Resolution on n-order functional - differential equations with operator coefficients and delay in Hilbert spaces

Chan Roath
Department of Scientific Research,
Ministry of Education, Youth and Sports,
Cambodia.
E-mail: chanroath@gmail.com

Introduction

Extensive development of differential equations theory related to the progress of natural science and technology. This theory was used in other fields such as physics, mechanics, engineering and economics. Particularly, the theory was applied in modern technology, involving phenomine like vibrating motion, automaticall system, remote control mechanics, electronic connections, etc.... . .

The differential-difference equation (DDE):

$$y'(x) = y(x-1). \tag{1}$$

was first reviewed in Condor city in 1771 in relation to the Euler problem, and had actually been studied in 1740. Later, common basic theory of differential equations with delayed variables and beginning handled problems was in Meuskic A.D.'s works "Differential Equations with delayed variables" in 1950. This theory has continuously progressed, as it was needed in developing natural science and technology.

The late in 1900s, many scholars have researched on abstract theory of functional-differential equations with delayed variables in Banach spaces. Many results on first order differential equations with delay are presented in the works of Dg. Xellon, R.G.Aliev, V.I. Xisan and others.

There is a book giving answer to many questions of modern theory of the functional-differential equation: the work of Dg. Xeisle. A research on a model of " the death of predator" and a study of Volterra vibrating provided as a common differential equations in the condition of preceded stage system.

In the late 1930s and 1940s, Minorsky worked on the problems related to the equilibrium situation of ship and automatically navigated operation.

This work showed clearly the importance of delayed mechanical systems of inverse direction.

Later, rapid progress of operator differential equation:

$$\sum_{j=0}^{m} A_j U^{(j)}(t) = f(t), \qquad m > 1 \tag{2}$$

in other spaces.

Equation (2) detailed on 1st and 2nd order with Cauchy Problem and could be solved.

These concepts appeared in the works of C.G.Kein, E. Xill and R. Fillipe, and the common results were directed from the researchs of E. Xille, R. Fillipe, K. Iosid, and T. Kato.

The first accomplishment was the existence theory for the Cauchy problem in equation (2) when m=1 and unbounded operator A in Banach space in term of semi-group operator.

Another accomplishment through T.Kato's work focused on the existence theory for the Cauchy problem :

$$X'(t) = A(t)X(t). \tag{3}$$

where $A(t)$ is an unbounded variable operator.

More class-operators-Cauchy Problems were developed by S.Agmon and L. Nirenberg; they formulated positive exponential asymptotic solutions. Later, A. Pasy extended this result to the applying it equations with negative exponential variable coefficients.

To sum it up, the above equations are functional-differential equations with operator coefficients and delay in Hilbert spaces in the form of :

$$D_t U(t) - \sum_{j=0}^{m} \left[A_j + A_j(t)\right] U(t - h_j - h_j(t)) = f(t) \tag{4}$$

which was first proposed by R.G. Aliev.

This work focussed on existence and uniqueness results for equation (4) under equilibrium conditions and asymptotic solution as $t \longrightarrow \infty$.

The main points to solve (4) were: Fourier Transform,Functional Analysis Methods, Inferential Methods, and the understanding of the previous stage of delay equations.

However, higher order of the equation types (4) have not been studied yet. Only the second-order equation:

$$D_t^2U(t) - \sum_{k=0}^{1}\sum_{j=0}^{m}\left[A_{kj} + A_{kj}(t)\right] S_{h_{kj}+h_{kj}(t)} D_t^k U(t) = f(t) \qquad (5)$$

has been analysed and solved by M. Asil in 1991.

Now we introduce the n-order functional-differential equation with operator coefficients and delay in Hilbert spaces:

$$L_{po}^n U(t) = f(t) \qquad (6)$$

$$D_t^n U(t) - \sum_{k=0}^{n-1}\sum_{j=0}^{m}\left[A_{kj} + A_{kj}(t)\right] S_{h_{k_j}+h_{k_j}(t)} D_t^k U(t) = f(t)$$

where $\quad L_{po}^n : X^{n,\alpha}_{R_+^{t_0}} \longrightarrow Y^{0,\alpha}_{R_+^{t_0}}, \quad D_t^k = \frac{d^k}{i^k dt^k}, \quad R_+^{t_0} = \{t \geq t_0\}$

$X^{n,\alpha}_{R_+^{t_0}}$ -Hilbert space, containing functions with norm:

$$\| U(t) \|^2 = \int_{t_0}^{\infty} \exp(2\alpha t)\left(\sum_{k=0}^{n-1} \| U^{(k)}(t) \|_x^2 + \| U^{(n)}(t) \|_y^2\right) dt,$$

$$t_0 \geq -\infty, \qquad \alpha \in \mathbb{R}$$

$Y^{0,\alpha}_{R_+^{t_0}}$ -Hilbert space, containing functions with norm:

$$\| U(t) \|^2 = \int_{t_0}^{\infty} \exp(2\alpha t) \| U(t) \|_y^2 \, dt, \quad t_0 \geq -\infty, \quad \alpha \in \mathbb{R}$$

$S_{h_{kj}+h_{kj}(t)} U(t) \triangleq U(t - h_{kj} - h_{kj}(t))$- operator translation.

A_{kj}	$-$abstract constant operator coefficient
$A_{kj}(t)$	$-$abstract variation operator coefficient
h_{kj}	$-$constant delayed variable
$h_{kj}(t)$	$-$variation delayed variable

Necessary and sufficient conditions for equation (6) to have unique solution are:

$$A_{kj}(t) \equiv 0, \quad h_{kj}(t) \equiv 0, \quad k \geq 0, \quad j \geq 0$$

In case the coefficients slightly variate, sufficiency is also satisfied.

We started with $t \geqslant t_0 > -\infty, \quad t \in \mathbb{R}$,a normal solution of (6) in entire Hilbert space.

(6) can be transformed to the first-order differential equation system (4)as in R.G. Aliev.

Initial Problem:

$x^{(n)}(t) = f(t, x(t), \ldots, x^{(n)}(t), x(t-\tau), x'(t-\tau), \ldots, x^{(n-1)}(t-\tau))$

$x^{(k)}(t_0 + 0) = \varphi_k(t_0)$

$x^{(k)}(t-\tau) = \varphi_(t-\tau), \quad t-\tau < t_0, \quad k = \overline{0, n-1}$

which involved with system of equations:

$\dot{x}_0(t) = x_1(t)$

$\dot{x}_1(t) = x_2(t)$

$\ldots\ldots$

$\ldots\ldots$

$\dot{x}_{n-2}(t) = x_{n-1}(t),$

$\dot{x}_{n-1}(t) = f(t, x_0(t), \ldots, x_{n-1}(t), x_0(t-\tau), \ldots, x_{n-1}(t-\tau))$

$x_k(t-\tau) = \varphi_k(t-\tau), \quad t-\tau < t_0, \quad k = \overline{0, n-1},$

is irreversible, meaning that result of first order differential equations systems are not equivalent to the initial nth-order functional-differential equations with operator coefficients and delay, as any given function $\varphi_k(t)$ for the n-order equation must correspond to the function $\varphi(t)$:

$$\varphi_{_k}(t) = \varphi^{(k)}(t), \quad k = \overline{0, n-1}.$$

Symbol and Definition

$\mathbf{X}, \mathbf{Y}$ - abstract Hilbert spaces with norms: $\|.\|_x, \quad \|.\|_y,$

$X \subset Y, \quad \|.\|_x \geq \|.\|_y, \quad \mathbb{R}_+^{t_0} = \{t \geq t_0\}, \quad \mathbb{R}_-^{t_0} = \{t \leq t_0\},$

$\mathbb{R}^n - n$-dimensional Euclidean space,

$L_p^n \equiv D_t^n - \sum_{k=0}^{n-1} \sum_{j=0}^{m} A_{kj} S_{h_{kj}} D_t^k,$

$L_{po}^n \equiv D_t^n - \sum_{k=0}^{n-1} \sum_{j=0}^{m} \left[A_{kj} + A_{kj}(t)\right] S_{h_{kj}+h_{kj}(t)} D_t^k$

$H(\mathbb{R})$ -set of absolute continuous functions h(t) with properties:

$$h'(t) \leq r < 1, \quad t \in \mathbb{R}$$

.

$\mathcal{L}(X, Y)$ - set of bounded operator from X to Y,

$\mathcal{L}_0(X, Y)$ - set of closed operator from X to Y,

$\mathcal{L}_\infty(X, Y)$ - set of strongly continuous operator from X to Y,

$\|.\|_x^{n,\alpha}$- symbol of norm in space $X_{R_+^{t_0}}^{n,\alpha}$,

$\|.\|_y^{0,\alpha}$- symbol of norm in space $Y_{R_+^{t_0}}^{0,\alpha}$,

$\chi_{_A}(S)$-characteristic function of operator A such that:

$$\|AU\|_y \leq S\|U\|_x + \chi_{_A}(S)\|U\|_y$$

$\tilde{U(\lambda)} = (\tilde{U(t)}) -$ Fourier transform of function U(t),

$\mathcal{F}(X, Y) -$ set of Fredholm operators from X to Y,

$\forall t \in \mathbb{R} \quad and \quad \lambda \in \mathbb{C}$, determined operator resolvent:

$$R_p(\lambda) \equiv (\lambda^n E - \sum_{k=0}^{n-1}\sum_{j=0}^{m} A_{kj}\lambda^k e^{-i\lambda h_{kj}})^{-1},$$

$$R_{po}(\lambda, t) \equiv \left(\lambda^n E - \sum_{k=0}^{n-1}\sum_{j=0}^{m} \left[A_{kj} + A_{kj}(t)\right] \lambda^k e^{-i\lambda(h_{kj}+h_{kj}(t))}\right)^{-1},$$

$$Condition \quad R^\alpha \quad (R^\alpha_-) \quad on \quad R(\lambda): \quad R(\lambda) - regular,$$

$$\|\lambda^k R(\lambda)\|_x = O(1), \quad k = \overline{0, n-1}, \quad \|\lambda^n R(\lambda)\|_y = O(1),$$

$$\mid \lambda \mid \to \infty, \quad \Im m\lambda = \alpha \quad (\Im m\lambda \leq \alpha)$$

Summary

This paper is concerned with the resolution of nth-order functional-differential equations with operator coefficients and delay in Hilbert spaces:

$$L^n_p U(t) = f(t) \qquad (1),$$

where $L^n_p \equiv D^n_t - \sum_{k=0}^{n-1}\sum_{j=0}^{m} A_{kj} S_{h_{kj}} D^k_t, \quad D^k_t = \frac{d^k}{i^k dt^k}$ and

$$L^n_{po} U(t) = f(t) \qquad (2)$$

where $L^n_{po} \equiv D^n_t - \sum_{k=0}^{n-1}\sum_{j=0}^{m} \left[A_{kj} + A_{kj}(t)\right] S_{h_{kj}+h_{kj}(t)} D^k_t$

Problem:

Determine the conditions on:

$R_p(\lambda), \quad R_{po}(\lambda, t), \quad A_{kj}, \quad A_{kj}(t), \quad h_{kj}, \quad h_{kj}(t), \quad k = \overline{0, n-1}, \quad j = \overline{0, m}$,

such that the operator:

$$L^n_p, \quad L^n_{po} \quad : X^{n,\alpha}_{R^{t_0}_+} \longrightarrow Y^{0,\alpha}_{R^{t_0}_+}, \quad t_0 \geqslant -\infty$$

be continuous. And determine the conditions under which is a Fredholm operator:

$$L^n_{po} \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow Y^{0,\alpha}_{\mathbb{R}}.$$

First step, solve the following problem:

$$L_p^n U(t) = f(t) \tag{1}$$

We prove the theorem of existence and uniqueness of equation (1) with operator coefficients $A_{kj} \in \mathcal{L}(X,Y)$ and constant delayed variable $h_{kj}, h_{kj} \geqslant 0, t_0 \geqslant -\infty, k = \overline{0, n-1}, j = \overline{0, m}$.

Theorem 1.1: The necessary and sufficient conditions for the operator

$$L_p^n \quad : X^{n,\alpha}_{R_+^{t_0}} \longrightarrow Y^{0,\alpha}_{R_+^{t_0}}, \quad t_0 = -\infty, \quad (t_0 > -\infty)$$

to be continuous and inverse, is that $R_p(\lambda)$ satisfies the condition $R^\alpha \quad (R_-^\alpha)$.

Theorem 1.2: If $R_p(\lambda)$ satisfies the condition R_-^α and$f(t) = 0$,

$$t \leq t_0, \quad f(t) \in L^2(\Delta, Y), \quad \Delta \in R_+^{t_0},$$

equation (2)has a unique solution $U(t)$, that satisfies $U(t) = 0, t \leq t_0$.

The following two examples are presented to illustrate the above theorems:

Example 1

$$U^n(t) - U(t) = f(t), \quad L_p^n = \frac{d^k}{i^k dt^k} - 1, \quad R_p(\lambda) = [(i\lambda)^n - 1]^{-1} \tag{1}$$

Example 2

$$\frac{\partial^2 U(t,x)}{\partial t^2} - \frac{\partial^4 U(t,x)}{\partial x^4} - bU(t-h,x) = f(t,x) \tag{2}$$

Next, solve the problem:

$$L_{po}^n U(t) = f(t) \tag{2}$$

by proving the theorem of existence and uniqueness of equation (2) with variation operator coefficients and variating delay under the hypothesis that $A_{kj}(t)$, $h_{kj}(t)$, $k = \overline{0, n-1}$, $j = \overline{0, m}$. one small enough.

Theorem 2.1: If the following conditions are satisfied:

$a). \quad A_{kj} \in \mathcal{L}_0(Y,Y)\bigcap \mathcal{L}_\infty(X,Y), \quad k=\overline{0,n-1}, \quad j=\overline{1,m}.$

$b). \quad R_p(\lambda)$ satisfies conditions:

$$R^\alpha, \quad (R_-^\alpha), \quad h_{kj}+h_{kj}(t)\geq 0, \quad k=\overline{0,n-1}, \quad j=\overline{0,m}.$$

$\exists\varepsilon>0, \quad if \quad \|A_{kj}(t)\|_y \leq \varepsilon, \quad |h_{kj}(t)|\leq\varepsilon, \quad t\in\mathbb{R}, \quad h_{kj}(t)\in H(\mathbb{R})$

$k=\overline{0,n-1}, \quad j=\overline{0,m},$ then the operator

$$L^n_{po} \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow \ Y^{0,\alpha}_{\mathbb{R}}, \quad t_0=-\infty, \quad (t_0>-\infty)$$

is continuous and inverse.

Theorem 2.2: If it satisfies the following conditions:

$a). \quad A_{kj} \in \mathcal{L}_0(Y,Y)\bigcap \mathcal{L}_\infty(X,Y), \quad k=\overline{0,n-1}, \quad j=\overline{1,m}.$

$b). \quad R_p(\lambda),$ satisfies the conditions: $R_-^\alpha, \quad h_{kj}+h_{kj}(t)\geq 0,$

$k=\overline{0,n-1}, \quad j=\overline{0,m}.$

$c). \quad f(t)=0, \quad t\leq t_0, \quad f(t)\in Y^{0,\alpha}_\Delta, \quad \Delta\subset R^{t_0}_+,$

$d). \quad h_{kj}(t)\in H(\mathbb{R}), \quad k=\overline{0,n-1}, \quad j=\overline{0,m}.$

$\exists\varepsilon>0, \quad if \quad \|A_{kj}(t)\|_y \leq \varepsilon, \quad |h_{kj}(t)|\leq\varepsilon, \quad t\in\mathbb{R}, \quad k=\overline{0,n-1},$

$j=\overline{0,m},$ then equation (2) has existed unique solution $U(t)$,

that satisfies $U(t)=0$, $t\leq t_0$.

The last, we find the condition of Fredholm operator

$$L^n_{po} \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow \ Y^{0,\alpha}_{\mathbb{R}}, \tag{3}$$

and condition that operator $(L^n_{po}-i\gamma)$ has zero indices, by proving the theorem of finite number of kernel and cokernel of operator L^n_{po}.

Theorem 3.1.1: If it satisfies the following conditions:

$a). A_{kj} \in \mathcal{L}_0(Y,Y)\bigcap \mathcal{L}_\infty(X,Y), \exists \lim\limits_{|t|\to\infty} \|A_{kj}(t)\|_y=0, \exists \lim\limits_{|t|\to\infty} |h_{kj}(t)|=0,$

$h_{kj}(t)\in H(\mathbb{R}), \quad k=\overline{0,n-1}, \quad j=\overline{0,m}.$

$b). \quad R_p(\lambda),$ satisfies the conditions: $R^\alpha,$

kernel of operator $L^n_{po} \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow \; Y^{0,\alpha}_{\mathbb{R}}$ is finite number.

Theorem 3.1.2: If it satisfies the following conditions:

$a). \quad A_{kj} + A_{kj}(t) \in \mathcal{L}_0(Y,Y) \bigcap \mathcal{L}_\infty(X,Y), \quad t \in \mathbb{R}, \quad k = \overline{0, n-1},$

$j = \overline{1,m}, \quad \exists \lim\limits_{|t|\to\infty} \|A_{kj}(t)\|_y = 0, \quad \exists \lim\limits_{|t|\to\infty} |h_{kj}(t)| = 0, \quad h_{kj}(t) \in H(\mathbb{R}),$

$A_{kj}(t)$ in term of $t \in \mathbb{R}, \quad k = \overline{0, n-1}, \quad j = \overline{0, m}.$

b). $R_p(\lambda)$ and $R_{po}(\lambda, t) \equiv [\lambda^n E - \sum\limits_{k=0}^{n-1} \lambda^k (A_{ko} + A_{ko}(t))]^{-1}.$

all finite, $\forall t \in \mathbb{R}$ satisfy the condition R^α,

Then cokernel of operator,

$$L^n_{po} \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow \; Y^{0,\alpha}_{\mathbb{R}}, \quad finite \quad number.$$

Consequences 3.1.1: If it satisfies the following conditions:

$a). \quad A_{kj} + A_{kj}(t) \in \mathcal{L}_0(Y,Y) \bigcap \mathcal{L}_\infty(X,Y), \quad \forall t \in \mathbb{R}, \quad k = \overline{0, n-1},$

$j = \overline{1,m}, \quad \exists \lim\limits_{|t|\to\infty} \|A_{kj}(t)\|_y = 0, \quad \exists \lim\limits_{|t|\to\infty} |h_{kj}(t)| = 0, \quad h_{kj}(t) \in H(\mathbb{R}),$

$A_{kj}(t), h_{kj}(t)$- continuous and in term of $t \in \mathbb{R}, k = \overline{0, n-1}, j = \overline{0, m}.$

b). $R_p(\lambda)$ and $R_{po}(\lambda, t) \equiv [\lambda^n E - \sum\limits_{k=0}^{n-1} \lambda^k (A_{ko} + A_{ko}(t))]^{-1}.$

all finite $t \in \mathbb{R}$ satisfy the condition R^α,

c). $f(t) \in Y^{0,\alpha}_R.$

$$\text{Then operator} \quad L^n_{po} \in F(X^{n,\alpha}_R, Y^{0,\alpha}_R).$$

The end, proving the theorem about the number of kernels and cokernels of operators

$$(L^n_{po} - i\gamma) \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow \; Y^{0,\alpha}_{\mathbb{R}}, \quad equal \quad to \quad zero.$$

Theorem 3.2.1: If it satisfies the following conditions:

$a).A_{kj} \in \mathcal{L}_0(Y,Y) \bigcap \mathcal{L}_\infty(X,Y), k = \overline{0, n-1}, j = \overline{1,m}, \exists \lim\limits_{|t|\to\infty} \|A_{kj}(t)\|_y = 0,$

$\exists \lim_{|t|\to\infty} |h_{kj}(t)| = 0, \quad h_{kj}(t) \in H(\mathbb{R}), \quad k = \overline{0, n-1}, \quad j = \overline{0, m}.$

$b). \quad R_p(\lambda, \gamma) \equiv [(\lambda^n - i\gamma)E - \sum_{k=0}^{n-1}\sum_{j=0}^{m} \lambda^k A_{kj} e^{-ih_{kj}}]^{-1},$

satisfies the conditions: R^α, and $\lim_{\gamma\to\infty} \|R_p(\lambda,\gamma)\|_x = 0, \quad \Im m\lambda = \alpha,$

Then kernel of operator $(L^n_{po} - i\gamma) \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow Y^{0,\alpha}_{\mathbb{R}},$ is equal to zero as γ is large enough.

Theorem 3.2.2: If it satisfies the following conditions:

$a). \quad A_{kj} + A_{kj}(t) \in \mathcal{L}_0(Y,Y) \bigcap \mathcal{L}_\infty(X,Y), \quad \forall t \in \mathbb{R}, \quad k = \overline{0, n-1},$

$j = \overline{1, m}, \quad \exists \lim_{|t|\to\infty} \|A_{kj}(t)\|_y = 0, \quad \exists \lim_{|t|\to\infty} |h_{kj}(t)| = 0, \quad h_{kj}(t) \in H(\mathbb{R}),$

$A_{kj}(t)$ -continuous and in term of $t \in \mathbb{R}, \quad k = \overline{0, n-1}, \quad j = \overline{0, m}.$

$b). \quad R_p(\lambda, \gamma) \equiv [(\lambda^n - i\gamma)E - \sum_{k=0}^{n-1}\sum_{j=0}^{m} \lambda^k A_{kj} e^{-i\lambda h_{kj}}]^{-1}$ and $R_{po}(\lambda, t, \gamma) \equiv$

$[(\lambda^n - i\gamma)E - \sum_{k=0}^{n-1} \lambda^k (A_{ko} + A_{ko}(t))]^{-1}.$

all finite, $\forall t \in \mathbb{R}$ satisfy the condition R^α, and

$\lim_{\gamma\to\infty} \|R_p(\lambda,\gamma)\|_x = 0, \quad \lim_{\gamma\to\infty} \|R_{po}(\lambda, t, \gamma)\|_x = 0, \quad \Im m\lambda = \alpha, \quad t \in \mathbb{R},$

c). $\quad f(t) \in Y^{0,\alpha}_R.$

Then the cokernel of operator $(L^n_{po} - i\gamma) \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow Y^{0,\alpha}_{\mathbb{R}},$

is equal to zero as γ is large enough.

Consequences 3.2.1: If it satisfies the following conditions:

$a). \quad A_{kj} + A_{kj}(t) \in \mathcal{L}_0(Y,Y) \bigcap \mathcal{L}_\infty(X,Y), \quad \forall t \in \mathbb{R}, \quad k = \overline{0, n-1},$

$j = \overline{1, m}, \quad \exists \lim_{|t|\to\infty} \|A_{kj}(t)\|_y = 0, \quad \exists \lim_{|t|\to\infty} |h_{kj}(t)| = 0, \quad h_{kj}(t) \in H(\mathbb{R}),$

$A_{kj}(t)$ -continuous and in term of $t \in \mathbb{R}$, $\quad k = \overline{0, n-1}$, $\quad j = \overline{0, m}$.

b). $\forall t \in \mathbb{R}$, $\quad R_p(\lambda, \gamma) \quad and \quad R_{po}(\lambda, t, \gamma)$ satisfy the condition R^α, and

$\exists \lim_{\gamma \to \infty} \|R_p(\lambda, \gamma)\|_x = 0, \exists \lim_{\gamma \to \infty} \|R_{po}(\lambda, t, \gamma)\|_x = 0, \Im m\lambda = \alpha, t \in \mathbb{R},$

$Then \quad indices \quad of \quad operator \quad (L^n_{po} - i\gamma) \quad : X^{n,\alpha}_{\mathbb{R}} \longrightarrow Y^{0,\alpha}_{\mathbb{R}}$,

is equal to zero .

References

[1] Agmon S., Nirenberg L. Properties of solutions of ordinary differential equations in Banach space. Comm.appl.math., 1963, V.XVI, p.121-139.

[2] Kato T. On linear differential equations in Banach spaces. Comm. On Pure and Appl. Math., 1956, V.9, p.479-486.

[3] Minorsky. Self-excited oscillations in dynamical systems prossessing retard actions. J.Appl.Mech.9 (1942), p67-71.

[4] Pasy A. Asymptotic expansions of the solutions of ordinary differential equations in Hilbert space. Arch.Rat.Mech. and Amal. 24.3 (1967), p. 193-218.

[5] Volterra. Sur la theorie mathematique des phenomenes hereditaires. J.Math., Pure and Appl. 7 (1928), p. 249-298.