

*MOVING FRAMES
FOR
LIE PSEUDOGRUUPS*

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WHAT IS NEW?

- Direct constructive algorithms for:
 - Invariant Maurer–Cartan forms
 - Structure equations
 - Moving frames
 - Differential invariants
 - Invariant differential forms
 - Invariant differential operators
 - Syzygies and recurrence formulas
- Basis Theorem for differential invariants

FURTHER APPLICATIONS:

- Symmetry groups of differential equations
- Vessiot method of group splitting
- Congruence of curves, surfaces, etc. in homogeneous spaces
- Invariant variational bicomplex:
 - Calculus of variations
 - Gauge theories
 - Riemannian submanifolds
 - Characteristic classes of foliations, Gelfand-Fuks cohomology

DIFFEOMORPHISM PSEUDO GROUP

| | |
|------------------------------------|---|
| M^m | m dimensional manifold |
| $\mathcal{D} = \mathcal{D}(M)$ | pseudogroup of local diffeomorphisms of M |
| $\mathcal{D}^n \subset J^n(M, M)$ | bundle of n th order jets, $0 \leq n \leq \infty$ |
| $\sigma(j_z^n \varphi) = z$ | source map |
| $\tau(j_z^n \varphi) = \varphi(z)$ | target map |

\mathcal{D}^∞ inherits the structure $\Omega^{r,s}$ of a *variational bicomplex* from $J^\infty(M, M)$:

Coordinates on \mathcal{D}^∞ : $(z^a, Z^b, Z_{c_1}^b, Z_{c_1 c_2}^b, \dots)$, where z^a, Z^b are local coordinates of M about the source and the target, and $Z_{c_1}^b, Z_{c_1 c_2}^b, \dots$ stand for the derivative variables.

Horizontal forms: dz^a

Contact forms: $\theta_{c_1 \dots c_p}^b = dZ_{c_1 \dots c_p}^a - \sum_i^m Z_{c_1 \dots c_p c_{p+1}}^a dz^{c_{p+1}}$

Split the exterior derivative $d = d_H + d_V$ into horizontal and vertical components.

Horizontal differential d_H :

$$\begin{aligned} d_H z^a &= dz^a, & d_H Z_{c_1 \dots c_p}^b &= Z_{c_1 \dots c_p c_{p+1}}^a dz^{c_{p+1}}, \\ d_H(dz^a) &= 0, & d_H \theta_{c_1 \dots c_p}^b &= dz^{c_{p+1}} \wedge \theta_{c_1 \dots c_p c_{p+1}}^b. \end{aligned}$$

Vertical differential d_V :

$$\begin{aligned} d_V z^a &= 0, & d_V Z_{c_1 \dots c_p}^b &= \theta_{c_1 \dots c_p}^b, \\ d_V(dz^a) &= 0, & d_V \theta_{c_1 \dots c_p}^b &= 0. \end{aligned}$$

Groupoid Structure: $j_{\varphi(z)}^n(\psi) \cdot j_z^n(\varphi) = j_z^n(\psi \circ \varphi)$

Important submanifolds of \mathcal{D}^n :

$$\begin{aligned} \mathcal{D}^n|_z &= \sigma^{-1}(z) \subset \mathcal{D}^n, \\ \mathcal{F}^n &= \cup_{z \in M} \mathcal{D}_z^n = \cup_{z \in M} \{g^n \in \mathcal{D}^n \mid \sigma(g^n) = \tau(g^n) = z\}. \end{aligned}$$

$\mathcal{F}^n \rightarrow M$ is a principal fiber bundle with the structure group $\mathrm{GL}^n(m)$.

MAURER-CARTAN FORMS FOR \mathcal{D}^∞

These are represented by invariant contact forms on \mathcal{D}^∞ .

\mathcal{D} acts on \mathcal{D}^n , $n \geq 0$, from both left and right by

$$L_\psi j_z^n \varphi = j_z^n(\psi \circ \varphi), \quad R_\psi j_z^n \varphi = j_{\psi^{-1}(z)}^n(\varphi \circ \psi).$$

These actions preserve the bicomplex structure of \mathcal{D}^∞ and so they commute with the horizontal and vertical differentials d_H, d_V .

Construction of right invariant forms on \mathcal{D}^∞ :

The target coordinate Z^b invariant under $R_\psi \implies$

$$\omega^b = d_H Z^b = \sum_c Z_c^b dz^c, \quad \mu^b = d_V Z^b = dZ^b - Z_c^b dz^c,$$

are also invariant under R_ψ .

Operators of invariant differentiation:

$$\mathbb{D}_{Z^a} = W_a^b \mathbb{D}_{z^b}, \quad \text{where}$$

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{p \geq 0} Z_{c_1 \dots c_p}^c \frac{\partial}{\partial Z_{c_1 \dots c_p}^c} \quad \text{and } W = (\nabla Z)^{-1}.$$

Right invariant coframe on \mathcal{D}^∞ :

$$\omega^a, \quad \mu_{b_1 \dots b_p}^a = \mathcal{L}_{\mathbb{D}_{z^{b_1}}} \cdots \mathcal{L}_{\mathbb{D}_{z^{b_p}}} \mu^a, \quad p \geq 0.$$

EXAMPLE: $M = \mathbb{R}$. As a coordinate space

$$\mathcal{D}^\infty(\mathbb{R}) = \{z, Z, Z_z, Z_{zz}, \dots, Z_{z^n}, \dots\}.$$

Now

$$\mathbb{D}_z = \frac{\partial}{\partial z} + Z_z \frac{\partial}{\partial Z} + Z_{zz} \frac{\partial}{\partial Z_z} + \cdots.$$

Basic right invariant horizontal form $\omega = d_H Z = Z_z dz$.

The dual total differentiation $\mathbb{D}_Z = \frac{1}{Z_z} \mathbb{D}_z$ commutes with the group action.

Right invariant Maurer-Cartan forms:

$$\begin{aligned}\mu &= \theta, & \mu_Z &= \mathcal{L}_{\mathbb{D}_Z} \mu = (Z_z)^{-1} \theta_z, \\ \mu_{ZZ} &= \mathcal{L}_{\mathbb{D}_Z}^2 \mu = (Z_z)^{-3} (Z_z \theta_{zz} - Z_{zz} \theta_z), \dots\end{aligned}$$

Similarly, the source coordinates z^a are invariant under L_ψ

$$\implies \tau^a = dz^a, \quad \vartheta^a = \sum_{c=1}^m W_c^a \theta^c \quad \text{are invariant under } L_\psi.$$

Left invariant coframe on \mathcal{D}^∞ :

$$dz^a, \quad \vartheta_{b_1 \dots b_p}^a = \mathcal{L}_{\mathbb{D}_{z^{b_1}}} \cdots \mathcal{L}_{\mathbb{D}_{z^{b_p}}} \vartheta^a, \quad p \geq 0.$$

Example: $M = \mathbb{R}$ again. Basic horizontal form $\tau = dz$. The total derivative \mathbb{D}_z commutes with the left action of $\mathcal{D}(\mathbb{R})$.

Left-invariant Maurer-Cartan forms

$$\begin{aligned}\vartheta &= (Z_z)^{-1} \theta, & \vartheta_z &= (Z_z)^{-2} (Z_z \theta_z - Z_{zz} \theta), \\ \vartheta_{zz} &= (Z_z)^{-3} (Z_z^2 \theta_{zz} - 2Z_z Z_{zz} \theta_z - (Z_z Z_{zzz} - 2Z_{zz}^2) \theta), \dots\end{aligned}$$

STRUCTURE EQUATIONS

Taylor series method: Write

$$\begin{aligned} Z^a[[h]] &= \sum_{|J| \geq 0} \frac{1}{|J|!} Z_J^a h^J, \\ \theta^a[[h]] &= \sum_{|J| \geq 0} \frac{1}{|J|!} \theta_J^a h^J, \\ \mu^\alpha[[H]] &= \sum_{|J| \geq 0} \frac{1}{|J|!} \mu_J^\alpha H^J. \end{aligned}$$

Then

$$\mu[Z[[h]] - Z] = \theta[[h]].$$

$$\left(f(z+h) = f \circ \varphi^{-1}(\varphi(z) + (\varphi(z+h) - \varphi(z))), \quad \text{and} \right.$$

$$\left. \frac{\partial}{\partial z^a} (f \circ \varphi^{-1}(z)) = (\mathbb{D}_{Z^a} f) \circ \varphi^{-1}(z). \right)$$

Write $H = Z[[h]] - Z$. Then

$$\begin{aligned} dH &= (\nabla_h Z[[h]] dz - \omega) + (\theta[[h]] - \theta), \\ d\theta[[h]] &= -\nabla\theta[[h]] \wedge dz. \end{aligned}$$

Apply d and eliminate $h \implies$

DIFFEOMORPHISM PSEUDOGROUP STRUCTURE EQUATIONS:

$$\begin{aligned} d\mu[[H]] &= \nabla_H \mu[[H]] \wedge (\mu[[H]] - dZ), \\ d\omega &= -\omega \wedge \mu[[0]]. \end{aligned}$$

Example: $M = \mathbb{R}$; invariant coframe $\omega, \mu, \mu_Z, \dots$

$$\begin{aligned} d\omega &= -\omega \wedge \mu_Z, \\ d\mu_{Z^n} &= -\mu_{Z^{n+1}} \wedge \omega + \sum_{i=0}^{n-1} \binom{n}{i} \mu_{Z^{i+1}} \wedge \mu_{Z^{n-i}}. \end{aligned}$$

PSEUDOGROUPS

$\mathcal{G} \subset \mathcal{D}$ is a *pseudogroup* if

1. $\text{id} \in \mathcal{G}$,
2. $\varphi, \psi \in \mathcal{G} \Rightarrow \varphi \circ \psi \in \mathcal{G}$ where defined,
3. $\varphi \in \mathcal{G} \Rightarrow \varphi^{-1} \in \mathcal{G}$.

\mathcal{G} is a *Lie* (or continuous) *pseudogroup* if, in addition, for all $n \geq N$,

4. $\mathcal{G}^n \subset \mathcal{D}^n$ is a subbundle,
5. $\mathcal{G}^{N+k} = \text{pr}^k \mathcal{G}^N$, $k \geq 1$,
6. $\varphi \in \mathcal{G} \iff j_z^n \varphi \in \mathcal{G}^n$.

REMARK: It follows that $\mathcal{G}_z^n \subset \mathcal{D}_z^n$ is a (finite dimensional) Lie group.

EXAMPLES OF LIE PSEUDOGROUPS.

1. Symmetry groups of Euler, Navier-Stokes, boundary layer, quasi-geostrophic equations and various other equations arising in fluid mechanics, magnetohydrodynamics, meteorology and geophysics.
2. Symmetry groups in gauge and field theories – Maxwell, Yang-Mills, conformal field theories, general relativity. Current/loop groups.
3. Symmetry groups of integrable equations in 2+1 dimensions – KP, Davey-Stewartson, and their variants.
4. Canonical transformations in Hamiltonian mechanics.
5. Configuration spaces:
 - a) $Diff(\Omega) \rightarrow$ compressible fluid flow
 - b) $Diff_{vol}(\Omega) \rightarrow$ incompressible fluid flow
 - c) Canonical transformations \rightarrow Maxwell-Vlasov

6. Transformations preserving a geometric structure:
 - a) Foliations
 - b) Symplectic/Poisson structures
 - c) Contact structures (quantomorphisms)
 - d) Complex manifolds/real hypersurfaces
 - e) G -structures
7. Image recognition – shape representation.
8. Finite dimensional Lie group actions.

INFINITESIMAL GENERATORS

A local vector field $\mathbf{v} \in \mathcal{X}(M)$ is a \mathcal{G} vector field if the flow $\Phi_t^{\mathbf{v}} \in \mathcal{G}$ for all fixed t on some interval about 0.

Let \mathcal{G}^n be given locally by $F_\alpha(z, Z^{(n)}) = 0$. Then a \mathcal{G} vector field \mathbf{v} satisfies

$$F_\alpha(z, \Phi_t^{\mathbf{v}(n)}) = 0 \quad \implies \quad L_\alpha(z, j_z^n \mathbf{v}) = 0.$$

These are the *infinitesimal determining equations* for \mathcal{G} .

Invariant coframe for \mathcal{G}^∞ : Simply pull back ω^i, μ_J^a defined on \mathcal{D}^∞ to \mathcal{G}^∞ .

Relations: On \mathcal{G}^∞ , the Maurer-Cartan forms μ_J^a satisfy the right invariant infinitesimal determining equations

$$L_\alpha(Z, \mu_J^a) = 0. \quad (1)$$

Structure equations for $\mathcal{G}^\infty \rightsquigarrow$ structure equations for \mathcal{D}^∞ modulo the relations (1).

The proof is based on the *Replacement Principle*:

If Ω is a right (or left) invariant form on \mathcal{D}^∞ and

$$\Omega_{j_z^\infty \text{id}} = \sum_a g_i(z) dz^a + \sum_{a,J} h_a^J(z) dZ_J^a,$$

then

$$\Omega = \sum_a g_i(Z) \omega^a + \sum_{a,J} h_a^J(Z) \mu_J^a.$$

$$(\text{or } \Omega = \sum_a g_i(z) dz^a + \sum_{a,J} h_a^J(z) \vartheta_J^a.)$$

EXAMPLE: The general symmetry transformation $\Psi_{F,G,H}$ of the KP equation

$$(u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx})_x + \frac{3}{4}\epsilon u_{yy} = 0, \quad \epsilon = \pm 1,$$

is given by

$$\begin{aligned} T &= F(t), & Y &= yF'(t)^{2/3} + G(t), \\ X &= xF'(t)^{1/3} - \frac{2}{9}\epsilon y^2 \frac{F''(t)}{F'(t)^{2/3}} - \frac{2}{3}\epsilon y \frac{G'(t)}{F'(t)^{1/3}} + H(t), \\ U &= \frac{u}{F'(t)^{2/3}} + \frac{2}{9}x \frac{F''(t)}{F'(t)^{5/3}} + \frac{4}{27}\epsilon y^2 \left(\frac{4}{3} \frac{F''(t)^2}{F'(t)^{8/3}} - \frac{F'''(t)}{F'(t)^{5/3}} \right) \\ &\quad + \frac{4}{9}\epsilon y \left(\frac{G'(t)F''(t)}{F'(t)^{7/3}} - \frac{G''(t)}{F'(t)^{4/3}} \right) + \frac{2}{3} \frac{H'(t)}{F'(t)} + \frac{2}{9}\epsilon \frac{G'(t)^2}{F'(t)^2}, \end{aligned}$$

with the composition rule $\Psi_{\hat{F},\hat{G},\hat{H}} = \Psi_{F,G,H} \circ \Psi_{f,g,h}$, where

$$\begin{aligned} \hat{F} &= F \circ f, & \hat{G} &= g(F' \circ f)^{2/3} + G \circ f, \\ \hat{H} &= h(F' \circ f)^{1/3} - \frac{2}{9}\epsilon \frac{g^2 F'' \circ f}{(F' \circ f)^{2/3}} - \frac{2}{3}\epsilon \frac{g G' \circ f}{(F' \circ f)^{1/3}} + H \circ f. \end{aligned}$$

The infinitesimal generators

$$\mathbf{v} = \tau(t, x, y, u)\partial_t + \xi(t, x, y, u)\partial_x + \eta(t, x, y, u)\partial_y + \varphi(t, x, y, u)\partial_u,$$

where

$$\tau = f(t), \quad \xi = \frac{1}{3}xf'(t) - \frac{2}{9}\epsilon y^2 f''(t) - \frac{2}{3}\epsilon yg'(t) + h(t),$$

$$\eta = \frac{2}{3}yf'(t) + g(t),$$

$$\varphi = -\frac{2}{3}uf'(t) + \frac{2}{9}xf''(t) - \frac{4}{27}\epsilon y^2 f'''(t) - \frac{4}{9}\epsilon yg''(t) + \frac{2}{3}h'(t),$$

satisfy

$$\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \xi_x - \frac{1}{3}\tau_t = 0, \quad \xi_y + \frac{2}{3}\epsilon\eta_t = 0,$$

$$\xi_u = 0, \quad \eta_x = 0, \quad \eta_y - \frac{2}{3}\tau_t = 0, \quad \eta_u = 0, \quad \varphi - \frac{2}{3}\xi_t + \frac{2}{3}u\tau_t = 0,$$

along with all their differential consequences.

The Maurer-Cartan forms satisfy the “lifted infinitesimal determining equations”

$$\mu_X^t = 0, \quad \mu_Y^t = 0, \quad \mu_U^t = 0, \quad \mu_X^x - \frac{1}{3}\mu_T^t = 0, \quad \mu_Y^x + \frac{2}{3}\epsilon\mu_T^y = 0,$$

$$\mu_U^x = 0, \quad \mu_X^y = 0, \quad \mu_Y^y - \frac{2}{3}\mu_T^t = 0, \quad \mu_U^y = 0,$$

$$\mu^u - \frac{2}{3}\mu_T^x + \frac{2}{3}U\mu_T^t = 0, \quad \dots$$

Basis of linearly independent Maurer–Cartan forms:

$$\begin{aligned}\omega^1 &:= \mu^t, & \omega^2 &:= \mu^x, & \omega^3 &:= \mu^y, & \omega^4 &:= \mu^u, \\ \omega^5 &:= \mu^t_T = \mu^t_{1,0,0,0}, & \omega^6 &:= \mu^y_T = \mu^y_{1,0,0,0}, \\ \alpha^i &:= \mu^u_{i,0,0,0}, & \beta^i &:= \mu^u_{i-1,1,0,0}, & \gamma^i &:= \mu^u_{i-1,0,1,0}.\end{aligned}$$

Structure equations for the invariant horizontal forms

$$\begin{aligned}d\sigma^t &= \omega^5 \wedge \sigma^t, \\ d\sigma^x &= \left(\frac{3}{2}\omega^4 + U\omega^5\right) \wedge \sigma^t + \frac{1}{3}\omega^5 \wedge \sigma^x - \frac{2}{3}\epsilon\omega^6 \wedge \sigma^y, \\ d\sigma^y &= \omega^6 \wedge \sigma^t + \frac{2}{3}\omega^5 \wedge \sigma^y, \\ d\sigma^u &= \alpha^1 \wedge \sigma^t + \beta^1 \wedge \sigma^x + \gamma^1 \wedge \sigma^y - \frac{2}{3}\omega^5 \wedge \sigma^u.\end{aligned}$$

Structure equations for the Maurer-Cartan forms

$$d\omega^1 = -\omega^5 \wedge \sigma^t,$$

$$d\omega^2 = -\left(\frac{3}{2}\omega^4 + U\omega^5\right) \wedge \sigma^t - \frac{1}{3}\omega^5 \wedge \sigma^x + \frac{2}{3}\epsilon\omega^6 \wedge \sigma^y,$$

$$d\omega^3 = -\omega^6 \wedge \sigma^t - \frac{2}{3}\omega^5 \wedge \sigma^y,$$

$$d\omega^4 = -\alpha^1 \wedge \sigma^t - \beta^1 \wedge \sigma^x - \gamma^1 \wedge \sigma^y + \frac{2}{3}\omega^5 \wedge \sigma^u,$$

$$d\omega^5 = -\frac{9}{2}\beta^1 \wedge \sigma^t,$$

$$d\omega^6 = -\frac{1}{3}\omega^5 \wedge \omega^6 + \frac{9}{4}\epsilon\gamma^1 \wedge \sigma^t - 3\beta^1 \wedge \sigma^y,$$

$$d\alpha^1 = -\frac{3}{2}\omega^4 \wedge \beta^1 - \frac{5}{3}\omega^5 \wedge \alpha^1 - U\omega^5 \wedge \beta^1 - \omega^6 \wedge \gamma^1 \\ - \alpha^2 \wedge \sigma^t - \beta^2 \wedge \sigma^x - \gamma^2 \wedge \sigma^y + 3\beta^1 \wedge \sigma^u,$$

$$d\beta^1 = -\omega^5 \wedge \beta^1 - \beta^2 \wedge \sigma^t,$$

$$d\gamma^1 = -\frac{4}{3}\omega^5 \wedge \gamma^1 + \frac{2}{3}\epsilon\omega^6 \wedge \beta^1 + \frac{4}{3}\epsilon\beta^2 \wedge \sigma^y - \gamma^2 \wedge \sigma^t,$$

$$d\alpha^2 = -3\omega^4 \wedge \beta^2 - \frac{8}{3}\omega^5 \wedge \alpha^2 - 2U\omega^5 \wedge \beta^2 - 2\omega^6 \wedge \gamma^2 + 9\alpha^1 \wedge \beta^1 \\ + 3\beta^2 \wedge \sigma^u - \alpha^3 \wedge \sigma^t - \beta^3 \wedge \sigma^x - \gamma^3 \wedge \sigma^y,$$

$$d\beta^2 = -2\omega^5 \wedge \beta^2 - \beta^3 \wedge \sigma^t,$$

$$d\gamma^2 = -\frac{7}{3}\omega^5 \wedge \gamma^2 + 2\epsilon\omega^6 \wedge \beta^2 - \frac{9}{2}\beta^1 \wedge \gamma^1 - \gamma^3 \wedge \sigma^t + \frac{4}{3}\epsilon\beta^3 \wedge \sigma^y,$$

⋮

Fix the values of the target coordinates T, X, Y, U to get

$$d\omega^1 = -\omega^1 \wedge \omega^5,$$

$$d\omega^2 = -\frac{3}{2}\omega^1 \wedge \omega^4 - U\omega^1 \wedge \omega^5 - \frac{1}{3}\omega^2 \wedge \omega^5 + \frac{2}{3}\epsilon\omega^3 \wedge \omega^6,$$

$$d\omega^3 = -\omega^1 \wedge \omega^6 - \frac{2}{3}\omega^3 \wedge \omega^5,$$

$$d\omega^4 = -\omega^1 \wedge \alpha^1 - \omega^2 \wedge \beta^1 - \omega^3 \wedge \gamma^1 + \frac{2}{3}\omega^4 \wedge \omega^5,$$

$$d\omega^5 = -\frac{9}{2}\omega^1 \wedge \beta^1,$$

$$d\omega^6 = \frac{9}{4}\epsilon\omega^1 \wedge \gamma^1 - 3\omega^3 \wedge \beta^1 - \frac{1}{3}\omega^5 \wedge \omega^6,$$

$$d\alpha^1 = -\omega^1 \wedge \alpha^2 - \omega^2 \wedge \beta^2 - \omega^3 \wedge \gamma^2 + \frac{3}{2}\omega^4 \wedge \beta^1 - \frac{5}{3}\omega^5 \wedge \alpha^1 \\ - U\omega^5 \wedge \beta^1 - \omega^6 \wedge \gamma^1,$$

$$d\beta^1 = -\omega^1 \wedge \beta^2 - \omega^5 \wedge \beta^1,$$

$$d\gamma^1 = -\omega^1 \wedge \gamma^2 + \frac{4}{3}\epsilon\omega^3 \wedge \beta^2 - \frac{4}{3}\omega^5 \wedge \gamma^1 + \frac{2}{3}\epsilon\omega^6 \wedge \beta^1,$$

$$d\alpha^2 = -\omega^1 \wedge \alpha^3 - \omega^2 \wedge \beta^3 - \omega^3 \wedge \gamma^3 - \frac{8}{3}\omega^5 \wedge \alpha^2 - 2U\omega^5 \wedge \beta^2 \\ - 2\omega^6 \wedge \gamma^2 + 9\alpha^1 \wedge \beta^1,$$

$$d\beta^2 = -\omega^1 \wedge \beta^3 - 2\omega^5 \wedge \beta^2,$$

$$d\gamma^2 = -\omega^1 \wedge \gamma^3 + \frac{4}{3}\epsilon\omega^3 \wedge \beta^3 - \frac{7}{3}\omega^5 \wedge \gamma^2 + 2\epsilon\omega^6 \wedge \beta^2 - \frac{9}{2}\beta^1 \wedge \gamma^1.$$

These correspond to the structure equations for the KP symmetry algebra obtained by Reid, Lisle, Boulton by Taylor series expansions.

EXTENDED JET BUNDLES

$J^n = J^n(M) = \{n\text{-jets of } p\text{-dimensional submanifolds of } M\}$.

Local coordinates on J^n : $(x^i, u^\alpha, u_{i_1}^\alpha, \dots, u_{i_1 \dots i_n}^\alpha)$.

\mathcal{D} acts on J^n in the usual fashion, and this action factors into an action of \mathcal{D}^n on J^n .

Let \mathcal{E}^n be the pull-back of $\mathcal{D}^n \rightarrow M$ under $\tilde{\pi}_o^n: J^n \rightarrow M$; hence $\mathbf{g}^n \in \mathcal{E}^n$ consists of a pair $\mathbf{g}^n = (z^n, g^n)$, where both $z^n \in J^n$, $g^n \in \mathcal{D}^n$ are based at the same point $z \in M$.

Source and target maps

$$\sigma(\mathbf{g}^n) = z^n, \quad \tau(\mathbf{g}^n) = g^n \cdot z^n.$$

\mathcal{D} acts on \mathcal{E}^n from the left by

$$L_\psi \mathbf{g}^n = (j_z^n \psi \cdot z^n, g^n \cdot j_{\psi(z)}^n \psi^{-1}).$$

Then $\tau(L_\psi \mathbf{g}^n) = \tau(\mathbf{g}^n)$ so that the target coordinates are \mathcal{D} invariant.

MOVING FRAMES

Let $\mathcal{H}^n \subset \mathcal{E}^n$ be the subbundle determined by the jets of transformations in a Lie pseudogroup \mathcal{G} .

A *moving frame* of order n is a \mathcal{G} -equivariant section

$$\rho: \mathcal{V} \rightarrow \mathcal{H}^n$$

defined on some open $\mathcal{V} \subset J^n$.

Hence $\sigma(\rho^n(z^n)) = z^n$ and $\rho^n(\psi \cdot z^n) = L_\psi \rho(z^n)$, $\psi \in \mathcal{G}$.

Existence of moving frames: *Isotropy subgroup* of z^n :

$$\mathcal{I}_{z^n}^n = \{g^n \in \mathcal{G}_z^n \mid g^n \cdot z^n = z^n\}.$$

\mathcal{G} acts *freely* at z^n if $\mathcal{I}_{z^n}^n = \{\text{id}_z^n\}$ and *locally freely* at z^n if $\mathcal{I}_{z^n}^n$ is a discrete subgroup of \mathcal{G}_z^n .

Remark: This is a slight generalization of the usual freeness of Lie group actions.

EXAMPLE: The group $(x, u) \rightarrow (x + a, u + bx^2 + cx + d)$ acts freely on $J^n(\mathbb{R}^2, 1)$ for $n \geq 0$ in the above sense, but in the usual Lie group terminology, the action is free only when $n \geq 2$.

Write $r_n = \dim \mathcal{G}^n|_z$. If \mathcal{G} acts freely at order n , then $r_n \leq \dim J^n$.

Alternate growth condition to the one provided by Spencer cohomology!

Theorem. *A local moving frame of order n exists in a neighborhood of $z^{(n)} \in J^n(M)$ if and only if \mathcal{G} acts locally freely at $z^{(n)}$.*

Theorem. *If \mathcal{G}^n acts locally freely at $z^n \in J^n$, then \mathcal{G}^l acts locally freely at any $z^l \in J^l$ with $\pi_n^l(z^l) = z^n$ for $l > n$.*

Construction: Choose a cross-section K for the action of \mathcal{G} on J^n . Define $\rho(z^n)$ by the condition $\tau(\rho(z^n)) \in K$.

By the invariance of τ , the components of $\tau \circ \rho$ contain a complete set of (local) differential invariants for the action of \mathcal{G} on J^n .

INVARIANTIZATION

Decompose $\Omega^*(\mathcal{E}^\infty) = \oplus_{i,j,k} \Omega^{i,j,k}$, where

i is the horizontal degree

j is the contact degree in J^∞

k is the contact degree in \mathcal{D}^∞

Let π_J be the projection $\pi_J: \Omega^* \rightarrow \oplus_{i,j} \Omega^{i,j,0}$.

π_J preserves \mathcal{D} invariance.

The *lift* of $\omega \in \Omega^*(J^\infty)$ is $\lambda(\omega) = \pi_J(\tau^*(\omega))$.

Let ρ be a moving frame for a pseudogroup \mathcal{G} .

The *invariantization* $\iota(\omega)$ of ω is $\iota(\omega) = \rho^*\lambda(\omega)$.

Theorem. *The invariantization of a (local) coframe on J^∞ produces an \mathcal{G} invariant (local) coframe on J^∞ .*

Structure equations: Formally

$$d\iota(\omega) = \iota(d\omega + \mathcal{L}_{\text{pr } \mathbf{v}}\omega). \quad (*)$$

(Write $\mathcal{L}_{\text{pr } \mathbf{v}}\omega = \zeta_{,b_1\dots b_r}^a \omega_a^{b_1\dots b_r}$. Then

$$\iota(\mathcal{L}_{\text{pr } \mathbf{v}}\omega) = \rho^*(\mu_{b_1\dots b_r}^a) \wedge \iota(\omega_a^{b_1\dots b_r}).)$$

The invariant Maurer-Cartan forms $\mu_{b_1\dots b_r}^a$ above are subject to the IDE for \mathcal{G} . This can be exploited in analyzing the structure of invariant objects for \mathcal{G} .

RECURRENCE FORMULAS.

Fix a coordinate cross section. Write

$$H^i = \iota(x^i), \quad I_{j_1 \dots j_r}^\alpha = \iota(u_{j_1 \dots j_r}^\alpha)$$

for the normalized differential invariants. Call an invariant *phantom* if it is a constant.

Let

$$\begin{aligned} \omega^i &= \pi_H \iota(dx^i), \\ \beta^i &= \pi_H \iota(\xi^i), \quad \psi_{j_1 \dots j_r}^\alpha = \pi_H \iota(\mathcal{L}_{\text{pr v}} u_{j_1 \dots j_r}^\alpha), \end{aligned}$$

and let \mathcal{D}_i be total differential operators dual to ω^i .

The horizontal component of (*) yields

$$\begin{aligned}(\mathcal{D}_j H^i) \omega^j &= \omega^i + \beta^i, \\ (\mathcal{D}_{j_{r+1}} I_{j_1 \dots j_r}^\alpha) \omega^{j_{r+1}} &= I_{j_1 \dots j_r j_{r+1}}^\alpha \omega^{j_{r+1}} + \psi_{j_1 \dots j_r}^\alpha.\end{aligned}$$

The above equations for phantom invariants can be solved for the independent horizontal invariantized Maurer-Cartan forms. Substitute the expression for these into the above equations for non-phantom invariants to derive the recurrence formulas

$$\begin{aligned}\mathcal{D}_j H^i &= \delta_j^i + P_j^i, \\ \mathcal{D}_{j_{r+1}} I_{j_1 \dots j_r}^\alpha &= I_{j_1 \dots j_r j_{r+1}}^\alpha + M_{j_1 \dots j_r, j_{r+1}}^\alpha.\end{aligned}$$

SYZYGIES.

Theorem. *A complete system of basic syzygies is given by*

$$\begin{aligned} \mathcal{D}_{j_1 \cdots j_r} I_{k_1 \cdots k_s}^\alpha &= c_{j_1 \cdots j_r k_1 \cdots k_s}^\alpha + M_{k_1 \cdots k_s, j_1 \cdots j_r}^\alpha, \\ \mathcal{D}_{j_1 \cdots j_r} I_{k_1 \cdots k_s l_1 \cdots l_t}^\alpha - \mathcal{D}_{k_1 \cdots k_s} I_{j_1 \cdots j_r l_1 \cdots l_t}^\alpha \\ &= M_{k_1 \cdots k_s l_1 \cdots l_t, j_1 \cdots j_r}^\alpha - M_{j_1 \cdots j_r l_1 \cdots l_t, k_1 \cdots k_s}^\alpha, \end{aligned}$$

where $c_{j_1 \cdots j_r k_1 \cdots k_s}^\alpha$ is a phantom differential invariant and $I_{j_1 \cdots j_r l_1 \cdots l_t}^\alpha, I_{k_1 \cdots k_s l_1 \cdots l_t}^\alpha$ are normalized differential invariants with $\{j_1 \cdots j_r\} \cap \{k_1 \cdots k_s\} = \emptyset$.

KP SYMMETRIES CONTINUED:

Horizontal invariantized Maurer-Cartan forms

$$\begin{aligned}\alpha_{ijkl} &= \pi_{H\iota}(\tau_{,ijkl}), & \beta_{ijkl} &= \pi_{H\iota}(\xi_{,ijkl}), \\ \gamma_{ijkl} &= \pi_{H\iota}(\eta_{,ijkl}), & \zeta_{ijkl} &= \pi_{H\iota}(\varphi_{,ijkl}),\end{aligned}$$

satisfy the lifted infinitesimal determining equations.

\implies

A basis for these is provided by

$$\alpha_{T^n}, \quad \beta_{T^n}, \quad \gamma_{T^n}, \quad n \geq 0.$$

Consequently,

$$\alpha_{T^n X^p Y^q U^r} = 0, \quad \text{if } (p, q, r) \neq (0, 0, 0);$$

$$\beta_{T^n X} = \frac{1}{3}\alpha_{T^{n+1}}, \quad \beta_{T^n Y} = -\frac{2}{3}\epsilon\gamma_{T^{n+1}},$$

$$\beta_{T^n Y Y} = -\frac{4}{9}\epsilon\beta_{T^{n+2}}, \quad \gamma_{T^n Y} = \frac{1}{3}\alpha_{T^{n+1}},$$

$$\beta_{T^n X^p Y^q U^r} = 0, \quad \gamma_{T^n X^p Y^q U^r} = 0, \quad \text{for all other choices of } (p, q, r);$$

$$\zeta_{T^n} = \frac{2}{3}\beta_{T^{n+1}} - \frac{2}{3}\sum_{s=0}^n \binom{n}{s} I_{s00} \alpha_{T^{n-s+1}},$$

$$\zeta_{T^n X} = \frac{2}{9}\alpha_{T^{n+2}} - \frac{2}{3}\sum_{s=0}^n \binom{n}{s} I_{s10} \alpha_{T^{n-s+1}},$$

$$\zeta_{T^n Y} = -\frac{4}{9}\epsilon\gamma_{T^{n+2}} - \frac{2}{3}\sum_{s=0}^n \binom{n}{s} I_{s01} \alpha_{T^{n-s+1}},$$

$$\zeta_{T^n Y Y} = -\frac{4}{27}\alpha_{T^{n+3}} - \frac{2}{3}\sum_{s=0}^n \binom{n}{s} I_{s02} \alpha_{T^{n-s+1}},$$

$$\zeta_{T^n X^p Y^q} = -\frac{2}{3}\sum_{s=0}^n \binom{n}{s} I_{spq} \alpha_{T^{n-s+1}}, \quad \text{for all other choices of } (p, q),$$

$$\zeta_{T^n U} = -\frac{2}{3}\alpha_{T^{n+1}}, \quad \zeta_{T^n X^p Y^q U^r} = 0, \quad \text{if } r \geq 2.$$

CORRECTION TERMS:

$$\begin{aligned}
\psi_{pqr} &= \frac{2}{9}\delta_{q1}\delta_{r0}\alpha_{T^{p+2}} - \frac{4}{9}\delta_{q0}\delta_{r1}\epsilon\gamma_{T^{p+2}} - \frac{8}{27}\delta_{q0}\delta_{r2}\epsilon\alpha_{T^{p+3}} \\
&\quad - \sum_{s=0}^p \binom{p}{s} \left(\frac{2+q+2r}{3} + \frac{p-s}{s+1} \right) I_{p-s,q,r}\alpha_{T^{s+1}} \\
&\quad + \frac{2}{9}\epsilon r(r-1) \sum_{s=0}^p \binom{p}{s} I_{p-s,q+1,r-2}\alpha_{T^{s+2}} \\
&\quad - \sum_{s=1}^p \binom{p}{s} I_{p-s,q+1,r}\beta_{T^s} - \sum_{s=1}^p \binom{p}{s} I_{p-s,q,r+1}\gamma_{T^s} \\
&\quad + \frac{2}{3}\epsilon r \sum_{s=0}^p \binom{p}{s} I_{p-s,q+1,r-1}\gamma_{T^{s+1}}
\end{aligned}$$

RECURRENCE FORMULAS:

$$\begin{aligned}
d_H H^1 &= \omega^t + \alpha, & d_H H^2 &= \omega^x + \beta, & d_H H^3 &= \omega^y + \gamma, \\
d_H I_{000} &= I_{100}\omega^t + I_{010}\omega^x + I_{001}\omega^y - \frac{2}{3}I_{000}\alpha_T + \frac{2}{3}\beta_T, \\
d_H I_{100} &= I_{200}\omega^t + I_{110}\omega^x + I_{101}\omega^y - \frac{5}{3}I_{100}\alpha_T - \frac{2}{3}I_{000}\alpha_{TT} \\
&\quad - I_{010}\beta_T + \frac{2}{3}\beta_{TT} - I_{001}\gamma_T, \\
d_H I_{010} &= I_{110}\omega^t + I_{020}\omega^x + I_{011}\omega^y - I_{010}\alpha_T + \frac{2}{9}\alpha_{TT}, \\
d_H I_{001} &= I_{101}\omega^t + I_{011}\omega^x + I_{002}\omega^y - \frac{4}{3}I_{001}\alpha_T \\
&\quad + \frac{2}{3}\epsilon I_{010}\gamma_T - \frac{4}{9}\epsilon\gamma_{TT}, \\
d_H I_{200} &= I_{300}\omega^t + I_{210}\omega^x + I_{201}\omega^y - \frac{8}{3}I_{200}\alpha_T - \frac{7}{3}I_{100}\alpha_{TT} \\
&\quad - \frac{2}{3}I_{000}\alpha_{TTT} - 2I_{110}\beta_T - I_{010}\beta_{TT} + \frac{2}{3}\beta_{TTT} \\
&\quad - 2I_{101}\gamma_T - I_{001}\gamma_{TT}, \\
d_H I_{110} &= I_{210}\omega^t + I_{120}\omega^x + I_{111}\omega^y - 2I_{110}\alpha_T - I_{010}\alpha_{TT} \\
&\quad + \frac{2}{9}\alpha_{TTT} - I_{020}\beta_T - I_{011}\gamma_T, \\
d_H I_{101} &= I_{201}\omega^t + I_{111}\omega^x + I_{102}\omega^y - \frac{7}{3}I_{101}\alpha_T - \frac{4}{3}I_{001}\alpha_{TT} \\
&\quad - I_{011}\beta_T + \left(\frac{2}{3}\epsilon I_{110} - I_{002}\right)\gamma_T + \frac{2}{3}\epsilon I_{010}\gamma_{TT} - \frac{4}{9}\epsilon\gamma_{TTT}, \\
d_H I_{020} &= I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y - \frac{4}{3}I_{020}\alpha_T, \\
d_H I_{011} &= I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y - \frac{5}{3}I_{011}\alpha_T + \frac{2}{3}\epsilon I_{020}\gamma_T, \\
&\quad \vdots
\end{aligned}$$

We impose normalizations of the invariants so that the resulting phantom recurrence relations can be solved for the basis of invariantized Maurer-Cartan forms. For this we let

$$\begin{aligned}
H^1 &\longmapsto 0, & H^2 &\longmapsto 0, & H^3 &\longmapsto 0, \\
I_{000} &\longmapsto 0, & I_{100} &\longmapsto 0, & I_{010} &\longmapsto 0, \\
I_{001} &\longmapsto 0, & I_{200} &\longmapsto 0, & I_{101} &\longmapsto 0, \\
I_{020} &\longmapsto 1, & I_{011} &\longmapsto 0, & I_{002} &\longmapsto 0, \\
I_{i,0,0} &\longmapsto 0, & I_{i-1,0,1} &\longmapsto 0, & I_{i-2,0,2} &\longmapsto 0, \quad \text{for all } i \geq 3.
\end{aligned}$$

These yield the following expressions for the basic invariantized Maurer-Cartan forms:

$$\begin{aligned}
\alpha &= -\omega^t, & \beta &= -\omega^x, & \gamma &= -\omega^y, \\
\alpha_T &= \frac{3}{4}(I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y), & \alpha_{TT} &= \frac{9}{2}(I_{110}\omega^t + \omega^x), \\
\alpha_{TTT} &= \frac{27}{8}\epsilon(I_{012}\omega^x + I_{003}\omega^y), \dots, \\
\beta_T &= 0, & \beta_{TT} &= -\frac{3}{2}I_{110}\omega^x, & \beta_{TTT} &= -\frac{3}{2}I_{210}\omega^x, \dots, \\
\gamma_T &= -\frac{3}{2}\epsilon(I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y), & \gamma_{TT} &= 0, \\
\gamma_{TTT} &= \frac{9}{4}\epsilon(-I_{110}I_{111}\omega^t + (I_{111} - I_{110}I_{021})\omega^x - I_{110}I_{012}\omega^y), \dots,
\end{aligned}$$

The higher order invariantized Maurer–Cartan forms can be recursively determined from the equations

$$\begin{aligned}
\alpha_{T^{p+3}} &= \frac{27}{8}\epsilon(I_{p12}\omega^x + I_{p03}\omega^y) + \frac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s}I_{p-2,1,0}\alpha_{T^{s+2}} \\
&\quad - \frac{27}{8}\epsilon\sum_{s=1}^p\binom{p}{s}I_{p-s,1,2}\beta_{T^s} + \frac{9}{2}\sum_{s=0}^{p-1}\binom{p}{s}I_{p-s,1,1}\gamma_{T^{s+1}} \\
&\quad - \frac{27}{8}\epsilon\sum_{s=1}^p\binom{p}{s}I_{p-s,0,3}\gamma_{T^s}, \\
\beta_{T^{p+1}} &= -\frac{3}{2}I_{p10}\omega^x + \frac{3}{2}\sum_{s=1}^{p-1}\binom{p}{s}I_{p-s,1,0}\beta_{T^s}, \\
\gamma_{T^{p+2}} &= \frac{9}{4}\epsilon I_{p11}\omega^x - \frac{9}{4}\epsilon\sum_{s=1}^{p-1}\binom{p}{s}I_{p-s,1,1}\beta_{T^s} + \frac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s}I_{p-s,1,0}\gamma_{T^{s+1}}.
\end{aligned}$$

Recurrence formulas between the differentiated and normalized invariants:

$$\begin{aligned}
\mathcal{D}_t I_{110} &= I_{210} - \frac{3}{2} I_{110} I_{120}, \\
\mathcal{D}_x I_{110} &= I_{120} - \frac{3}{2} I_{110} I_{030} + \frac{3}{4} \epsilon I_{012}, \\
\mathcal{D}_y I_{110} &= I_{111} - \frac{3}{2} I_{110} I_{021} + \frac{3}{4} \epsilon I_{003}, \\
\mathcal{D}_t I_{210} &= I_{310} - \frac{9}{4} I_{210} I_{120} + \frac{3}{2} \epsilon I_{111}^2 + \frac{9}{8} I_{111} I_{003} + 12 I_{110}^2, \\
\mathcal{D}_x I_{210} &= I_{220} - \frac{9}{4} I_{210} I_{030} + \frac{3}{4} \epsilon I_{112} + \frac{3}{2} \epsilon I_{111} I_{021} + \frac{9}{8} I_{003} I_{021} + \frac{27}{2} I_{110}, \\
\mathcal{D}_y I_{210} &= I_{211} - \frac{9}{4} I_{210} I_{021} + \frac{3}{2} \epsilon I_{111} I_{012} + \frac{3}{4} \epsilon I_{103} + \frac{9}{8} I_{012} I_{003}, \\
\mathcal{D}_t I_{120} &= I_{220} + \frac{3}{2} \epsilon I_{111} I_{021} - \frac{7}{4} I_{120}^2 + 6 I_{110}, \\
\mathcal{D}_x I_{120} &= I_{130} - \frac{7}{4} I_{120} I_{030} + \frac{3}{2} \epsilon I_{021}^2 + 6, \\
\mathcal{D}_y I_{120} &= I_{121} - \frac{7}{4} I_{120} I_{021} + \frac{3}{2} \epsilon I_{021} I_{012}, \\
\mathcal{D}_t I_{111} &= I_{211} - (3 I_{120} - \frac{3}{2} \epsilon I_{012}) I_{111}, \\
\mathcal{D}_x I_{111} &= I_{121} - (I_{120} - \frac{3}{2} \epsilon I_{012}) I_{021} - 2 I_{111} I_{030}, \\
\mathcal{D}_y I_{111} &= I_{112} - (I_{120} - \frac{3}{2} \epsilon I_{012}) I_{012} - 2 I_{111} I_{021}, \\
&\vdots
\end{aligned}$$

Syzygies for $I_{110}, I_{030}, I_{021}, I_{012}, I_{003}$:

$$\mathcal{D}_y I_{012} - \mathcal{D}_x I_{003} + \frac{3}{4} I_{012} I_{021} - 2I_{030} I_{003} = 0,$$

$$\mathcal{D}_x I_{021} - \mathcal{D}_y I_{030} + \frac{5}{4} I_{021} I_{030} = 0,$$

$$\mathcal{D}_y I_{021} - \mathcal{D}_x I_{012} - \frac{1}{2} I_{021}^2 - \frac{3}{4} I_{012} I_{030} - 2\epsilon = 0,$$

$$\begin{aligned} \mathcal{D}_x \mathcal{D}_x I_{110} - \mathcal{D}_t I_{030} - \frac{3}{4} \epsilon \mathcal{D}_y I_{021} + \frac{3}{2} I_{110} \mathcal{D}_x I_{030} + 2I_{030} \mathcal{D}_x I_{110} \\ - \frac{9}{8} \epsilon I_{021}^2 + \frac{3}{16} \epsilon I_{030} I_{012} + \frac{3}{4} I_{030}^2 I_{110} - \frac{9}{2} = 0, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_x \mathcal{D}_y I_{110} - \mathcal{D}_t I_{021} + I_{030} \mathcal{D}_y I_{110} + I_{021} \mathcal{D}_x I_{110} + \frac{3}{2} I_{110} \mathcal{D}_y I_{030} \\ - \frac{3}{4} \epsilon \mathcal{D}_x I_{003} - \frac{9}{8} I_{110} I_{030} I_{021} - \frac{3}{4} \epsilon I_{030} I_{003} - \frac{9}{8} \epsilon I_{021} I_{012} = 0, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_x \mathcal{D}_y I_{110} - \mathcal{D}_t I_{021} + I_{030} \mathcal{D}_y I_{110} + I_{021} \mathcal{D}_x I_{110} + \frac{3}{2} I_{110} \mathcal{D}_x I_{021} \\ + \frac{3}{4} I_{110} I_{030} I_{021} - \frac{3}{4} \epsilon \mathcal{D}_x I_{003} - \frac{9}{8} \epsilon I_{021} I_{012} - \frac{3}{4} \epsilon I_{030} I_{003} = 0, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_y \mathcal{D}_y I_{110} - \mathcal{D}_t I_{012} + \frac{3}{2} I_{021} \mathcal{D}_y I_{110} - \frac{3}{4} I_{012} \mathcal{D}_x I_{110} \\ + \left(\frac{3}{2} \mathcal{D}_x I_{012} + \frac{3}{4} I_{021}^2 + \epsilon \right) I_{110} - \frac{3}{4} \epsilon \mathcal{D}_y I_{003} - \frac{15}{16} \epsilon I_{012}^2 = 0, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_y \mathcal{D}_y \mathcal{D}_y I_{110} - \mathcal{D}_t \mathcal{D}_x I_{003} - 2I_{030} \mathcal{D}_t I_{003} + 3I_{021} \mathcal{D}_t I_{012} - 2I_{003} \mathcal{D}_t I_{030} \\ + \left(2\mathcal{D}_y I_{021} - \frac{7}{4} I_{021}^2 \right) \mathcal{D}_y I_{110} + \left(\frac{21}{16} I_{012} I_{021} - \frac{5}{4} \mathcal{D}_y I_{012} \right) \mathcal{D}_x I_{110} \\ + \left(\frac{3}{2} \mathcal{D}_y \mathcal{D}_y I_{021} - \frac{15}{4} I_{021} \mathcal{D}_y I_{021} - \frac{15}{8} \mathcal{D}_y I_{012} I_{030} + \frac{63}{32} I_{021} I_{012} I_{030} \right. \\ \left. + \frac{3}{4} I_{021}^3 + 6\epsilon I_{021} \right) I_{110} - \frac{3}{4} \epsilon \mathcal{D}_y \mathcal{D}_y I_{003} + \frac{9}{8} \epsilon I_{021} \mathcal{D}_y I_{003} - \frac{57}{16} \epsilon I_{012} \mathcal{D}_y I_{012} \\ \left. + \frac{3}{4} \epsilon I_{003} \mathcal{D}_y I_{021} - \frac{3}{8} \epsilon I_{021}^2 I_{003} - \frac{3}{2} I_{003} + \frac{9}{64} \epsilon I_{021} I_{012}^2 = 0. \end{aligned}$$

Theorem. *The differential invariants I_{110} , I_{021} , I_{003} form a generating set for the algebra of differential invariants for the KP symmetry pseudogroup.*

Specifically,

$$I_{110} = u_{xx}^{-3/2} \left(u_{tx} + \frac{3}{2} u u_{xx} + \frac{3}{2} u_x^2 + \frac{3}{4} \epsilon u_{yy} \right),$$

$$I_{021} = u_{xx}^{-5/2} (u_{xx} u_{xxy} - u_{xy} u_{xxx}),$$

$$I_{003} = u_{xx}^{-5} (u_{xx}^3 u_{yyy} - 3u_{xx}^2 u_{xy} u_{xyy} + 3u_{xx} u_{xy}^2 u_{xxy} - u_{xy}^3 u_{xxx}).$$

Invariant differential operators:

$$\mathcal{D}_t = u_{xx}^{-3/4} D_t + \frac{3}{4} u_{xx}^{-11/4} (2u u_{xx}^2 - \epsilon u_{xy}^2) D_x + \frac{3}{2} \epsilon u_{xy} u_{xx}^{-7/4} D_y,$$

$$\mathcal{D}_x = u_{xx}^{-1/4} D_x,$$

$$\mathcal{D}_y = -u_{xx}^{-3/2} u_{xy} D_x + u_{xx}^{-1/2} D_y.$$