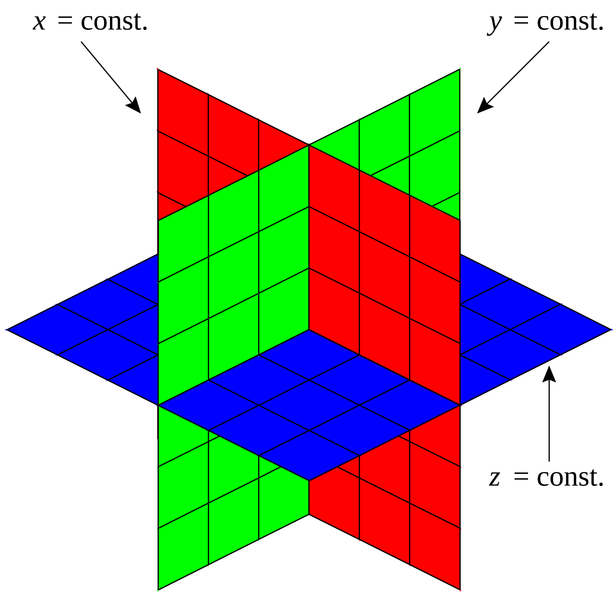
Oblate Spheroidal



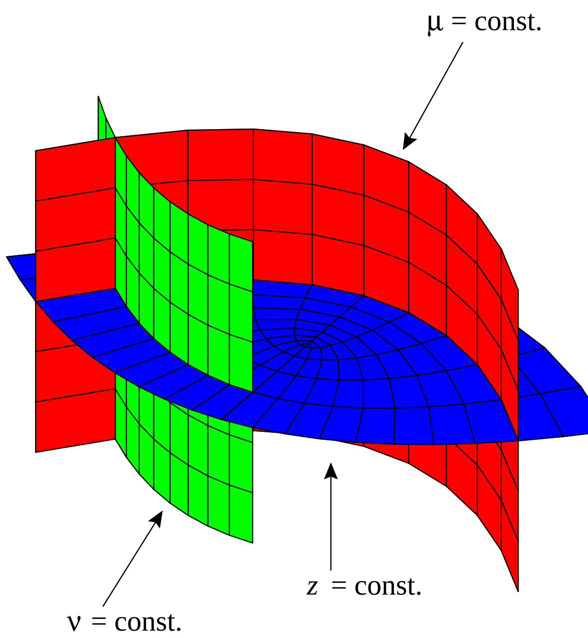
Parabolic



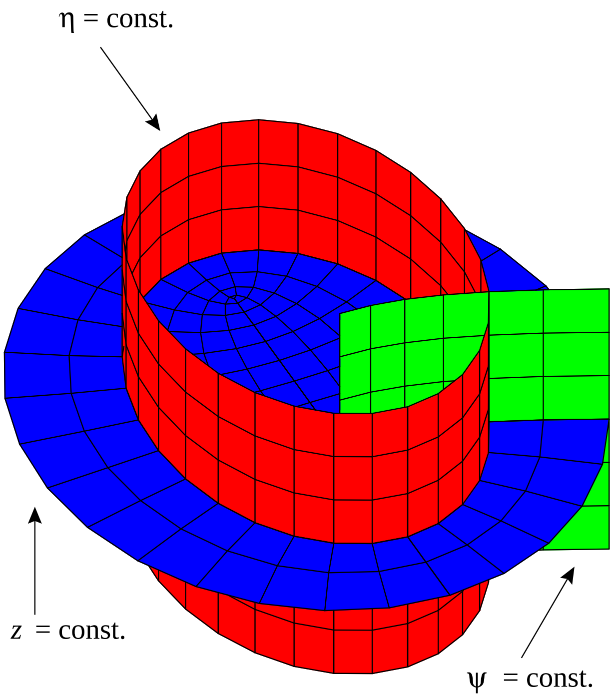
Circular Cylindrical



Cartesian



Parabolic Cylindrical



Elliptic-hyperbolic

Problem 1. We wish to classify **geometrically** the Killing two-tensors generating orthogonal coordinate webs.

Problem 2. Use the solution to Problem 1 in solving the Hamilton-Jacobi (Schrödinger) equation.

3 "What is geometry?"

The relationships between different approaches to geometry can be described as follows [22, 13]:

$$\begin{array}{ccc} \text{Euclidean Geometry} & \stackrel{\text{generalization}}{\longrightarrow} & \text{Klein Geometries} \\ \downarrow \text{generalization} & & \text{generalization} \downarrow \\ \text{Riemannian Geometry} & \stackrel{\text{generalization}}{\longrightarrow} & \text{Cartan Geometries} \end{array} \tag{11}$$

4 Geometry "à la Cartan"

The model for $\mathbb{E}^n$:

$$\text{Frame} = (\boldsymbol{x}, E_1, E_2, \dots, E_n),$$

where $\boldsymbol{x} \in \mathbb{E}^n$ and $E_1, \dots, E_n$ is a non-coordinate orthonormal frame (following [11]) from $\mathbb{E}^n$.

We work in the principal fiber bundle:

$$\boldsymbol{x} : SE(n) \to SE(n)/SO(n) \simeq \mathbb{E}^n.$$

(In fact, this is a geometry "im Sinne von Klein"!)

Example: Let us solve Problem 1 for the case $\mathcal{K}_0^2(\mathbb{E}^2)$. Let $\boldsymbol{K} \in \mathcal{K}_0^2(\mathbb{E}^2)$ (different from the metric) and $\{E^1, E^2\}$ be the *rigid moving frame* of orthonormal eigenforms of $\boldsymbol{K}$. Then the quadratic differential form

$$\boldsymbol{g} = \delta_{ab} E^a \odot E^b, \quad a, b = 1, 2$$

is the pull-back under $\boldsymbol{x}$ of the usual metric on $\mathbb{E}^2$, while the Killing tensor $\boldsymbol{K}$ assumes the form

$$K_{ab} = \lambda_a \delta_{ab} E^a \odot E^b, \quad a, b = 1, 2.$$

The equations for structure functions $C^c{}_{ab}, a, b, c = 1, 2$ are given by

$$[E_a, E_b] = C^c{}_{ab} E_c \quad \text{or} \quad dE^a = \frac{1}{2} C^a{}_{bc} E^b \wedge E^c. \tag{12}$$

We introduce the connection coefficients Γ as follows:

$$\nabla_{E_a} E_b = \Gamma_{ab}{}^c E_c, \quad \nabla_{E_c} E^b = -\Gamma_{cd}{}^b E^d, \tag{13}$$

where ∇ denotes the Levi-Civita connection associated with the metric $\boldsymbol{g}$.

Remark 2. The choice of the connection is not arbitrary. As is well-known from Riemannian geometry, given a connection ∇ on a manifold M one can parallel propagate frames. For any path τ between two points of M parallel transport along τ defines a linear mapping $L(\tau)$ between the tangent spaces of two points. This linear map is an *isometry* if the connection ∇ is a Levi-Civita connection. Clearly, the linear map $L(\tau)$ induced by a Levi-Civita connection ∇ maps orthonormal frames to orthonormal frames.

The vanishing of the torsion tensor $T^a{}_{bc}$ is given by

$$T^a{}_{bc} = \Gamma_{bc}{}^a - \Gamma_{cb}{}^a - C^a{}_{bc} = 0, \tag{14}$$

while the components of the Riemann curvature tensor $R^a{}_{bcd}$ with respect to the moving frame are defined as follows:

$$R^a{}_{bcd} = E_c\Gamma_{db}{}^a + \Gamma_{db}{}^e\Gamma_{ce}{}^a - E_d\Gamma_{cb}{}^a - \Gamma_{cb}{}^e\Gamma^a_{de} - C^e{}_{cd}\Gamma_{eb}{}^a, \tag{15}$$

respectively. We now define a one-form valued matrix $\omega^a{}_b$ called the *connection one-form* by

$$\omega^a{}_b := \Gamma_{cb}{}^a E^c. \tag{16}$$

Further, we can define

$$\omega_{ab} := g_{ad}\omega^d{}_b.$$

On account of the above the connection one-forms, ω_{ab} are obviously skew-symmetric. They satisfy Mourer-Cartan's structural equations,

$$dE^a + \omega^a{}_b \wedge E^b = 0, \tag{17}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Theta^a{}_b, \tag{18}$$

where we have introduced the *curvature two-form*

$$\Theta^a{}_b := (1/2)R^a{}_{bcd}E^c \wedge E^d.$$

In addition, for a $(0,2)$ Killing tensor $\boldsymbol{K}$ we have the Killing tensor equation:

$$\nabla_{(c}K_{ab)} = 0, \tag{19}$$

where ∇_c denotes the covariant derivative defined by

$$\nabla_c K_{ab} := E_c K_{ab} - K_{db} \Gamma_{ca}{}^d - K_{ad} \Gamma_{cb}{}^d. \tag{20}$$

It is easy to see that the integrability conditions yield

$$E^a \wedge dE^a = 0, a, b = 1, 2.$$

Hence by the Frobenius theorem there exist functions f, g and variables u, v such that

$$E^1 = f du, \quad E^2 = g dv.$$

. . . .

Ultimately, we find the canonical forms of the four orbits that represent the four coordinate webs by considering the following three *isometrically invariant* cases (see [23] and the references therein), namely

$$\begin{array}{ll} \text{Case 1} & \lambda_1 \text{ and } \lambda_2 \text{ are constant} \\ \text{Case 2} & \lambda_1 \text{ is constant } (\lambda_2 \text{ is constant}) \\ \text{Case 3} & \lambda_1 \text{ and } \lambda_2 \text{ are not constant} \end{array} \tag{21}$$

The last Case 3 leads to two orbits corresponding to elliptic-hyperbolic and polar coordinates respectively.

Question: Can we solve in the same way Problem 1 for the case $\mathcal{K}_0^2(\mathbb{E}^3)$? Unfortunately, - impossible (application of the Frobenius theorem yields PDEs which are unsolvable by exact methods). However, there is a *better* way!

5 "The Eisenhart code"

Recall that in 1934 Eisenhart [7] *did* solve Problem 1 for the case $\mathcal{K}_0^2(\mathbb{E}^3)$. How did he do that? He (implicitly) employed the method of moving frames:

"We assume that a_{ij} is such that these vector-fields are normal and that the hypersurfaces are taken as parametric; and we *write the fundamental form thus*

$$\textit{(1.4)} \qquad ds^2 = e_1 H_1^2 (dx_1)^2 + \cdots + e_n H_n^2 (dx_n)^2,$$

where e's are plus or minus one as the case may be..."

Eisenhart found necessary and sufficient condition for the geodesic equations determined by the Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} g^{ij}(\boldsymbol{q}) p_i p_j \tag{22}$$

be *orthogonally separable* (i.e., admit the Stäckel form):

"A necessary and sufficient condition that the fundamental quadratic form of V_n can be given the Stäckel form is that the equations of geodesic admit $n-1$ independent quadratic first integrals, that the roots of the characteristic equations (1.2) for each of these integrals be simple, that (1.11) hold, and that the vector-fields determined by (1.3) be normal and be the same vector-fields for all the first integrals..."

In addition, Eisenhart also determined that the solution to the Killing tensor equation $[\boldsymbol{K}, \boldsymbol{g}] = 0$ in the moving frame is given by

$$c_1\boldsymbol{g} + c_2\boldsymbol{K}_1 + c_3\boldsymbol{K}_2, \tag{23}$$

where c_1, c_2, c_3 are arbitrary constants. Note that the Killing tensors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ share the same normal (surface-forming) eigenvectors. Killing tensors with distinct eigenvalues and normal eigenvectors (eigenforms) are called *characteristic Killing tensors (CKT).*

6 Canonical forms for the case $\mathcal{K}_0^2(\mathbb{E}^3)$

We have 11 canonical forms corresponding to the 11 orthogonal coordinate webs determined by characteristic Killing tensors for the orbits in the vector space $\mathcal{K}_0^2(\mathbb{E}^3)$ generated by the group action $SE(3) \circlearrowright \mathcal{K}_0^2(\mathbb{E}^3)$. They are

Cartesian: (x, y, z) (24)

$$\begin{cases} x = x,\ y = y,\ z = z \\ -\infty < x, y, z < \infty \\ \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(0, 1, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Circular cylindrical: (r, θ, z) (25)

$$\begin{cases} x = r\cos\theta,\ y = r\sin\theta,\ z = z \\ r \geqslant 0, \quad 0 \leqslant \theta < 2\pi, \quad -\infty < z < \infty \\ \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(0, r^4, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Parabolic cylindrical: (μ, ν, z) (26)

$$\begin{cases} x = \frac{1}{2}(\mu^2 - \nu^2),\ y = \mu\nu,\ z = z \\ \mu \geqslant 0, \quad -\infty < \nu < \infty, \quad -\infty < z < \infty \\ \mathrm{d}s^2 = (\mu^2 + \nu^2)(\mathrm{d}\mu^2 + \mathrm{d}\nu^2) + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(\nu^2 g_{11}, -\mu^2 g_{22}, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Elliptic-hyperbolic: (η, ψ, z) (27)

$$\begin{cases} x = a\cosh\eta\cos\psi,\ y = a\sinh\eta\sin\psi,\ z = z \\ \eta \geqslant 0, \quad 0 \leqslant \psi < 2\pi, \quad -\infty < z < \infty, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\cosh^2\eta - \cos^2\psi)(\mathrm{d}\eta^2 + \mathrm{d}\psi^2) + \mathrm{d}z^2 \\ K_1^{ij} = \mathrm{diag}(a^2\cos^2\psi\, g_{11}, a^2\cosh^2\eta\, g_{22}, 0) \\ K_2^{ij} = \mathrm{diag}(0, 0, 1) \end{cases}$$

Spherical: (r, θ, ϕ) (28)

$$\begin{cases} x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \\ r \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \phi < 2\pi \\ \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \\ K_1^{ij} = \mathrm{diag}(0, r^4, r^4\sin^2\theta) \\ K_2^{ij} = \mathrm{diag}(0, 0, r^4\sin^4\theta) \end{cases}$$

Prolate spheroidal: (η, θ, ψ) (29)

$$\begin{cases} x = a\sinh\eta\sin\theta\cos\psi,\ y = a\sinh\eta\sin\theta\sin\psi,\ z = a\cosh\eta\cos\theta \\ \eta \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \psi < 2\pi, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\sinh^2\eta + \sin^2\theta)(\mathrm{d}\eta^2 + \mathrm{d}\theta^2) + a^2\sinh^2\eta\sin^2\theta\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(-a^2\sin^2\theta\, g_{11}, a^2\sinh^2\eta\, g_{22}, a^2(\sinh^2\eta - \sin^2\theta) g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, a^2\sinh^2\eta\sin^2\theta\, g_{33}) \end{cases}$$

Oblate spheroidal: (η, θ, ψ) (30)

$$\begin{cases} x = a\cosh\eta\sin\theta\cos\psi,\ y = a\cosh\eta\sin\theta\sin\psi,\ z = a\sinh\eta\cos\theta \\ \eta \geqslant 0, \quad 0 \leqslant \theta < \pi, \quad 0 \leqslant \psi < 2\pi, \quad a > 0 \\ \mathrm{d}s^2 = a^2(\cosh^2\eta - \sin^2\theta)(\mathrm{d}\eta^2 + \mathrm{d}\theta^2) + a^2\cosh^2\eta\sin^2\theta\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(a^2\sin^2\theta\, g_{11}, a^2\cosh^2\eta\, g_{22}, a^2(\cosh^2\eta + \sin^2\theta) g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, a^2\cosh^2\eta\sin^2\theta\, g_{33}) \end{cases}$$

Parabolic: (μ, ν, ψ) (31)

$$\begin{cases} x = \mu\nu\cos\psi,\ y = \mu\nu\sin\psi,\ z = \frac{1}{2}(\mu^2 - \nu^2) \\ \mu \geqslant 0, \quad \nu \geqslant 0, \quad 0 \leqslant \psi < 2\pi \\ \mathrm{d}s^2 = (\mu^2 + \nu^2)(\mathrm{d}\mu^2 + \mathrm{d}\nu^2) + \mu^2\nu^2\,\mathrm{d}\psi^2 \\ K_1^{ij} = \mathrm{diag}\big(-\nu^2 g_{11}, \mu^2 g_{22}, (\mu^2 - \nu^2) g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(0, 0, \mu^2\nu^2 g_{33}) \end{cases}$$

Conical: (r, θ, λ) (32)

$$\begin{cases} x^2 = \left(\dfrac{r\theta\lambda}{bc}\right)^2,\ y^2 = \dfrac{r^2(\theta^2 - b^2)(b^2 - \lambda^2)}{b^2(c^2 - b^2)},\ z^2 = \dfrac{r^2(c^2 - \theta^2)(c^2 - \lambda^2)}{b^2(c^2 - b^2)} \\ r \geqslant 0, \quad b^2 < \theta^2 < c^2, \quad 0 < \lambda^2 < b^2, \\ \mathrm{d}s^2 = \mathrm{d}r^2 + \dfrac{r^2(\theta^2 - \lambda^2)}{(\theta^2 - b^2)(c^2 - \theta^2)}\,\mathrm{d}\theta^2 + \dfrac{r^2(\theta^2 - \lambda^2)}{(b^2 - \lambda^2)(c^2 - \lambda^2)}\,\mathrm{d}\lambda^2 \\ K_1^{ij} = \mathrm{diag}(0, r^2\lambda^2 g_{22}, r^2\theta^2 g_{33}) \\ K_2^{ij} = \mathrm{diag}(0, r^2 g_{22}, r^2 g_{33}) \end{cases}$$

Paraboloidal: (μ, ν, λ) (33)

$$\begin{cases} x^2 = \dfrac{4(\mu - b)(b - \nu)(b - \lambda)}{b - c},\ y^2 = \dfrac{4(\mu - c)(c - \nu)(\lambda - c)}{b - c}, \\ \qquad z = \mu + \nu + \lambda - b - c \\ 0 < \nu < c < \lambda < b < \mu < \infty \\ \mathrm{d}s^2 = \dfrac{(\mu - \nu)(\mu - \lambda)}{(\mu - b)(\mu - c)}\,\mathrm{d}\mu^2 + \dfrac{(\mu - \nu)(\lambda - \nu)}{(b - \nu)(c - \nu)}\,\mathrm{d}\nu^2 \\ \qquad + \dfrac{(\lambda - \nu)(\mu - \lambda)}{(b - \lambda)(\lambda - c)}\,\mathrm{d}\lambda^2 \\ K_1^{ij} = \mathrm{diag}\big(2(\nu + \lambda) g_{11}, 2(\lambda + \mu) g_{22}, 2(\mu + \nu) g_{33}\big) \\ K_2^{ij} = \mathrm{diag}(-4\nu\lambda g_{11}, -4\lambda\mu g_{22}, -4\mu\nu g_{33}) \end{cases}$$

$$
\text{Ellipsoidal: } (\eta,\theta,\lambda) \quad (34) \qquad
\begin{cases}
x^2 = \dfrac{(a-\eta)(a-\theta)(a-\lambda)}{(a-b)(a-c)},\ y^2 = \dfrac{(b-\eta)(b-\theta)(b-\lambda)}{(b-a)(b-c)}, \\
\qquad z^2 = \dfrac{(c-\eta)(c-\theta)(c-\lambda)}{(c-a)(c-b)} \\
a > \eta > b > \theta > c > \lambda \\
\mathrm{d}s^2 = \dfrac{(\eta-\theta)(\eta-\lambda)}{4(a-\eta)(b-\eta)(c-\eta)}\,\mathrm{d}\eta^2 + \dfrac{(\theta-\eta)(\theta-\lambda)}{4(a-\theta)(b-\theta)(c-\theta)}\,\mathrm{d}\theta^2 \\
\qquad + \dfrac{(\lambda-\eta)(\lambda-\theta)}{4(a-\lambda)(b-\lambda)(c-\lambda)}\,\mathrm{d}\lambda^2 \\
K_1^{ij} = \mathrm{diag}\big(-(\theta+\lambda)g_{11}, -(\lambda+\eta)g_{22}, -(\eta+\theta)g_{33}\big) \\
K_2^{ij} = \mathrm{diag}(\theta\lambda g_{11}, \lambda\eta g_{22}, \eta\theta g_{33})
\end{cases}
$$

For each of the eleven separable coordinate systems, we give the components of the corresponding CKT with respect to Cartesian coordinates and any restrictions on the Killing tensor parameters [12]

1. Cartesian web

$$K^{ij} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{35}$$

2. Circular cylindrical web

$$K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 xy & 0 \\ -c_3 xy & a_1 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{36}$$

3. Parabolic cylindrical web

$$K^{ij} = \begin{pmatrix} a_1 & b_{23} y & 0 \\ b_{23} y & a_1 - 2b_{23} x & 0 \\ 0 & 0 & a_3 \end{pmatrix} \tag{37}$$

4. Elliptic-hyperbolic web

$$K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 xy & 0 \\ -c_3 xy & a_2 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \frac{a_1 - a_2}{c_3} > 0 \tag{38}$$

5. Spherical web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_1 + c_2 x^2 + c_2 y^2 \end{pmatrix} \tag{39}$$

6. Prolate spheroidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} > 0 \tag{40}$$

7. Oblate spheroidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 xz \\ -c_3 xy & a_1 + c_3 x^2 + c_2 z^2 & -c_2 yz \\ -c_2 xz & -c_2 yz & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} < 0 \tag{41}$$

8. Parabolic web

$$K^{ij} = \begin{pmatrix} a_1 - 2b_{12} z + c_3 y^2 & -c_3 xy & b_{12} x \\ -c_3 xy & a_1 - 2b_{12} z + c_3 x^2 & b_{12} y \\ b_{12} x & b_{12} y & a_1 \end{pmatrix} \tag{42}$$

9. Conical web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 zx \\ -c_3 xy & a_1 + c_3 x^2 + c_1 z^2 & -c_1 yz \\ -c_2 zx & -c_1 yz & a_1 + c_1 y^2 + c_2 x^2 \end{pmatrix} \tag{43}$$

10. Paraboloidal web

$$K^{ij} = \begin{pmatrix} a_1 - 2b_{12} z + c_3 y^2 & -c_3 xy & b_{12} x \\ -c_3 xy & a_2 + 2b_{21} z + c_3 x^2 & -b_{21} y \\ b_{12} x & -b_{21} y & a_3 \end{pmatrix}, \tag{44}$$
$$b_{12}[b_{12}b_{21} + c_3(a_2 - a_3)] + b_{21}[b_{12}b_{21} + c_3(a_1 - a_3)] = 0$$

11. Ellipsoidal web

$$K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 xy & -c_2 zx \\ -c_3 xy & a_2 + c_3 x^2 + c_1 z^2 & -c_1 yz \\ -c_2 zx & -c_1 yz & a_3 + c_1 y^2 + c_2 x^2 \end{pmatrix}, \tag{45}$$
$$(a_1 - a_2)c_1 c_2 + (a_2 - a_3)c_2 c_3 + (a_3 - a_1)c_3 c_1 = 0$$

To check whether or not at given Killing tensor $\boldsymbol{K} \in \mathcal{K}^2_0(\mathbb{E}^3)$ has normal eigenvectors we employ (after making sure that the eigenvalues are distinct) the *Tonolo-Schouten-Nijenhuis conditions*:

$$N^{\ell}{}_{[jk} g_{i]\ell} = 0, \tag{46a}$$
$$N^{\ell}{}_{[jk} K_{i]\ell} = 0, \tag{46b}$$
$$N^{\ell}{}_{[jk} K_{i]m} K^{m}{}_{\ell} = 0 \tag{46c}$$

where $N^i{}_{jk}$ are the components of the Nijenhuis tensor of K^{ij} given by

$$N^i{}_{jk} = K^i{}_{\ell} K^{\ell}{}_{[j,k]} + K^{\ell}{}_{[j} K^i{}_{k],\ell}. \tag{47}$$

We remark that the TSN conditions (46a–46c) yield 10 quadratic, 35 cubic and 84 quartic equations, respectively, in the Killing tensor parameters (see [12] for more details).

7 Non-canonical characteristic Killing tensors in the vector space $\mathcal{K}_0^2(\mathbb{E}^3)$

Consider a Hamiltonian system defined by a natural Hamiltonian:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} g^{ij}(\boldsymbol{q}) p_i p_j + V(\boldsymbol{q}), \tag{48}$$

where g^{ij} are the components of the metric of $(M, \boldsymbol{g})$. A more general theorem has been formulated and proven by Benenti [1]:

Theorem 1. *The Hamiltonian system defined by (48) is orthogonally separable if and only if there exists a valence-two Killing tensor $\boldsymbol{K}$ with (i) pointwise simple and real eigenvalues, (ii) orthogonally integrable (normal) eigenvectors and (iii) such that*

$$d(\boldsymbol{K}\, dV) = 0. \quad \textit{(a.k.a. "Bertrand-Darboux equations").} \tag{49}$$

Often a Killing tensor(s) $\boldsymbol{K}$ satisfying the conditions of Theorem 1 is *not* in its canonical form!

Conclusion: Therefore to solve Problems 1 and 2 we need to employ a more general version of the moving frames method (i.e., that goes beyond the method that was used by Eisenhart), which will allow us not only to classify, but also transform a given characteristic Killing tensor to its respective canonical form.

Note that the group $SE(3)$ (or $SE(2)$ in the case $\mathcal{K}_0^2(\mathbb{E}^2)$) acts transitively in the bundle of orthonormal frames of eigenvectors of Killing tensors of $\mathcal{K}_0^2(\mathbb{E}^3)$. Thus one can try to solve Problems 1 and 2 *in the group*, rather than in the frames.

8 Geometric construction of moving frames (i.e., moving frames "à la Fels & Olver")

From the first lecture given by Peter Olver at the IMA Summer Program [20, 8, 9, 16]:

$$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$$

K - cross-section to the group orbits

$\mathcal{O}_z$ - orbit through $z \in M$

$k = K \cap \mathcal{O}_z$ - unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the *fundamental invariants*

$g \in G$ - *unique* group element mapping k to z

$\Rightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Example: The orbit problem $SE(2) \circlearrowright \mathcal{K}_0^2(\mathbb{E}^2)$ [27, 18]

Group action $SE(2) \circlearrowright \mathbb{E}^2$:

$$\begin{aligned} \tilde{x}_1 &= x_1 \cos p_3 - x_2 \sin p_3 + p_1, \\ \tilde{x}_2 &= x_1 \sin p_3 + x_2 \cos p_3 + p_2, \end{aligned} \tag{50}$$

where p_1, p_2 and p_3 are the parameters of the isometry group $SE(2)$.
Group action $SE(2) \circlearrowright \mathcal{K}_0^2(\mathbb{E}^2)$:

$$\begin{aligned} \tilde{\beta_1} &= \beta_1 \cos^2 p_3 - 2\beta_3 \cos p_3 \sin p_3 + \beta_2 \sin^2 p_3 - 2p_2\beta_4 \cos p_3 - 2p_2\beta_5 \sin p_3 \\ &\quad +\beta_6 p_2^2, \\ \tilde{\beta_2} &= \beta_1 \sin^2 p_3 - 2\beta_3 \cos p_3 \sin p_3 + \beta_2 \cos^2 p_3 - 2p_1\beta_5 \cos p_3 + 2p_1\beta_4 \sin p_3 \\ &\quad +\beta_6 p_1^2, \\ \tilde{\beta_3} &= (\beta_1 - \beta_2) \sin p_3 \cos p_3 + \beta_3(\cos^2 p_3 - \sin^2 p_3) + (p_1\beta_4 + p_2\beta_5) \cos p_3 \\ &\quad +(p_1\beta_5 - p_2\beta_4) \sin p_3 - \beta_6 p_1 p_2, \\ \tilde{\beta_4} &= \beta_4 \cos p_3 + \beta_5 \sin p_3 - \beta_6 p_2, \\ \tilde{\beta_5} &= \beta_5 \cos p_3 - \beta_4 \sin p_3 - \beta_6 p_1, \\ \tilde{\beta_6} &= \beta_6. \end{aligned} \tag{51}$$

Infinitesimal generators of the group action in $\mathcal{K}_0^2(\mathbb{E}^2)$:

$$\begin{aligned} \boldsymbol{V}_1 &= -2\beta_5 \frac{\partial}{\partial \beta_2} - \beta_4 \frac{\partial}{\partial \beta_3} + \beta_6 \frac{\partial}{\partial \beta_5}, \\ \boldsymbol{V}_2 &= 2\beta_4 \frac{\partial}{\partial \beta_1} - \beta_5 \frac{\partial}{\partial \beta_3} + \beta_6 \frac{\partial}{\partial \beta_4}, \\ \boldsymbol{V}_3 &= -2\beta_3 \Big(\frac{\partial}{\partial \beta_1} - \frac{\partial}{\partial \beta_2} \Big) + (\beta_1 - \beta_2) \frac{\partial}{\partial \beta_3} + \beta_5 \frac{\partial}{\partial \beta_4} - \beta_4 \frac{\partial}{\partial \beta_5}. \end{aligned} \tag{52}$$

Cross-section K:

$$K = \{\beta_3 = \beta_4 = \beta_5 = 0\}. \tag{53}$$

The moving frame map $\rho : \mathcal{K}_0^2(\mathbb{E}^2) \to SE(2)$ for the the *normalization equations* corresponding to the cross-section (53):

$$\tilde{\beta}_3 = \tilde{\beta}_4 = \tilde{\beta}_5 = 0. \tag{54}$$

Indeed, solving (54) for the group parameters p_1, p_2 and p_3, we get

$$\begin{aligned} p_1 &= \frac{\beta_5 \cos p_3 - \beta_4 \sin p_3}{\beta_6} \\ p_2 &= \frac{\beta_4 \cos p_3 + \beta_5 \sin p_3}{\beta_6} \\ p_3 &= \frac{1}{2} \arctan \frac{2(\beta_3\beta_6 + \beta_4\beta_5)}{\beta_6(\beta_1 - \beta_2) - \beta_4^2 + \beta_5^2} \end{aligned} \tag{55}$$

Fundamental invariants:

$$\begin{aligned} \Delta_1 &= \beta_6 \\ \Delta_2 &= \beta_6(\beta_1 + \beta_2) - \beta_4^2 - \beta_5^2 \\ \Delta_3 &= (\beta_6(\beta_1 - \beta_2) - \beta_4^2 + \beta_5^2)^2 + 4(\beta_6\beta_3 + \beta_4\beta_5)^2 \end{aligned} \tag{56}$$

Classification:

$$\begin{aligned} \text{Cartesian (C)}: &\quad \Delta_1 = 0 \quad \Delta_3 = 0 \\ \text{Polar (P)}: &\quad \Delta_1 \neq 0 \quad \Delta_3 = 0 \\ \text{Parabolic (PB)}: &\quad \Delta_1 = 0 \quad \Delta_3 \neq 0 \\ \text{Elliptic-hyperbolic (EH)}: &\quad \Delta_1 \neq 0, \quad \Delta_3 \neq 0 \end{aligned} \tag{57}$$

Note that the "Cartesian" orbits are one-dimensional, "parabolic" — two-dimensional, while "elliptic-hyperbolic" and "polar" — three-dimensional.

Example: The orbit problem $SE(3) \circlearrowright \mathcal{K}_0^2(\mathbb{E}^3)$[12]

<u>Group action</u> $SE(3) \circlearrowright \mathbb{E}^3$:

$$x^i = \lambda_j{}^i \tilde{x}^j + \delta^i, \tag{58}$$

where $\lambda_j{}^i \in SO(3)$ and $\delta^i \in \mathbb{R}^3$.

<u>Group action</u> $SE(2) \circlearrowright \mathcal{K}_0^2(\mathbb{E}^2)$:

$$\begin{aligned}
\tilde{A}^{ij} &= A^{k\ell}\lambda^i{}_k\lambda^j{}_\ell + 2B^{k\ell}\lambda^{(i}{}_k\mu^{j)}{}_\ell + C^{k\ell}\mu^i{}_k\mu^j{}_\ell, \\
\tilde{B}^{ij} &= B^{k\ell}\lambda^i{}_k\lambda^j{}_\ell + C^{k\ell}\lambda^j{}_\ell\mu^i{}_k, \\
\tilde{C}^{ij} &= C^{k\ell}\lambda^i{}_k\lambda^j{}_\ell,
\end{aligned} \tag{59}$$

where

$$\mu^j{}_i = \epsilon^k{}_{\ell i}\lambda^j{}_k\delta^\ell. \tag{60}$$

<u>Infinitesimal generators:</u>

$$\boldsymbol{U}_i = 2\epsilon^{(j}{}_{i\ell}B^{k)\ell}\frac{\partial}{\partial A^{jk}} + \epsilon^j{}_{i\ell}C^{k\ell}\frac{\partial}{\partial B^{jk}} \tag{61}$$

$$\begin{aligned}
\boldsymbol{V}_i = {} & (\epsilon^j{}_{\ell i}A^{\ell k} + \epsilon^k{}_{\ell i}A^{j\ell})\frac{\partial}{\partial A^{jk}} + (\epsilon^j{}_{\ell i}B^{\ell k} + \epsilon^k{}_{\ell i}B^{j\ell})\frac{\partial}{\partial B^{jk}} \\
& + (\epsilon^j{}_{\ell i}C^{\ell k} + \epsilon^k{}_{\ell i}C^{j\ell})\frac{\partial}{\partial C^{jk}},
\end{aligned} \tag{62}$$

for $i = 1, 2, 3$.

Fundamental invariants:

$$
\begin{aligned}
\Delta_1 &= B_i{}^i,\ \ \Delta_2 = C_i{}^i,\ \ \Delta_3 = B^{ij}C_{ij},\ \ \Delta_4 = C^{ij}C_{ij},\ \ \Delta_5 = B^{ij}B_{ji} + A^{ij}C_{ij},\\
\Delta_6 &= B^{ij}C_j{}^kC_{ki}, \quad \Delta_7 = C^{ij}C_j{}^kC_{ki}, \quad \Delta_8 = C^{ij}[B_j{}^k(B_{ik} + 2B_{ki}) + A_j{}^kC_{ki}],\\
\Delta_9 &= \epsilon_{ikm}\epsilon_{j\ell n}B^{ij}B^{k\ell}B^{mn} - 2(B_i{}^{[i}B_j{}^{j]} + A^{ij}C_{ij})B_k{}^k + 6B^{ij}A_j{}^kC_{ki},\\
\Delta_{10} &= B^{ij}(B_i{}^kC_{kj} - 2B_j{}^kC_{ki}) - (B^{ij}B_{ij} + A^{ij}C_{ij})C_k{}^k + A_i{}^iC_j{}^{[j}C_k{}^{k]},\\
\Delta_{11} &= \epsilon_{i\ell m}\epsilon_{jkp}B^{ij}B^{k\ell}C^{mn}C_n{}^p + B^{ij}[B_{ij}C^{k\ell}C_{k\ell} - C_j{}^k(C_k{}^\ell B_{i\ell} + 4C_{[k}{}^\ell B_{\ell]i})]\\
&\quad + A^{ij}C_{ij}C_k{}^{[k}C_\ell{}^{\ell]},\\
\Delta_{12} &= A_i{}^i[(C_j{}^jC_k{}^k + 3C^{jk}C_{jk})C_\ell{}^\ell - 4C^{jk}C_k{}^\ell C_{\ell j}] - 6A^{ij}C_{ij}C^{k\ell}C_{k\ell}\\
&\quad + 6B^{ij}\{B_{ij}C^{k\ell}C_{k\ell} - C_j{}^k[(B_{ik} - 2B_{ki})C_\ell{}^\ell + 4C_k{}^\ell B_{\ell i}]\}\\
&\quad + 12\epsilon_{i\ell m}\epsilon_{jkp}B^{ij}B^{k\ell}C^{mn}C_n{}^p,\\
\Delta_{13} &= A^{ij}(B_{ij}C_k{}^{[k}C_\ell{}^{\ell]} + B_j{}^kC_k{}^\ell C_{\ell i} - 2C_{i(j}B_{k)}{}^kC_\ell{}^\ell)\\
&\quad + A_i{}^iC^{jk}(B_{jk}C_\ell{}^\ell - B_k{}^\ell C_{\ell j}) - B^{ij}[B_{ij}B^{k\ell}C_{k\ell} + 2C_j{}^kB_{ki}B_\ell{}^\ell\\
&\quad + B_j{}^kB_{ik}C_\ell{}^\ell - (B_j{}^kB_{i\ell} + B_i{}^kB_{\ell j})C_k{}^\ell],\\
\Delta_{14} &= 4A_i{}^{[i}A_j{}^{j]}C_k{}^{[k}C_\ell{}^{\ell]} + 8A^{ij}(A_j{}^kC_{k[i}C_{\ell]}{}^\ell + A_k{}^kC_{[j}{}^\ell C_{\ell]i}) + A^{ij}C_{ij}(A^{k\ell}C_{k\ell}\\
&\quad + 4B_k{}^{[k}B_\ell{}^{\ell]}) + 4C^{ij}B_j{}^kA_k{}^\ell B_{i\ell} + 16A^{ij}C_j{}^kB_{[k}{}^\ell B_{\ell]i},\\
\Delta_{15} &= A^{ij}C_{ij}[(C_k{}^kC_\ell{}^\ell - 3C^{k\ell}C_{k\ell})C_m{}^m + 2C^{k\ell}C_\ell{}^mC_{mk}]\\
&\quad - 6A^{ij}C_j{}^kC_{ki}C_\ell{}^{[\ell}C_m{}^{m]} - 12C^{ij}B_j{}^k(C_k{}^\ell B_{i[\ell}C_{m]}{}^m + 2B_k{}^\ell C_{i[\ell}C_{m]}{}^m).
\end{aligned}
\tag{63}
$$

Classification:

The classification in this case is very complicated [12]. In brief, we divide the 11 orthogonal coordinate webs into three groups, namely "translational", "rotational" and "asymmetric", - according to their geometric properties. Note that the fundamental invariants given by (63) distinguish only between *asymmetric* webs, namely paraboloidal, ellipsoidal and conical (they are generated by the elements of $\mathcal{K}_0^2(\mathbb{E}^3)$ that belong to six-dimensional orbits). To classify the remaining eight orthogonal coordinate webs we employ the following concept.

Definition 2. *We say that a CKT $\boldsymbol{K} \in \mathcal{K}^2(\mathbb{E}^3)$ is translational (rotational) if it admits a translational (rotational) Killing vector $\boldsymbol{V}$: $\mathcal{L}_{\boldsymbol{V}}(\boldsymbol{K}) = 0$.*[2]

In order to employ this concept in our classification, we classify the elements of the vector space $\mathcal{K}_0^1(\mathbb{E}^3)$ first in terms of the algebraic invariants of the group action: $SE(3) \circlearrowleft \mathcal{K}_0^1(\mathbb{E}^3)$ (i.e., distinguish between the "rotational" and "translational" Killing tensors).

[2]The circular cylindrical tensor (36) also admits a rotational Killing vector and can therefore be considered as both translational and rotational.

Transformations to canonical forms: The moving frames map!

Now we can solve Problems 1 and 2 in $\mathbb{E}^3$. Let a Hamiltonian system be given by the Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} g^{ij}(\boldsymbol{q}) p_i p_j + V(\boldsymbol{q}), \quad i, j = 1, 2, 3. \tag{64}$$

Given the potential V, we can employ the procedure above to answer the questions

- Whether or not the system defined by (64) is orthogonally-separable, or multi-separable.
- If it is, we can determine what systems of orthogonal coordinates the corresponding Hamilton-Jacobi equation separates in and find the transformations from the given, to the separable coordinates, thus ultimately solving the Hamiltonian system in question by quadratures.

The procedure described above has been implemented by Joshua Horwood into the `KillingTensor` computer algebra package.

9 $\boldsymbol{K}$ (Killing tensor) vs V (potential)

Many problems of Hamiltonian mechanics that amount to the study of the Bertrand-Darboux equations:

$$\mathrm{d}(\boldsymbol{K} \mathrm{d} V) = 0$$

can now be solved via the invariants of the Killing tensor(s) $\boldsymbol{K}$, rather than manipulations with the potential V, stemming from the celebrated paper of 1901 by Darboux [3]. They are redundant!

Example: [2nd Integrable Case of Yatsun [18]]

Consider a Hamiltonian system with two degrees of freedom defined in $\mathbb{E}^2$ by the following Hamiltonian:

$$\begin{aligned} H(\boldsymbol{q},\boldsymbol{p}) &= \frac{1}{2}(p_1^2+p_2^2) - 2(q_1^4 + 2q_1^2q_2^2 + \frac{2\lambda}{g_2}q_2^4) \\ &\quad +4(q_1^3+q_1q_2^2) - 2(q_1^2+q_2^2). \end{aligned} \tag{65}$$

It is known that the Hamiltonian system defined by (65) is completely integrable if $g^2 = 2\lambda$ admitting in this case the following additional first integral independent of (65), which is quadratic in the momenta:

$$\begin{aligned} F(\boldsymbol{q},\boldsymbol{p}) &= \left(q_2^2 + \frac{3}{4}\right)p_1^2 - (2q_1-1)q_2p_1p_2 + (q_1-1)q_1p_2^2 - 3q_1^4 - \\ &\quad 2q_1^2q_2^2 + q_2^4 + 6q_1^3 + 2q_1q_2^2 - 3q_1^2. \end{aligned} \tag{66}$$

Observe that the Killing tensor $\boldsymbol{K}$ determined by (66) is given by

$$\begin{aligned} \boldsymbol{K} &= \left(\frac{3}{4} + q_2^2\right)\partial_1 \odot \partial_1 + \left(\frac{1}{2}q_2 - q_1q_2\right)\partial_1 \odot \partial_2 \\ &\quad + \left(-q_1 + q_1^2\right)\partial_2 \odot \partial_2. \end{aligned} \tag{67}$$

$$\beta_1 = \frac{3}{4}, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = -\frac{1}{2}, \quad \beta_5 = 0, \quad \beta_6 = 1.$$

Substituting this data into the formulas for Δ_1 and Δ_3 (56), we obtain

$$\Delta_1 = 1 \neq 0, \quad \Delta_3 = \frac{1}{4} \neq 0,$$

which immediately shows that the Killing tensor (67) generates elliptic-hyperbolic coordinates. Next, we compute the moving frames map (55):

$$p_1 = -\frac{1}{2}, \quad p_2 = p_3 = 0.$$

we conclude therefore that the system determined by (65) is orthogonally integrable with respect to *shifted* elliptic-hyperbolic coordinates:

$$\begin{cases} q_1 = \frac{1}{2} + \cosh u \cos v, \\ q_2 = \sinh u \sin v. \end{cases} \tag{68}$$

Thus, we have solved the problem *without solving the Bertrand-Darboux PDE*!

10 An application to the superintegrability theory

The theory of superintegrable Hamiltonian systems has its origins in earlier papers by Pavel Winternitz and collaborators [10, 17]. A general structure and classification theory for superintegrable systems (both classical and quantum) defined in two- and three-dimensional spaces has been developed in a number of recent papers by Ernie Kalnins, Jonathan Kress and Willard Miller (see [14, 15] and the relevant references therein).

It follows that any Killing tensor with normal eigenvectors admitting a Killing vector $\boldsymbol{V} = \boldsymbol{R}_3$, that is

$$\mathcal{L}_{\boldsymbol{R}_3}(\boldsymbol{K}) = 0 \tag{69}$$

has the form [12]

$$K_R^{ij} = \begin{pmatrix} a_1 - 2b_{12}z + c_2z^2 + c_3y^2 & -c_3xy & b_{12}x - c_2xz \\ -c_3xy & a_1 - 2b_{12}z + c_3x^2 + c_2z^2 & b_{12}y - c_2yz \\ b_{12}x - c_2xz & b_{12}y - c_2yz & a_3 + c_2x^2 + c_2y^2 \end{pmatrix}. \tag{70}$$

We define the subspace $\mathcal{K}_R^2(\mathbb{E}^3)$ of $\mathcal{K}_0^2(\mathbb{E}^3)$ to be the set of all Killing tensors of the form (70) and shall refer to this subspace as the *space of rotational Killing tensors*. We remark that the form of the general rotational Killing tensor (70) is also invariant under subgroup of translations about the z-axis and that all canonical rotational CKTs (39)–(42) are special cases of (70).

Problem 3. What is the most general potential V compatible via

$$\mathrm{d}(\boldsymbol{K}\mathrm{d}V) = 0$$

with the generic rotational Killing tensor given by (70)?

Answer [24] (see also [17, 2]):

$$\boldsymbol{V}_{GCM} = \frac{\varphi(y/x)}{x^2 + y^2}. \tag{71}$$

1. Functionally independent first integrals of (71):

$$
\begin{aligned}
H &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V_{GCM} \\
F_1 &= p_1^2(y^2 + z^2) - 2yzp_2p_3 + \\
&\quad p_2^2(x^2 + z^2) - 2xzp_1p_3 + \\
&\quad p_3^2(x^2 + y^2) - 2xyp_1p_2 + \\
&\quad \frac{\varphi(y/x)(1 + z^2)}{x^2 + y^2} \qquad (72) \\
F_2 &= x^2p_2^2 - xyp_1p_2 - y^2p_1^2 - 2\varphi(y/x) \\
F_3 &= \frac{1}{2}p_3^2 \\
F_4 &= 2xp_1p_3 - 2yp_2p_3 + 2z(p_2^2 - p_1^2) + \frac{\varphi(y/x)}{2(x^2 + y^2)}
\end{aligned}
$$

Hence the Hamiltonian system defined by (71) is maximally superintegrable for *any* φ.

2. Separation of variables:

- Spherical
- Circular cylindrical
- Rotational parabolic
- Oblate spheroidal
- Prolate spheroidal

3. A connection with the Calogero-Moser potential:

$$V_{CM} = \frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} \qquad (73)$$

For

$$\varphi(t) = 2(1 + t^2)\left[\frac{3 + t^2}{(3 - t^2)^2} + 1\right], \qquad (74)$$

where $t = y/x$ the potential (71) is reducible to the Calogero-Moser potential (73).

We conclude therefore that there is an infinite number of *maximally superintegrable* potentials that can have an *arbitrary* number of constants.

---

* * *

Thus we have demonstrated that the Hamilton-Jacobi theory of orthogonal separation of variables (including the study of multi-separable and superintegrable systems) is deeply rooted in the Cartan geometry via invariant theory, moving frames method and the equivalence problem.

* * *

"Every mathematical discipline goes through three periods of development: the naive, the formal, and the critical." David Hilbert.

References

[1] S. Benenti, *Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation,* J. Math. Phys. **38** (1997), 6578–6602.

[2] S. Benenti, C. Chanu and G. Rastelli,*The super-separability of the three-body inverse-square Calogero system*, J. Math. Phys. **41** (2000), 4654–4678.

[3] G. Darboux, *Sur un probléme de mècanique*, Arch. Néerlandaises Sci. **6** (1901), 371–376.

[4] R. P. Delong, Jr., *Killing Tensors and the Hamilton-Jacobi Equation.* - PhD thesis, University of Minnesota: 1982.

[5] M. Eastwood, *Representations via overdetermined systems* Contemp. Math., AMS **368** (2005), 201–210.

[6] M. Eastwood, *Higher symmetries of the Laplacian*, Ann. of Math. **161** (2005), no. 3, 1645–1665.

[7] L. P. Eisenhart, *Separable systems of Stäckel*, Ann. of Math. **35** (1934), 284–305.

[8] M. Fels and P.J. Olver, *Moving coframes. I. A practical algorithm*, Acta. Appl. Math. **51** (1998), 161–213.

[9] M. Fels and P.J. Olver, *Moving coframes. II. Regularization and theoretical foundations*, Acta. Appl. Math. **55** (1999), 127–208

[10] I. Friš, V. Mandrosov, Ya. A. Smorodinsky, M. Uhliř and P. Winternitz, *On higher order symmetries in quantum mechanics,* Phys. Lett. **16** (1965), 354–356.

[11] P. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry,* Duke Math. J. **41** (1974), 775–814.

[12] J. T. Horwood, R. G. McLenaghan and R. G. Smirnov, *Invariant classification of orthogonally separable Hamiltonians systems in Euclidean space,* Comm. Math. Phys. **259** (2005), 679–709.

[13] T. A. Ivey and J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Forms.* - Providence: AMS, 2003.

[14] E. G. Kalnins, J. M. Kress and W. Miller, Jr., *Second order superintegrable systems in conformally flat spaces. IV. The classical 3D Stäckel transform and 3D classification theory*, J. Math. Phys. **47** (2006), 043514.

[15] E. G. Kalnins, J. M. Kress and W. Miller, Jr., *Second order superintegrable systems in conformally flat spaces. V: 2D and 3D quantum systems,* 37 pages, to appear in J. Math. Phys. (2006).

[16] I. A. Kogan, *Two algorithms for a moving frame construction,* Canad. J. Math. **55** (2003), 266–291.

[17] A. A. Makarov, Ya. A. Smorodinsky, Kh. Valiev and P. Winternitz, *A systematic approach for nonrelativistic systems with dynamical symmetries*, Nuovo Cim. **52** (1967) 1061–1084.

[18] R.G. McLenaghan, R.G. Smirnov and D. The, *Group invariant classification of separable Hamiltonian systems in the Euclidean plane and the $O(4)$-symmetric Yang-Mills theories of Yatsun*, J. Math. Phys. **43** (2002), 1422–1440.

[19] A. G. Nikitin and O. I. Prylypko, *Generalized Killing tensors and symmetry of Klein-Gordon equations*, www.arxiv.org/abs/math-ph/0506002, 1990.

[20] P.J. Olver, *Classical Invariant Theory* - Cambridge: Cambridge University Press, 1999

[21] J. A. Schouten, *Über Differentalkomitanten zweier kontravarianter Grössen,* Proc. Kon. Ned. Akad. Amsterdam **43** (1940), 449–452.

[22] R. W. Sharpe, *Differential Geometry. Cartan's Generalization of Klein's Erlangen Program.* - Springer, 1996.

[23] R. G. Smirnov, *The classical Bertrand-Darboux problem*, www.arxiv.org: math-ph/0604038, to appear in Fund. Appl. Math. (in Russian, 2006).

[24] R. G. Smirnov and P. Winternitz, *A class of superintegrable potentials of Calogero type,* www.arxiv.org: math-ph/0606006, to appear in J. Math. Phys. (2006).

[25] M. Takeuchi, *Killing tensor fields on spaces of constant curvature,* Tsukuba J. Math. **7** (1983), 233–255.

[26] G. Thompson, *Killing tensors in spaces of constant curvature*, J. Math. Phys. **27** (1986), 2693–2699.

[27] P. Winternitz and I. Friš, *Invariant expansions of relativistic amplitudes and subgroups of the proper Lorenz group* , Soviet J. Nuclear Phys. **1** (1965), 636–643.