Separation of variables for systems of first-order PDE's: the Dirac equation in two-dimensional manifolds

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People involved in separation of Dirac Equation:

Wheeler, Chandrasekhar, Carter, McLenaghan,
Spindel, Miller, Shapovalov, Kalnins,
Bagrov, Shishkin, Villalba, Fels,
Kamran,

and apologies...

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The Dirac equation in dimension two

Following the notations of [Fatibene, McLenaghan,Smith], let us consider on a Riemannian two-dimensional manifold the Dirac equation

$$D\psi - \lambda\psi = i\tilde{\gamma}^{\mu}\nabla_{\mu}\psi - \lambda\psi = 0$$

with

$$\tilde{\gamma}^{\mu} = \gamma^{a} e^{\mu}_{\ a}$$

where

$$\gamma^{1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

such that

$$\gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a} = 2\eta_{ab} I$$

is the Dirac representation, $(e^{\mu}_{\ a})$ a spin frame associated to the metric

$$g_{\mu\nu} = e_{\mu}^{\ a}\eta_{ab}e_{\nu}^{\ b}$$

$\eta_{ab} = \delta_{ab}$ and

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi,$$

with

$$\Gamma_\mu = \frac{1}{8}[\gamma_a, \gamma_b]\Gamma_\mu^{ab} = \frac{1}{4}\gamma_a \gamma_b \Gamma_\mu^{ab},$$

where $\Gamma_\mu^{ab} = e_\alpha{}^a(\Gamma^\alpha_{\beta\mu} e^{\beta b} + \partial_\mu e^{\alpha b})$ and therefore

$$\Gamma_\mu^{ab} = -\Gamma_\mu^{ba}.$$

We have

$$\tilde{\gamma}^1 = \begin{pmatrix} e^1{}_1 & -ie^1{}_2 \\ ie^1{}_2 & -e^1{}_1 \end{pmatrix}, \qquad \tilde{\gamma}^2 = \begin{pmatrix} e^2{}_1 & -ie^2{}_2 \\ ie^2{}_2 & -e^2{}_1 \end{pmatrix}$$

and

$$\Gamma_\mu = \frac{1}{2}\begin{pmatrix} 0 & -i\Gamma_\mu^{12} \\ -i\Gamma_\mu^{12} & 0 \end{pmatrix}$$

moreover, by putting

$$\begin{cases} X = -2e^\mu{}_2 \Gamma_\mu^{12} \\ Y = -2e^\mu{}_1 \Gamma_\mu^{12} \end{cases}$$

we have

$$i\tilde{\gamma}^\mu \Gamma_\mu = \frac{1}{4}\begin{pmatrix} iX & -Y \\ Y & -iX \end{pmatrix}.$$

Then, the Dirac equation becomes

$$\mathbf{D}\psi - \lambda\psi = \tilde{\mathbf{A}}\partial_x\psi + \tilde{\mathbf{B}}\partial_y\psi + \tilde{\mathbf{C}}\psi - \lambda\psi = 0$$

where

$$\tilde{\mathbf{A}} = \begin{pmatrix} ie^1{}_1 & e^1{}_2 \\ -e^1{}_2 & -ie^1{}_1 \end{pmatrix}$$

$$\tilde{\mathbf{B}} = \begin{pmatrix} ie^2{}_1 & e^2{}_2 \\ -e^2{}_2 & -ie^2{}_1 \end{pmatrix}$$

$$\tilde{\mathbf{C}} = i\tilde{\gamma}^\mu\Gamma_\mu = \frac{1}{4}\begin{pmatrix} iX & -Y \\ Y & -iX \end{pmatrix}$$

and

$$\begin{cases} X = -2e^\mu{}_2\Gamma^{12}_\mu = -2\Gamma_{212} \\ Y = -2e^\mu{}_1\Gamma^{12}_\mu = -2\Gamma_{112}. \end{cases}$$

Orthonormal moving frame

$$\begin{cases} E_a = e^{\mu}{}_a \dfrac{\partial}{\partial x^{\mu}} \\ E^a = \hat{e}_{\mu}{}^a dx^{\mu} \end{cases}$$

The structure functions C^a_{bc} are defined by

$$\begin{cases} dE^a = \dfrac{1}{2} C^a_{bc} E^b \wedge E^c \\ [E_a, E_b] = C^c_{ab} E_c \end{cases}$$

and connection coefficients by

$$\begin{cases} \nabla_{E_a} E_b = \Gamma^c_{ba} E_c \\ \nabla_{E_a} E^b = -\Gamma^b_{ca} E^c \end{cases}$$

with

$$\Gamma_{abc} = \frac{1}{2} \left(C_{cab} - C_{abc} - C_{bca} \right)$$

then, Γ_{abc} are defined by the spin frame only.

Coordinates separating Dirac equation are associated with separation of Hamilton-Jacobi and Schrödinger equations via Klein-Gordon equation [Miller, Kalnins, Smith, ...]
and not [Varaksin, Shirokof, ...].
Coordinates (q^i) separate additively H-J equation $g^{ij}\partial_i W \partial_j W = h$ associated to $H = g^{ij} p_i p_j$ (i.e. there exists a complete solution $W = W_1(q^1) + \cdots + W_n(q^n)$) if and only if the Levi-Civita equations hold:

$$\partial_i H \partial_j H \partial^{ij} H + \partial^i H \partial^j H \partial_{ij} H - \partial_i H \partial^j H \partial_j^i H - \partial^i H \partial_j H \partial_i^j H = 0$$

where indices are not summed and $\partial_i = \partial/\partial q^i$, $\partial^i = \partial/\partial p_i$.
If q^i is such that

$$\frac{\partial_i H}{\partial^i H} = Q^j(q^k) p_j$$

it is said of first class and if $\partial_i H = 0$ it is ignorable. Ignorable coordinates are associated with Killing vectors (isometries). In HJ theory of SoV, first class coordinates always can be transformed into ignorable without changing the separability structure.

Multiplicative separation of eigenvalue-type PDE systems

Let (x, y) a (local) coordinate system on a two dimensional manifold and

$$\psi = \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{pmatrix}.$$

Let **D** the operator defined by

$$\mathbf{D} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \partial_x + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \partial_y + \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where A_i, B_i and C_i are functions of (x, y), such that

$$\mathbf{D}\,\psi = \begin{pmatrix} A_1\partial_x\psi_1 + A_2\partial_x\psi_2 + B_1\partial_y\psi_1 + B_2\partial_y\psi_2 + C_1\psi_1 + C_2\psi_2 \\ A_3\partial_x\psi_1 + A_4\partial_x\psi_2 + B_3\partial_y\psi_1 + B_4\partial_y\psi_2 + C_3\psi_1 + C_4\psi_2 \end{pmatrix}.$$

Let $\lambda \neq 0$ a real or complex number, we consider the equation

$$\mathbf{D}\psi - \lambda\psi = 0.$$

If we assume multiplicative separation for ψ, i.e.

$$\psi_i = a_i(x)b_i(y)$$

then the equation $\mathbf{D}\psi - \lambda\psi = 0$ becomes

$$\begin{cases} A_1\dot{a}_1 b_1 + A_2\dot{a}_2 b_2 + B_1 a_1\dot{b}_1 + B_2 a_2\dot{b}_2 + (C_1 - \lambda)a_1 b_1 + C_2 a_2 b_2 = 0 \\ A_3\dot{a}_1 b_1 + A_4\dot{a}_2 b_2 + B_3 a_1\dot{b}_1 + B_4 a_2\dot{b}_2 + C_3 a_1 b_1 + (C_4 - \lambda)a_2 b_2 = 0 \end{cases}$$

hence, we define separability of $\mathbf{D}$ as follows:

The operator **D** is separate in (x, y) if there exist nonzero functions $R_i(x, y)a_r b_s$ such that the above equations can be written as:

$$\begin{cases} R_1 a_r b_s (E_1^x + E_1^y) = 0 \\ R_2 a_t b_u (E_2^x + E_2^y) = 0 \end{cases}$$

where $E_i^x(x, a_j, \dot{a}_j)$, $E_i^y(y, b_j, \dot{b}_j)$.

$$E_i^x = \mu_i = -E_i^y$$

provide the separation constants μ_i.

Operators directely generated by separated equations

In the Miller-Shapovalov theory first-order symmetry operators were employed to characterize separation, but later some separation was found [Fels, Kamran 1989] associated only to second-order symmetry operators. In all cases the operators were strictly related to separated equations for Dirac operator.
Following these paths we can build eigenvalue-type operators **L** making use of the terms E_i^x and E_i^y so that $\mathbf{L}\psi = \mu\psi$ with $\mu(\mu_i)$ constant.
Searching for first-order operators, we impose $\mu_1 = \mu_2 = \mu$ in the separated equations.
The requirement that the operator **L** is independent of λ (otherwise we are considering some kind of "fixed energy separation") imposes the employement, for the construction of **L**, of those terms only among E_i^x or E_i^y independent of λ.
We require $[\mathbf{L}, \mathbf{D}] = 0$.

1- Three kinds of separation

Separation ansatz

$$\psi = \begin{pmatrix} a_1(x)b_1(y) \\ a_2(x)b_2(y) \end{pmatrix}$$

I. $a_1 \neq a_2$ and $b_1 \neq b_2$.

II. $a_1 = a_2 = a$ and $b_1 \neq b_2$ (or vice-versa).

III. $a_1 = a_2 = a$ and $b_1 = cb_2 = b$ (c constant).

We do not consider reduced separation such as assuming from the beginning $a_1 = e^x$, $a_1 = x^2 - 1$, etc.

Type I

We assume at first $R_i = 1$. Because of the term $\lambda\psi_i$, one at least of the indices $a_l b_s$ factorizing the $i-th$ equation must be equal to i. Then, all possible schemes of separation arise by multiplying the system by the matrix

$$\mathbf{T} = \begin{pmatrix} \dfrac{1}{a_l b_m} & 0 \\ 0 & \dfrac{1}{a_p b_q} \end{pmatrix}$$

where one of l, m must be 1 and one of p, q must be 2. According to the allowed 4-tuples of indices (l, m, p, q), all the possible separation schemes are the following nine:

$$\begin{cases} (1,1,1,2) & (1,1,2,1) & (1,1,2,2) \\ (1,2,1,2) & (1,2,2,1) & (1,2,2,2) \\ (2,1,1,2) & (2,1,2,1) & (2,1,2,2) \end{cases}$$

The separation conditions $(\mathbf{D} - \lambda)_i \psi = a_l b_m (E_i^x + E_i^y)$ impose restrictions on functions A_i, B_i and C_i that, corresponding to the different cases, determine respectively the separate form of $\mathbf{D}$:

$$\mathbf{D}_1 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_2 : \begin{pmatrix} A_1(x) & 0 \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_3 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y +$$

$$+ \begin{pmatrix} C_{11}(x) + C_{12}(y) & 0 \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix},$$

$$\mathbf{D}_4 : \begin{pmatrix} 0 & A_2(x) \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_5 : \begin{pmatrix} 0 & A_2(x) \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_6 : \begin{pmatrix} 0 & A_2(x) \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & 0 \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_2(x) \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix},$$

$$\mathbf{D}_7 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ B_3(y) & 0 \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ C_3(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_8 : \begin{pmatrix} A_1(x) & 0 \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ C_3(x) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_9 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} 0 & B_2(y) \\ 0 & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(x) & C_2(y) \\ 0 & C_{41}(x) + C_{42}(y) \end{pmatrix}.$$

After exchanging x and y some of the previous operators coincide, namely $\mathbf{D}_1 \equiv \mathbf{D}_2$, $\mathbf{D}_4 \equiv \mathbf{D}_8$, $\mathbf{D}_5 \equiv \mathbf{D}_7$, $\mathbf{D}_6 \equiv \mathbf{D}_9$ and $\mathbf{D}_3 \equiv \mathbf{D}_3$. Moreover, by introducing the operator

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

such that

$$\mathbf{J} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

we observe that $\mathbf{D}_1\psi$ is of the same form as $\mathbf{J}\mathbf{D}_9\mathbf{J}\psi$, then we can consider $\mathbf{D}_1$ and $\mathbf{D}_9$ as equivalent in the following discussion. Thus we have the four distinct classes represented by $\mathbf{D}_1$, $\mathbf{D}_3$, $\mathbf{D}_4$ and $\mathbf{D}_5$. By writing $\mathbf{T}(\mathbf{D}_i\psi - \lambda\psi)$ as

$$\begin{pmatrix} E_x^1 + E_y^1 \\ E_x^2 + E_y^2 \end{pmatrix}$$

and separating them as

$$E_x^i = \mu_i = -E_y^i,$$

I) The class $\mathbf{D}_1$:
Let us consider $\mathbf{D}_1\psi - \lambda\psi = 0$, assuming here and in the following that $\lambda \neq 0$, the first component can be separated as

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 - \lambda a_1 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 = -\mu_1 b_1 \end{cases}$$

or

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 - \lambda b_1 = -\mu_1 b_1 \end{cases}$$

according to alternative grouping of λ with terms in x or y. The second component reads

$$\begin{cases} A_4(x)\dot{a}_2 + C_4(x)a_2 - \lambda a_2 = \mu_2 a_1 \\ B_3(y)\dot{b}_1 + C_3(y)b_1 = -\mu_2 b_2 \end{cases}$$

where μ_1 and μ_2 are the separation constants.
Separated equations can be decoupled by integrating a_1 and b_1 from the first two and substituting the results in the last. The solutions a_i, b_i are in all cases given by first-order ODE's.

By using the only terms independent of λ and assuming $\mu_1 = \mu_2$ one can try to build an operator $\mathbf{L}$. However, this is impossible under the assumption of independence among a_i and b_i. The same for higher-order operators, when we can assume $\mu_1 \neq \mu_2$. It follows that no eigenvalue-operator is associated with the $\mathbf{D}_1$ separation scheme.

II) The class $\mathbf{D}_3$.
By considering equation $\mathbf{D}_3\psi - \lambda\psi = 0$ we obtain the following four systems of separation equations, according to the possible different groupings of λ:

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 - \lambda a_1 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 = -\mu_1 b_1 \end{cases}$$

$$\begin{cases} A_1(x)\dot{a}_1 + C_{11}(x)a_1 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_{12}(y)b_1 - \lambda b_1 = -\mu_1 b_1 \end{cases}$$

$$\begin{cases} A_4(x)\dot{a}_2 + C_{41}(x)a_2 - \lambda a_2 = \mu_2 a_2 \\ B_4(y)\dot{b}_2 + C_{42}(y)b_2 = -\mu_2 b_2 \end{cases}$$

$$\begin{cases} A_4(x)\dot{a}_2 + C_{41}(x)a_2 = \mu_2 a_2 \\ B_4(y)\dot{b}_2 + C_{42}(y)b_2 - \lambda b_2 = -\mu_2 b_2 \end{cases}$$

All equations are decoupled in a_i, b_i and solutions are always given by first-order ODE's.

By putting $\mu_1 = \mu_2 = \mu$ we can obtain from the previous systems the following couples of equations respectively, suitable for the construction of operators:

$$\begin{cases} (B_1\partial_y + C_{12})b_1 = -\mu b_1 \\ (B_4\partial_y + C_{42})b_2 = -\mu b_2 \end{cases}$$

$$\begin{cases} (B_1\partial_y + C_{12})b_1 = -\mu b_1 \\ (A_4\partial_x + C_{41})a_2 = \mu a_2 \end{cases}$$

$$\begin{cases} (A_1\partial_x + C_{11})a_1 = \mu a_1 \\ (B_4\partial_y + C_{42})b_2 = -\mu b_2 \end{cases}$$

$$\begin{cases} (A_1\partial_x + C_{11})a_1 = \mu a_1 \\ (A_4\partial_x + C_{41})a_2 = \mu a_2 \end{cases}$$

The corresponding operators of the form $\mathbf{L}\psi = \mu\psi$ are then respectively

$$\mathbf{L}_1 = -\begin{pmatrix} B_1\partial_y + C_{12} & 0 \\ 0 & B_4\partial_y + C_{42} \end{pmatrix}$$

$$\mathbf{L}_2 = \begin{pmatrix} -B_1\partial_y - C_{12} & 0 \\ 0 & A_4\partial_x + C_{41} \end{pmatrix}$$

$$\mathbf{L}_3 = \begin{pmatrix} A_1\partial_x + C_{11} & 0 \\ 0 & -B_4\partial_y - C_{42} \end{pmatrix}$$

$$\mathbf{L}_4 = \begin{pmatrix} A_1\partial_x + C_{11} & 0 \\ 0 & A_4\partial_x + C_{41} \end{pmatrix}$$

An easy computation shows that all these operators commute with $\mathbf{D}_3$ and between themselves when applied to some generic ψ. The same holds for the powers of the $\mathbf{L}_i$.

III) The class $\mathbf{D}_4$.
We have for $\mathbf{D}_4\psi - \lambda\psi = 0$ the following separated equations:

$$\begin{cases} A_2(x)\dot{a}_2 + C_2(x)a_2 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_1(y)b_1 - \lambda b_1 = -\mu_1 b_2 \end{cases}$$

$$\begin{cases} A_4(x)\dot{a}_2 + C_4(x)a_2 - \lambda a_2 = \mu_2 a_1 \\ B_3(y)\dot{b}_1 + C_3(y)b_1 = -\mu_2 b_2 \end{cases}$$

The equations can be decoupled by substitution.

The only couple of equations suitable to define an operator **L** is

$$\begin{cases} (A_2\partial_x + C_2)a_2 = \mu_1 a_1 \\ (B_3\partial_y + C_3)b_1 = -\mu_2 b_2 \end{cases}$$

It follows

$$\mathbf{L} = \begin{pmatrix} 0 & A_2\partial_x + C_2 \\ -B_3\partial_y - C_3 & 0 \end{pmatrix}$$

However, the requirement $[\mathbf{L}, \mathbf{D}_4]\psi = 0$ for every ψ implies $B_1 = B_3 = A_2 = A_4 = 0$. Then, no differential operator of this kind can exist provided a_i and b_i are independent. Moreover, no higer-order operator can be constructed. No operator can be associated with $\mathbf{D}_4$ assuming type I separation.

IV) The class $\mathbf{D}_5$.
The separation of $\mathbf{D}_5\psi - \lambda\psi = 0$ is given by

$$\begin{cases} A_2(x)\dot{a}_2 + C_2(x)a_2 = \mu_1 a_1 \\ B_1(y)\dot{b}_1 + C_1(y)b_1 - \lambda b_1 = -\mu_1 b_2 \end{cases}$$

$$\begin{cases} A_3(x)\dot{a}_1 + C_3(x)a_1 = \mu_2 a_2 \\ B_4(y)\dot{b}_2 + C_4(y)b_2 - \lambda b_2 = -\mu_2 b_1 \end{cases}$$

Equations in a_i can be decoupled by substituting, for instance, a_1 and $\dot{a}_1$ given by the first equation of the first system and its derivative with respect to x into the left-hand term of the remaining equation yielding a second-order ODE in a_2 only. Solutions for a_i are therefore given by second-order ODE's. The same procedure allows us to decouple b_i.

It is easy to verify that no first-order operator independent of λ can be defined by using the previous equations.
Higher order operators:
we have immediately

$$\begin{cases}(A_2\partial_x + C_2)(A_3\partial_x + C_3)a_1 = \mu_1\mu_2 a_1\\ (A_3\partial_x + C_3)(A_2\partial_x + C_2)a_2 = \mu_1\mu_2 a_2\end{cases}$$

Then, for the operator

$$\mathbf{L}_5 = \begin{pmatrix}(A_2\partial_x + C_2)(A_3\partial_x + C_3) & 0\\ 0 & (A_3\partial_x + C_3)(A_2\partial_x + C_2)\end{pmatrix}$$

we have

$$\mathbf{L}_5\psi = \mu_1\mu_2\psi$$

and for its powers

$$\mathbf{L}_5^h\psi = (\mu_1\mu_2)^h\psi$$

for any positive integer h. We have $[\mathbf{D}_5, \mathbf{L}_5] = 0$, as do the powers $\mathbf{L}_5^h$.
If we call μ the eigenvalue of $\mathbf{L}$ associated with ψ, from $\mu = \mu_1\mu_2$ we obtain as separation constants, for example, μ_1 and μ/μ_1.

The only cases allowing complete type I separation for the functions a_i, b_j occur for the class $\mathbf{D}_3$, with four independent first-order operators and their powers, and for the class $\mathbf{D}_5$, where there is one order-two operator and its powers.

Let now $R_i \neq 1$. The separated equations obtained from $\mathbf{D}$ are of the same form as seen above, except for the terms including λ which becomes now λ/R_i, but this do not change the above analysis. Recalling that each R_i depends on only one coordinate, we see that operators $\mathbf{L}$ built as above from separated equations do not contain the functions R_i, moreover, terms in $\mathbf{L}$ arising from the same separate equation depend on the coordinate different from that appearing in the factor R_i. Then, the operators $\mathbf{L}$ are the same we obtained above discussing the case $R_i = 1$, while the operators $\mathbf{D}$ are consequently in the form

$$\mathbf{D}_k = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{D}'_k$$

where the $\mathbf{D}'_k$ are the operators seen in the previous section.
We analyze now the cases corresponding to the existence of operators associated with separation.

Let $\mathbf{D} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{D}_3$. According to the different operators $\mathbf{L}_1, \ldots, \mathbf{L}_4$ we have respectively

$$R_1(x), R_2(x)$$

$$R_1(x), R_2(y)$$

$$R_1(y), R_2(x)$$

$$R_1(y), R_2(y)$$

Since $\mathbf{D}$ and $\mathbf{L}_i$ are diagonalized simultaneously, it is easy to verify that $\mathbf{L}_i\mathbf{D} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{L}_i\mathbf{D}_3$ and $\mathbf{D}\mathbf{L}_i = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{D}_3\mathbf{L}_i$. Then the operators $\mathbf{L}_i$ still commute with $\mathbf{D}$.

For $\mathbf{D} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \mathbf{D}_4$ we should have

$$R_1(y),\ R_2(x)$$

By imposing $[\mathbf{L}_i, \mathbf{D}] = 0$, again we have that there is no differential operator $\mathbf{L}_i$ of the required form satisfying the previous equation, whatever R_i can be.
In a similar way we can see for $\mathbf{D} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} D_5$ that for the associated operator $\mathbf{L}_5$ described above we must have

$$R_1(y),\ R_2(y)$$

and again $[\mathbf{L}_5, \mathbf{D}] = 0$.
Then: in dimension two, the separated forms of the operator $\mathbf{D}$ associated with commuting differential operators are given by $\mathbf{D}_3$ and $\mathbf{D}_5$.

The Dirac case

By considering all possible commuting cases for the separation of Type I, we have
1) $(\mathbf{D}_3, \mathbf{L}_i)$.
The separation scheme $\mathbf{D}_3$ is never associated with the separation of the Dirac equation in the Dirac representation, being associated with singular spin frames: since in any case $e_2^1 = e_2^2 = 0$, the matrix (e_a^μ) is singular everywhere.

2)($\mathbf{D}_5, \mathbf{L}_5$): $R_1(y)$, $R_2(y)$.
Proceding as above, we have the separability conditions

$$\begin{cases} e_1^1 = e_2^2 = 0 \\ ie_1^2 = R_1(y)B_1(y) \\ e_2^1 = R_1(y)A_2(x) \\ -e_2^1 = R_2(y)A_3(x) \\ -ie_1^2 = R_2(y)B_4(y) \\ iX = 4R_1(y)C_1(y) \\ -Y = 4R_1(y)C_2(x) \\ Y = 4R_2(y)C_3(x) \\ -iX = 4R_2(y)C_4(y) \end{cases}$$

from which we can derive $R_2 = cR_1$, where c is a constant, and $A_2 = -cA_3$, $B_1 = -cB_4$, $C_1 = -cC_4$ and $C_2 = -cC_3$. The system is therefore equivalent to

$$\begin{cases} e_1^1 = e_2^2 = 0 \\ ie_1^2 = R_1(y)B_1(y) \\ e_2^1 = R_1(y)A_2(x) \\ iX = 4R_1(y)C_1(y) \\ -Y = 4R_1(y)C_2(x). \end{cases}$$

We obtain for the only allowed symmetry operators the following expressions

$$\mathbf{L}_5 = -\frac{1}{c}\begin{pmatrix} (A_2\partial_x + C_2(x))^2 & 0 \\ 0 & (A_2\partial_x + C_2(x))^2 \end{pmatrix}$$

Given the previous expression for the components of the spin frame in these coordinates, the quantities $X = -2\Gamma_{122}$ and $Y = -2\Gamma_{121}$ can be computed from the e_a^μ. Moreover, since e_a^μ are assumed to be real functions: $B_1(y) = i\bar{B}_1(y)$. We have

$$\begin{cases} X = 4A_2(x)(R_1(y)\bar{B}_1(y))^{-1}\dfrac{d(R_1(y)\bar{B}_1(y))}{dy} \\ Y = 4iR_1(y)B_1(y)A_2(x)^{-1}\dfrac{dA_2(x)}{dx} \end{cases}$$

which is consistent with the above conditions iff

$$\begin{cases} A_2 = k \\ C_2(x) = 0 \\ C_1 = i\bar{C}_1 \\ \bar{C}_1(y) = (R_1(y)\bar{B}_1(y))^{-1}\dfrac{d(R_1(y)\bar{B}_1(y))}{dy} \end{cases}$$

with k constant, $\bar{C}_1(y)$ and $\bar{B}_1$ real functions.

Then

$$\mathbf{L}_5 = -\frac{1}{c}\begin{pmatrix}\partial_x^2 & 0 \\ 0 & \partial_x^2\end{pmatrix}.$$

Remark:
The operator

$$L = \begin{pmatrix}\partial_x & 0 \\ 0 & \partial_x\end{pmatrix}$$

that commutes with D_5, it is not a good symmetry operator in this case. Indeed, it is not built from the separated equations and it implies $a_1 = a_2$ (up to a constant factor that can be absorbed by b_i).

The components of the metric tensor are

$$\begin{cases} g_{11} = (kR_1(y))^{-2} \\ g_{22} = (R_1(y)\bar{B}_1(y))^{-2} \end{cases}$$

that, after rescaling of y, corresponds to polar or cartesian coordinates on the euclidean plane. The differential equations for the a_i are

$$\begin{cases} \dot{a}_2 = \mu_1 a_1 \\ -\dot{a}_1 = \mu_2 a_2 \end{cases}$$

Solutions for a_i in the simplest case $k = 1$, are

$$\begin{cases} a_1 = c_1^1 e^{-\sqrt{-\mu_1\mu_2}x} + c_1^2 e^{\sqrt{-\mu_1\mu_2}x} \\ a_2 = \sqrt{-\mu_1/\mu_2}\left(-c_1^1 e^{-\sqrt{-\mu_1\mu_2}x} + c_2^1 e^{\sqrt{-\mu_1\mu_2}x}\right) \end{cases}$$

For b_i the separated equations are

$$\begin{cases} i\bar{B}_1(y)\dot{b}_1 + i\bar{C}_1(y)b_1 - R_1(y)^{-1}\lambda b_1 = -\mu_1 b_2 \\ -\frac{i}{c}\bar{B}_1(y)\dot{b}_2 - \frac{i}{c}\bar{C}_1(y)b_2 - R_1(y)^{-1}\lambda b_2 = -\mu_2 b_1 \end{cases}$$

with $\bar{C}_1(y) = (R_1(y)\bar{B}_1(y))^{-1}\dfrac{d(R_1(y)\bar{B}_1(y))}{dy}$. The system can be decoupled as we did for the a_i.
If also $\bar{B}_1 = -c\bar{B}_4 = 1$, $R_1 = 1$, (Cartesian coordinates):

$$\begin{cases} b_1 = d_1^1 e^{yr_1} + d_1^2 e^{yr_2} \\ b_2 = d_2^1 e^{yr_1} + d_2^2 e^{yr_2} \end{cases}$$

$$\begin{cases} d_2^1 = \dfrac{\lambda - r_1}{\mu_1} d_1^1 \\ d_2^2 = \dfrac{\lambda - r_2}{\mu_1} d_1^2 \end{cases}$$

where r_i are roots of $r^2 - c\lambda(1-c)r - c(\lambda^2 + \mu_1\mu_2) = 0$.

Type II

Let us assume $a_1 = a_2$ and $b_1 \neq b_2$. We have

$$\mathbf{D}\psi - \lambda\psi = \begin{pmatrix} A_1 \dot{a} b_1 + A_2 \dot{a} b_2 + B_1 a \dot{b}_1 + B_2 a \dot{b}_2 + (C_1 - \lambda) a b_1 + C_2 a b_2 \\ A_3 \dot{a} b_1 + A_4 \dot{a} b_2 + B_3 a \dot{b}_1 + B_4 a \dot{b}_2 + C_3 a b_1 + (C_4 - \lambda) a b_2 \end{pmatrix} = 0.$$

We now can divide each row of by the two functions ab_i, $i = 1, 2$ only, obtaining the following four separated operators

$$\mathbf{D}_1 : \begin{pmatrix} A_1(x) & 0 \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & B_2(y) \\ B_3(y) & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_{11}(x) + C_{12}(y) & C_2(y) \\ C_{31}(x) + C_{32}(y) & C_4(y) \end{pmatrix},$$

$$\mathbf{D}_2 : \begin{pmatrix} A_1(x) & 0 \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & B_2(y) \\ B_3(y) & B_4(y) \end{pmatrix} \partial_y +$$

$$+ \begin{pmatrix} C_{11}(x) + C_{12}(y) & C_2(y) \\ C_3(y) & C_{41}(x) + C_{42}(y) \end{pmatrix},$$

$$\mathbf{D}_3 : \begin{pmatrix} 0 & A_2(x) \\ A_3(x) & 0 \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & B_2(y) \\ B_3(y) & B_4(y) \end{pmatrix} \partial_y +$$

$$+ \begin{pmatrix} C_1(y) & C_{21}(x) + C_{22}(y) \\ C_{31}(x) + C_{32}(y) & C_4(x) \end{pmatrix},$$

$$\mathbf{D}_4 : \begin{pmatrix} 0 & A_3(x) \\ 0 & A_4(x) \end{pmatrix} \partial_x + \begin{pmatrix} B_1(y) & B_2(y) \\ B_3(y) & B_4(y) \end{pmatrix} \partial_y + \begin{pmatrix} C_1(y) & C_{21}(x) + C_{22}(y) \\ C_3(y) & C_{41}(x) + C_{42}(y) \end{pmatrix}.$$

Moreover, as for the type I separation, in order to obtain the most general separated equations we must consider each of the D_i multipliyed by the matrix

$$\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

where,
for D_1: $R_1(x)$ or $R_1(y)$ and $R_2(y)$;
for D_2: $R_1(x)$ or $R_1(y)$ and $R_2(x)$ or $R_2(y)$;
for D_3: $R_1(y)$ and $R_2(y)$;
for D_4: $R_1(y)$ and $R_2(x)$ or $R_2(y)$.

From D_1 we have the following systems of separated ODE,

$$\begin{cases} A_1\dot{a} + (C_{11} - \lambda)a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + C_2(y)b_2 + C_{12}b_1 = -\mu b_1 \end{cases}$$

$$\begin{cases} A_1\dot{a} + C_{11}a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + C_2(y)b_2 + (C_{12} - \lambda)b_1 = -\mu_1 b_1 \end{cases}$$

for the first component, corresponding to the possible grouping of the parameter λ; and

$$\begin{cases} A_3\dot{a} + C_{31}a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + (C_4(y) - \lambda)b_2 + C_{32}b_1 = -\mu_2 b_1. \end{cases}$$

Let us consider the case D_2. We have now $e_2^1 = 0$. All the possible separated equations are

$$\begin{cases} A_1\dot{a} + (C_1^1 - \lambda)a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + C_2(y)b_2 + C_{12}b_1 = -\mu_1 b_1 \end{cases}$$

$$\begin{cases} A_1\dot{a} + C_{11}a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + C_2(y)b_2 + (C_{12} - \lambda)b_1 = -\mu_1 b_1 \end{cases}$$

for the separation of the first component, and

$$\begin{cases} A_4\dot{a} + (C_{41} - \lambda)a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + C_3 b_1 + C_{42}b_2 = -\mu_2 b_2 \end{cases}$$

$$\begin{cases} A_4\dot{a} + C_{41}a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + C_3(y)b_1 + (C_{42} - \lambda)b_2 = -\mu_2 b_2 \end{cases}$$

for the second component, according to the two possible groupings of λ.

Let us consider the operator D_3. the separated equations are

$$\begin{cases} A_2\dot{a} + C_{21}a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + (C_1(y) - \lambda)b_1 + C_{22} = -\mu_1 b_2 \end{cases}$$

$$\begin{cases} A_3\dot{a} + C_{31}a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + (C_4(y) - \lambda)b_2 + C_{32}b_1 = -\mu_2 b_1 \end{cases}$$

Let us consider at last the operator D_4.
The separated equations are

$$\begin{cases} A_2\dot{a} + C_{21}a = \mu_1 a \\ B_1\dot{b}_1 + B_2\dot{b}_2 + (C_1(y) - \lambda)b_1 + C_{22} = -\mu_1 b_2 \end{cases}$$

for the separation of the first component, and

$$\begin{cases} A_4\dot{a} + (C_{41} - \lambda)a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + C_3(y)b_1 + C_{42}b_2 = -\mu_2 b_2 \end{cases}$$

$$\begin{cases} A_4\dot{a} + C_{41}a = \mu_2 a \\ B_3\dot{b}_1 + B_4\dot{b}_2 + C_3(y)b_1 + (C_{42} - \lambda)b_2 = -\mu_2 b_2 \end{cases}$$

for the second.

The Dirac case, type II

(Work in progress)
It is possible to see that to D_1 always correspond singular spin frames; due to the expression of D_1 we have that $e_1^1 = e_2^1 = 0$.
In the case D_4 we have immediately $e_1^1 = e_2^1 = 0$ and all related spin frames are necessarily singular.
Then, no separable Dirac equation can correspond to the separation schemes D_1 and D_4.

The operator D_2:
In this case we have $e_2^1 = 0$.
Let $R_1(x) = -R_2(y) = c$, then for example we can have

$$L_{IV}^2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} \partial_x + \begin{pmatrix} C_{11} & 0 \\ 0 & C_{41} \end{pmatrix}$$

which commutes with D_2 for nonsingular spin frames if and only if $A_1 = A_4$, $C_{41} = C_{11}$.

We have the conditions

$$\begin{cases} e_1^1 = -icA_1 \\ e_2^1 = 0 \\ e_1^2 = -icB_1 \\ e_2^2 = cB_2 \\ X = -i4c(C_{11} + C_{12}) \\ Y = -4cC_2(y) \\ B_2 = B_3 \\ B_1 = B_4 \\ C_2 = C_3 \\ C_{12} = C_{42} \end{cases}$$

where c is a constant. It follows

$$\begin{cases} A_1(x) = i\bar{A}_1(x) \\ B_1(y) = i\bar{B}_1(y) \\ X = 0 \\ Y = -4c\bar{B}_1 \dfrac{\dot{\bar{A}}_1}{\bar{A}_1} \end{cases}$$

Therefore, we have:

$$\begin{cases} A_1 = A_4 \\ C_{41} = C_{11} \\ C_1 = 0 \\ \bar{A}_1 = k_1 e^{k_2 x} \\ C_2 = 4k_2 \bar{B}_1/4. \end{cases}$$

where k_i are constants. The covariant components of the metric tensor:

$$\begin{cases} g_{11} = \dfrac{B_2^2 + \bar{B}_1^2}{c^2 \bar{A}_1^2 B_2^2} \\ g_{12} = -\dfrac{\bar{B}_1}{c^2 \bar{A}_1 B_2^2} \\ g_{22} = \dfrac{1}{c^2 B_2^2} \end{cases}$$

The Levi-Civita equations are always satisfied and x is of first-class.
By putting, for example, $\bar{B}_1 = c_1$, $B_2 = -(c_2 y + c_3)^{-1}$, c_i constants, the Riemann tensor is zero.

Let now $R_1 = R_2 = 1$
We can build the following four second-order commuting with D_2 operators:

$$L^2_{VI} = \begin{pmatrix} L_a L_b & 0 \\ 0 & L_c L_d \end{pmatrix}$$

where $a, b, c, d = 1, 2,\ a \neq b,\ c \neq d$ and $L_1 = A_1 \partial_x + C_{11},\ L_2 = A_4 \partial_x + C_{41}$. The operators commute if: $i)\, A_1 = 0, C_{11} = k$ for constant k, and
$ii)\, A_4 = -A_1,\ C_{41} = -C_{11} + k$.

The only case corresponding to nonsingular spin frames is *ii* and we have the following conditions:

$$\begin{cases} e_1^1 = -iA_1 \\ e_2^1 = 0 \\ e_1^2 = -iB_1 \\ e_2^2 = -B_2 \\ X = -i4(C_{11} + C_{12}) \\ Y = -4C_2(y) \\ C_{42} = -C_{12} + ik/4 \\ C_3 = -C_2 \\ B_3 = -B_2 \\ B_4 = -B_1 \end{cases}$$

Proceeding as above, we obtain

$$\begin{cases} A_1(x) = i\bar{A}_1(x) \\ B_1(y) = i\bar{B}_1(y) \\ X = 0 \\ Y = -4\bar{B}_1 \dfrac{\dot{\bar{A}}_1}{\bar{A}_1} \end{cases}$$

Therefore, we have:

$$\begin{cases} A_4 = -A_1 \\ C_{41} = -C_{11} + k \\ C_{41} = k \\ C_{42} = -i\dfrac{k}{4} \\ C_1 = 0 \\ C_2 = 4\bar{B}_1/4 \\ \bar{A}_1 = k_1 e^{k_2 x}. \end{cases}$$

where k_i are constant.

The covariant components of the metric tensor:

$$\begin{cases} g_{11} = \dfrac{B_2^2 + \bar{B}_1^2}{k_1^2 e^{2k_2 x} B_2^2} \\ g_{12} = -\dfrac{\bar{B}_1}{k_1 e^{k_2 x} B_2^2} \\ g_{22} = \dfrac{1}{B_2^2} \end{cases}$$

The Levi-Civita equations are always satisfied and x is of first class. By chosing for example $\bar{B}_1 = y$, $B_2 = y(c_1 + c_2 ln(y))^{-1}$, the Riemann tensor is zero.

The operator D_3:
$R_1 = R_2 = 1$. We have $e^1_1 = 0$
For second-order operators we have

$$L^3_{III} = \begin{pmatrix} L_a L_b & 0 \\ 0 & L_c L_d \end{pmatrix}$$

where $a, b, c, d = 1, 2$, $a \neq b$, $c \neq d$ and $L_1 = A_2\partial_x + C_{21}$, $L_2 = A_3\partial_x + C_{31}$. The operators commute and related spin frames are nonsingular if:
$i)\, A_3 = -A_2,\ C_{31} = -C_{21} + k$.

It follows in all this cases

$$\begin{cases} e_1^1 = 0 \\ e_2^1 = A_2 \\ e_1^2 = -iB_1 \\ e_2^2 = B_1 \\ X = -i4C_1(y) \\ Y = -4(C_{21} + C_{22}) \\ C_4(y) = -C_1(y) \\ C_{32} = -C_{22} - k \\ B_4 = -B_1 \\ B_3 = -B_2 \end{cases}$$

$$\begin{cases} B_1 = i\bar{B}_1 \\ X = 4A_2\dot{\bar{B}}_1/\bar{B}_1 \\ Y = -4(\dot{\bar{B}}_1 B_2 - \dot{B}_2 + \dot{A}_2/A_2) \end{cases}$$

Therefore, we have:

$$\begin{cases} A_2(x) = 1 \\ C_1 = i\bar{C}_1 \\ \bar{C}_1 = \dot{\bar{B}}_1/\bar{B}_1 \\ C_{21} = 0 \\ C_{22} = (-\dot{\bar{B}}_2 + B_2\dot{\bar{B}}_1). \end{cases}$$

The covariant components of the metric tensor:

$$\begin{cases} g_{11} = \dfrac{B_2^2 + \bar{B}_1^2}{\bar{B}_1^2} \\ g_{12} = -\dfrac{B_2}{\bar{B}_1^2} \\ g_{22} = \dfrac{1}{\bar{B}_1^2} \end{cases}$$

The Levi-Civita equations are always satisfied and x is ignorable. The Riemann tensor is zero for suitable values of the parameters.

Type III

$a_1 = a_2$, $b_1 = cb_2$, where c is a constant.
Work in progress.

Conclusions

The operators of above, even if not the most general possible, are a probe to investigate the properties of the separation of variables of the Dirac equation. We can see how coordinates and spin frames are related to each other, what kind of metrics are generated by the separation conditions and how they are related to Hamilton-Jacobi separation of variables.
We can find explicitely the transformation from separable to pseudo-Cartesian coordinates (Horwood-McLenaghan).
Although the physical Dirac equation is in dimension four, separation in two dimension can occur after reduction by symmetries (Kerr solution).

Future directions:
- Equivalence of separated coordinates (comparison between Hamilton-Jacobi and Dirac).
- $\lambda = 0$ case.
- Symmetry operators depending on λ (Fixed-energy separation).
- Comparison with operators obtained from geometry (Killing vectors, symmetry operators of higher orders).
- Third and higher order operators.
- The Lorentzian case.
- Dimension four.